Rudolf E. Kalman: Father of Mathematical Systems Theory

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Abstract

This paper is dedicated to the memory of Professor Rudolf E. Kalman.

Professor Rudolf Kalman was a friend and mentor to both of us whose profound insights deeply influenced our research. His contributions to Systems Theory are far too numerous to describe in detail in such a short note, so we will be content with outlining some of the key highlights.

The field of Mathematical Systems Theory was essentially created by Professor Rudolf Kalman, and has had a huge impact on the way all modern systems and control is being carried out today. Indeed, the contribution to science and technology is so great that we feel it should be put into the context of the history of science. In his book, *The Structure of Scientific Revolutions,* Thomas Kuhn made a distinction between *normal science*, the science where incremental progress is made, and *revolutionary science*. Part of the distinction is based on comparing puzzle solving (which normal science usually does), and the concept of a *paradigm shift*. Examples of such paradigm shifts include the Copernican revolution, the Newtonian revolution, general relativity, and quantum mechanics, each of which changed the universe of discourse. Kuhn argues that such revolutions occur when there are "anomalies" (i.e., crises) when explanations based on current science break down.

We believe that in the field of systems and control, there was such a crisis in the 1950's. There were several indicators of these anomalies:

1. Internal Stability: Feedback control systems designed from an external (input/output) point of view failed to recognize the presence of internal instabilities.

2. The approach to design of multi-input/multi-output systems was essentially a reduction to a sequence of single-input/single-output systems through a decoupling technique.

3. The attempts to deal with the Wiener filtering problem in the nonstationary situation (Zadeh-Ragazzini) leading to some analogue of the Wiener-Hopf equation was not very successful since no procedure analogous to spectral factorization was available.

The resulting scientific revolution, in the way theories of systems and control developed, was almost single-handedly carried out by Rudolf Kalman. There are several key points to be sketched. Firstly, there was the reconciliation of the external and internal point of view of systems; namely, the introduction of the fundamental problem of *realization* (representation) theory of input/output maps and the concomitant fact that the input/output map captures the reachable and observable

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part of the underlying dynamical system. The dynamical viewpoint is made explicit, and Kalman employs the mathematics of algebra and differential equations; complex function theory (transfer functions) plays a secondary role in the Kalman approach. The adoption of a new language, namely algebra and differential equations, immediately puts the multi-input/ multi-output systems in the same footing as single-input/ single-output systems. Further, since most of the problems in Kalman's set-up could be reduced to those involving matrices (Kalman's famous notation was (F, G, H)), and the computer was just becoming a very powerful computational tool widely available to the general control community, this approach was perfect for taking advantage of the computer revolution. In a similar way, the Wiener filtering problem is reformulated as a problem in the theory of conditional Markov (Gaussian) processes by realizing a finite-dimensional Gaussian process as a linear function of a Gauss-Markov process. In the stationary situation, the spectral density of the output process is now restricted to rational spectral densities but the nonstationary situation can be handled effortlessly. Moreover, the equilibrium properties of the filter, namely the map from the observations to the conditional distribution of the "state" given the observations can be related to the intrinsic structural properties of the Gauss-Markov process, namely, reachability, observability, and their relation to stability. There has been no greater contribution to signals and systems than the ubiquitous Kalman filter. Further, the external view of stability, leading to the positive real lemma can be reconciled with the internal view of stability theory, namely Lyapunov theory.

This change in viewpoint, the shift to algebra, differential equations and Markov processes (the concept of Markovianity is intimately related to the concept of state) makes it natural to connect to the calculus of variations (especially the Caratheodory viewpoint: Hamilton-Jacobi Theory and dynamic programming) and leads to the formulation of the Linear Quadratic, Gaussian (LQG) problem both over a finite and infinite time. Its solution leading to the separation theorem for Stochastic Control where there is a separation between filtering and optimal control is surely one of the crowning achievements of this new radical approach for control and systems.

The Kalman revolution also brought in concepts of geometric invariant theory when looking at *families of systems* and their global properties. More precisely, Kalman described systems in an invariant (coordinate free) manner, which led him to consider the symmetries and invariants of linear time invariant systems. When when considers $n \times n$ matrices, it is not very fruitful to treat them simply as an array of n^2 numbers, but instead in terms of the action of $GL(n)$ acting on $M(n)$, the linear space of $n \times n$ matrices. In algebraic terms this leads to canonical forms (i.e., Jordan forms), and to the decomposition of $\mathbb{R}^n(\mathbb{C}^n)$ into a direct sum of invariant subspaces. The dimensions of the invariant subspaces are the invariants of the systems. A deeper question is whether the structure of a variety can be induced on the quotient space via the equivalence relation induced by the group action. The famous work of Mumford shows that, in order to do this, the nilpotent part of the decomposition of the matrix into a semi-simple plus nilpotent matrix needs to be thrown out. A more difficult question is to carry through this *geometric invariant theory* when we have pairs of matrices, (F, G) under the action of an algebraic group. It turns out that the space of reachable pairs is precisely the set of stable points (in the sense of geometric invariant theory) under the action of $GL(n)$ (change of basis in the state space), and thus the quotient has the nice structure of algebraic variety in the single-input case, and quasi-projective in the multi-input case. This work rules out the existence of global continuous canonical forms in the multi-input setting, in contrast to the single-input case in which one has, for example, the control canonical form. Finally, the Kalman quotient space has the property of being a *moduli space*, i.e., a universal parameter space of linear time invariant systems.

A somewhat different question, again raised by Kalman, is to understand the invariants of the system (F, G) under the action of the semidirect product of $GL(n), GL(m)$ and the feedback group which leads to the Kronecker invariants. The algebro-geometric interpretation relates the Kronecker invariants to the decomposition of a vector bundle over complex projective line into a direct sum of line bundles. These issues are at the heart of problems of control (feedback, modeling and identification). The structural questions and the question of trying to unify automata theory and linear systems theory led Kalman to the definition of a linear system as a $K[z]$-module, and the study of linear systems is reduced to the study of modules (for example, over principal ideal of domains). Kalman had also proposed at about the same time the study of systems over rings, which became a rich area of research. One of the underlying motivations was to reduce computations for systems to computer algebra, as well as getting a deeper view of digital signal processing, and algebraic coding theory. It also revealed how far one could push the algebraic point of view. So for example, one could do pole placement for systems over a principal ideal domain (e.g., modeling systems with one delay), but note over systems over polynomial rings in two or more variables (modeling systems with multiple non-commensurate delays).

There are of course numerous other aspects of Rudolf Kalman's work that deserve mention. For example, Kalman's work on regression led to the study of the decomposition structure of a covariance matrix into a positive semidefinite matrix plus a diagonal matrix. There is the important message here. *One needs to identify the noise.* This work has had a profound impact on econometrics, one of Kalman's greatest interests.

The fruits of the Kalman Revolution continue. ***Researchers and practitioners do systems theory the Kalman way.*** We conclude by listing some of the issues that arise from the Kalman point of view several of which arise in the study of complex networks.

1. What should systems theory for systems which have both a temporal and spatial structure, namely, Systems on graphs with Markov structure look like?
2. What is the invariant way of thinking about systems evolving on networks modeled as weighted graphs?
3. What are the "proper" notions of robustness of networks (weighted graphs)?
4. Can one formulate a natural geometric structure on families of graphs, and develop some universal parameter (moduli) space?

One can only look in awe at the remarkable career of the greatest systems scientist of our time. Rest in peace Rudy. Your work continues.

Biographical Sketch: Sanjoy Mitter

Sanjoy K. Mitter joined MIT in 1969 where he has been a Professor of Electrical Engineering since 1973. He was the Director of the MIT Laboratory for Information and Decision Systems from 1981 to 1999. He was also a Professor of Mathematics at the Scuola Normale, Pisa, Italy from 1986 to 1996. He has held visiting positions at Imperial College, London; University of

Groningen, Holland; INRIA, France; Tata Institute of Fundamental Research, India and ETH, Zürich, Switzerland; and several American universities. He was the Ulam Scholar at Los Alamos National Laboratories in April 2012 and the John von Neumann Visiting Professor in Mathematics at the Technical University of Munich, Germany from May-June 2012. He was the McKay Professor at the University of California, Berkeley in March 2000, and held the Russell-Severance-Springer Chair in Fall 2003. Dr. Mitter is a Fellow of the IEEE and IFAC. He was elected to the National Academy of Engineering in 1988. He was elected a Foreign Member of Istituto Veneto di Scienze, Lettere ed Arti in 2003 and a Foreign Fellow of the Indian National Academy of Engineering in 2015. He is the winner of the IEEE Control Systems Award for 2000, and was awarded the AACC Richard E. Bellman Control Heritage Award for 2007. He was the recipient of the IEEE Eric E. Sumner Award for 2015. His current research interests are communication and control in a networked environment, the relationship of statistical and quantum physics to information theory, and control and autonomy and adaptiveness for integrative organization.

Biographical Sketch: Allen Tannenbaum

Allen Tannenbaum is an applied mathematician and presently Distinguished Professor of Computer Science and Applied Mathematics & Statistics at the State University of New York at Stony Brook. He is also Investigator of Medical Physics at Memorial Sloan Kettering Cancer Center in New York City. He has held a number of other positions in the United States, Israel (Weizmann Institute), and Canada (McGill University). Tannenbaum's work has won several awards including IEEE Fellow, O. Hugo Schuck Award of the American Automatic Control Council in 2007 (shared with S. Dambreville and Y. Rathi), and the George Taylor Award for Distinguished Research from the University of Minnesota in 1997. He has given numerous plenary talks at major conferences including the American Mathematical Society, Society for Industrial and Applied Mathematics(SIAM) Conference on Control in 1998, IEEE Conference on Decision and Control of the IEEE Control Systems Society in 2000, and the International Symposium on the Mathematical Theory of Networks and Systems (MTNS) in 2012. His current research interests are in complex networks, the biology of cancer, and quantum information theory.