Journal of Mathematical Systems, Estimation, and Control
Vol. 6, No. 2, 1996, pp. 1–26

Linear Systems over Noetherian Rings in the Behavioural Approach*

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Abstract

In this paper the properties of linear systems in the behavioural approach are analyzed in the case when the alphabet set is a finitely generated module over a Noetherian ring. This study is motivated by some possible applications to the theory of convolutional codes and of parametrized systems. The first class of systems that is considered is the class of controllable systems. Controllable systems admit a nice image representation and this seems to be of some use for the theory of convolutional codes. The second part of the paper is devoted to the analysis of autonomous systems, which constitute a special class of non controllable systems. In this class of systems it is possible to find a natural extension of the classical Rouchaleau-Kalman-Wyman theorem to the behavioural approach. Finally the realizability of a system, that corresponds to the fact that the canonical state space is a finitely generated module, is analyzed. This concept is strictly connected with the problem of representing the trellis diagram of a convolutional code.

Key words: Noetherian rings and modules, linear systems over rings, behavioural approach, state space, controllable systems

AMS Subject Classifications: 93B25, 93C25, 93C55

1 Introduction

When, in the early '70s, systems theory showed to be a very powerful tool for the study of dynamical systems, many researchers tried to extend the results found in this field to more general frameworks, such as systems over rings. These attempts were motivated both by pure theoretical reasons and by some possible applications to real world problems such as for instance

*Received July 18, 1994; received in final form December 23, 1994. Summary appeared in Volume 6, Number 2, 1996.

parametrized systems [4, 13], time delay systems [15], distributed systems [3] and the coding theory [5].

The strict relationship between system theory and coding theory was first pointed out by Forney in his study of convolutional codes [11]. Actually it was noted that an encoder for a convolutional code can be seen as a linear system over a finite field. Consequently much knowledge on linear systems could be used in the study of convolutional codes. Since codes over finite fields can be sometimes too restrictive, it seemed natural to develop a theory of convolutional codes also over groups and rings. This produced a remarkable contribution to linear systems over rings.

In the last years the behavioural approach to dynamical systems has been the object of much investigation. Actually, this approach constitutes an alternative framework for modelling phenomena that seems to be more effective when there exists an unclear distinction between causes and effects (see [25]). Also optimal control techniques [26] and modelling procedures [24, 14, 1] have been proposed in this setup.

Very recently it has been also realized that the behavioural approach to system theory is a very useful framework where one can develop a more general theory of convolutional codes (see [12, 10]). Therefore it seems quite interesting to extend this approach, originally introduced for systems over general fields and in particular over the real field, to more general structures, such as groups or rings. In [19, 12] it is shown that many properties of linear systems over fields hold true even for systems over noncommutative groups. The most important result in this sense is the fact that even in this generality it is possible to define a canonical state space group. It is shown moreover how the state group can be connected with the trellis diagrams of the convolutional code and how it can be used for the synthesis of a canonical encoder.

Another interesting field in which systems over rings in the behavioural approach could be applied is the theory of parametrized systems. These are the systems whose behaviours are not fixed but depend on a set of parameters. It seems that the study of this class of systems will give some light to the structure of time varying systems. A subsequent paper will be devoted to this kind of questions.

In this paper we will study linear systems in the behavioural approach when the signal alphabet is a finitely generated module over a Noetherian ring. The choice of Noetherian rings is motivated by various reasons. First Noetherian property seems to be sufficiently general to cover all rings of common interest. Only dealing with parametrized systems more general rings could be needed. Another reason is connected with the canonical state space and the problem of realizability. As shown in this paper, under weak hypotheses, the canonical state space of a systems over a Noetherian ring is a finitely generated module and this provides a nice representation of the so called trellis diagram, representation that can be useful in practice. Finally the Noetherian property seems to be the weakest requirement allowing to characterize systems admitting an image representation by the controllability property as it happens for systems over fields [25].

The most relevant ring in the theory of convolutional code is the ring of integers $\mathbb{Z}$, since alphabet sets that are often used in this context are finite Abelian groups. These can be always considered finite generated modules over $\mathbb{Z}$. In any case in this paper we focus our attention to questions related to the finitely generated structure of the signal alphabet. This can be useful even for finite groups. Actually, when the alphabet set is a finite group and contains a large number of elements, it can be useful to analyse if it can be efficiently generated by a small number of elements. This can simplify the analysis of the properties of the code and can be useful for the synthesis of encoders, decoders and trellis diagrams.

We give now a brief outline of the paper. In section two we present the definition of linear system over a Noetherian ring and the definitions of complete and strongly complete system. In section three we list various possible definitions of controllability which appear in literature and we show how they are related each other in our context. We show moreover that any complete and controllable system admits an image representation and finally we introduce the concept of controllable subsystem. In section four we introduce the concepts of autonomous system and of finitely generated system and we investigate the relations between these two properties. We also present an interesting result showing that there exists a strict relationship between an autonomous system over a Noetherian domain and a corresponding linear system over the field of fractions. This seems to give a first extension to the behavioural approach of the classical Rouchaleau-Kalman-Wyman theorem [5, 21, 20]. In section five we find some conditions ensuring the realizability of a system, i.e. ensuring that its canonical state space module is finitely generated. Note that the fact that the canonical state space is a finitely generated module seems to be the natural translation of the realizability problem as it has been proposed in the classical systems over rings setup [21] to the behavioural approach.

Finally note that the proof of the results presented in this paper are essentially different from the proof of the analogous results for systems over fields. There are two main reasons. First in the context of systems over fields there exists a very powerful tool that is the theory of duality of vector spaces and such a theory can be extended from vector spaces to modules only partially. Moreover, the existence of the Smith canonical form for polynomial matrices with coefficients over a field simplifies drastically the proofs of the results on systems over fields.

2 Basic Definitions

In this section some basic concepts of behavioural theory of dynamical systems will be recalled and then some more specific ideas for systems over rings in this approach will be introduced.

Before it is necessary to introduce some notation. In this paper only commutative Noetherian rings will be considered. A commutative ring R is Noetherian if it satisfies the ascending chain condition, i.e. $\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \mathcal{I}_3 \subseteq \cdots$ is an increasing sequence of ideals in R, then there exists $n \in \mathbb{N}$ such that $\mathcal{I}_n = \mathcal{I}_{n+1} = \mathcal{I}_{n+2} = \cdots$. It can be seen that R is Noetherian if and only if each ideal in R is finite generated. For the elementary results on commutative Noetherian rings needed in this paper, refer to [2]. Note only that if W is a finitely generated module over a Noetherian ring, then W satisfies the ascending chain condition on its submodules and that every submodule of W is finitely generated.

Given a ring R, $x \in R$ is said to be a zero-divisor, if there exists $y \neq 0$ in R such that $xy = 0$. If R is a domain, then no nonzero element of R is zero-divisor.

The symbol $R[z, z^{-1}]$ we will denote the ring of all Laurent polynomials with coefficients in R, i.e. the ring of polynomials for which positive and negative powers of the indeterminate z are allowed. More formally the ring $R[z, z^{-1}]$ can be considered the ring of fraction of $R[z]$ with respect to the multiplicatively closed set $S = \{z^i : i \in \mathbb{N}\}$. It follows from Hilbert basis theorem and from prop. 7.3 in [2] that $R[z, z^{-1}]$ is Noetherian, if R is.

A dynamical system is defined as a triple $\Sigma = (T, W, \mathcal{B})$, where T is the time set, W is the signal alphabet, i.e. the space where the signals take their values and finally $\mathcal{B}$ is a subset of the set W^T of all the signals and it describes the dynamics of the system simply specifying what are the signals that are allowed. In this paper only a particular kind of dynamical systems will be considered, i.e. *linear shift-invariant* systems over Noetherian rings.

More precisely a linear shift-invariant system over a Noetherian ring R is a dynamical system such that:

- The time set T is the set of integers $\mathbb{Z}$. Dynamical systems whose time set is $\mathbb{Z}$ are called discrete.
- The signal alphabet W is a finite generated module over a Noetherian ring R.
- On the set of all signals $W^{\mathbb{Z}}$ can be introduced a module structure over the ring $R[z, z^{-1}]$ of the Laurent polynomials over R as follows:

  Both the sum of two signals in $W^{\mathbb{Z}}$ and the product of an element of R and a signal in $W^{\mathbb{Z}}$ are done pointwise while for every $w \in W^{\mathbb{Z}}$,

$z^h w \in W^{\mathbb{Z}}$ is defined as

$$(z^h w)(t) := w(t+h).$$

The product of a Laurent polynomial in $R[z, z^{-1}]$ and a signal in $W^{\mathbb{Z}}$ is defined by extending the previous definitions by linearity. More precisely if $p = \sum_{i=l}^{L} p_i z^i$ is a polynomial in $R[z, z^{-1}]$ and $w \in W^{\mathbb{Z}}$, then

$$(pw)(t) = \sum_{i=l}^{L} p_i w(t+i).$$

The behaviour of a linear shift-invariant system over R is a $R[z, z^{-1}]$ submodule of $W^{\mathbb{Z}}$.

In case that R is a field, we obtain the linear shift-invariant systems introduced by Willems in [25].

An important property of linear shift-invariant systems that is useful to consider is the completeness.

Definition 1 *Let $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be any dynamical system. Then*

1. *Σ is complete if*

$$w \in \mathcal{B} \quad \Leftrightarrow \quad w_{|I} \in \mathcal{B}_{|I} \text{ for all finite } I \subseteq \mathbb{Z}.$$

2. *Σ is L-complete with $L \in \mathbb{N}$ if*

$$w \in \mathcal{B} \quad \Leftrightarrow \quad w_{|[t,t+L]} \in \mathcal{B}_{|[t,t+L]} \text{ for all } t \in \mathbb{Z}.$$

3. *Σ is strongly complete if Σ is L-complete for some $L \in \mathbb{N}$.*

It is clear that a complete linear shift-invariant system is determined by the countable family of finitely generated R-submodules $\mathcal{B}_{|[-n,n]} \subseteq W^{2n+1}, n = 0, 1, 2, ...$, while if Σ is a L-complete linear shift-invariant system, then it is determined by the finitely generated R-submodule $\mathcal{B}_{|[0,L]} \subseteq W^{L+1}$.

As shown in [25], for linear shift-invariant systems over fields, completeness and strongly completeness are equivalent. This assertion can be generalized for systems over rings whose signal alphabet W is a module satisfying the descending chain condition, i.e. $\mathcal{M}_1 \supseteq \mathcal{M}_2 \supseteq \mathcal{M}_3 \supseteq \cdots$ is an decreasing sequence of submodule of W, then there exists $n \in \mathbb{N}$ such that $\mathcal{M}_n = \mathcal{M}_{n+1} = \mathcal{M}_{n+2} = \cdots$. Examples of such modules are given by finite dimensional vector spaces or by finite Abelian groups that can be considered modules over the ring $\mathbb{Z}$. The proof of this equivalence is essentially similar to the one given in [23] for systems over fields.

Proposition 1 *Let $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant complete system over a ring R. If the module W satisfies the descending chain condition, then Σ is strongly complete.*

Proof: Consider the modules

$$\mathcal{M}_n := \{w(n) : w \in \mathcal{B}, w_{|[0,n-1]} = 0\}.$$

It is clear that $\mathcal{M}_1 \supseteq \mathcal{M}_2 \supseteq \mathcal{M}_3 \supseteq \cdots$ and so there exists $N \in \mathbb{N}$ such that $\mathcal{M}_N = \mathcal{M}_{N+1} = \mathcal{M}_{N+2} = \cdots$. We want to show that Σ is N-complete. To this purpose it is enough to show that if $w_{|[t,t+N]} \in \mathcal{B}_{|[t,t+N]}$ for all $t = 0, 1, \ldots, n$, then we have that $w_{|[0,n+N]} \in \mathcal{B}_{|[0,n+N]}$. We show this by induction on n. For $n = 0$ this is true. Suppose that the assertion is true for $n \Leftrightarrow 1$ and suppose that $w_{|[t,t+N]} \in \mathcal{B}_{|[t,t+N]}$ for all $t = 0, 1, \ldots, n$. Then, by induction, we have that $w_{|[0,n+N-1]} \in \mathcal{B}_{|[0,n+N-1]}$ and so there exists $w_1 \in \mathcal{B}$ such that $w_{1|[0,n+N-1]} = w_{|[0,n+N-1]}$. On the other hand, since $w_{|[n,n+N]} \in \mathcal{B}_{|[n,n+N]}$, then there exists $w_2 \in \mathcal{B}$ such that $w_{2|[n,n+N]} = w_{|[n,n+N]}$. Let $w' := w_2 \Leftrightarrow w_1 \in \mathcal{B}$. Then $w'_{|[n,n+N-1]} = 0$. Since $\mathcal{M}_N = \mathcal{M}_{N+n}$, then there exists $\bar{w} \in \mathcal{B}$ such that $\bar{w}_{|[0,n+N-1]} = 0$ and $\bar{w}(n+N) = w'(n+N)$. Let $w'' := w_1 + \bar{w} \in \mathcal{B}$. Then $w''_{|[0,n+N-1]} = w_{1|[0,n+N-1]} = w_{|[0,n+N-1]}$ and $w''(n+N) = w_1(n+N) + \bar{w}(n+N) = w_1(n+N) + w'(n+N) = w_1(n+N) + w_2(n+N) \Leftrightarrow w_1(n+N) = w_2(n+N)$. We can argue that $w''_{|[0,n+N]} = w_{|[0,n+N]}$ and so $w_{|[0,n+N]} \in \mathcal{B}_{|[0,n+N]}$. ■

Completeness and strongly completeness are not equivalent in general even for systems over the principal ideal domain $\mathbb{Z}$, as shown in [7]. In any case it seems that complete systems that are not strongly complete seem to be very pathological and very difficult to characterize.

3 Controllable Systems

We begin the study of linear shift-invariant systems over a Noetherian ring by analyzing the class of controllable systems. The notion of controllability considered in this paper is not connected with a state space realization, but is a property of the system itself. This property has been first introduced by Willems in [25]. Other notions of controllability have been introduced also in [12, 19] and they are not always equivalent. In this section various definitions of controllability will be presented and the connections between them will be analyzed.

Definition 2 *Let $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant system over a ring R. Then*

1. *Σ is zero-controllable if for all $w \in \mathcal{B}$, there exist $k \in \mathbb{N}$ and $w' \in \mathcal{B}$ such that*
$$w'_{|(-\infty,0]} = w_{|(-\infty,0]},\ w'_{|[k,+\infty)} = 0.$$

2. *Σ is symmetric controllable if for all $w \in \mathcal{B}$, there exist $h, k \in \mathbb{N}$ and $w', w'' \in \mathcal{B}$ such that*

$$\left\{ \begin{array}{ll} w'_{|(-\infty,0]} = w_{|(-\infty,0]}, & w'_{|[k,+\infty)} = 0, \\ w''_{|(-\infty,-h]} = 0, & w''_{|[0,+\infty)} = w_{|[0,+\infty)}. \end{array} \right.$$

3. *Σ is strongly zero-controllable if there exists $k \in \mathbb{N}$ such that for all $w \in \mathcal{B}$, there exists $w' \in \mathcal{B}$ such that*

$$w'_{|(-\infty,0]} = w_{|(-\infty,0]},\ w'_{|[k,+\infty)} = 0.$$

4. *Σ is controllable if for all $w_1, w_2 \in \mathcal{B}$, there exists $k \in \mathbb{N}$ and $w \in \mathcal{B}$ such that*

$$w_{|(-\infty,0]} = w_{1|(-\infty,0]},\ w_{|[k,+\infty)} = (z^{-k}w_2)_{|[k,+\infty)}.$$

5. *Σ is strongly controllable if there exists $k \in \mathbb{N}$ such that for all $w_1, w_2 \in \mathcal{B}$, there exists $w \in \mathcal{B}$ such that*

$$w_{|(-\infty,0]} = w_{1|(-\infty,0]},\ w_{|[k,+\infty)} = (z^{-k}w_2)_{|[k,+\infty)}.$$

It can be seen that strong zero-controllability and strong controllability are equivalent and so they will not be distinguished. Moreover strong controllability implies controllability that implies symmetric controllability that finally implies zero-controllability. This is summarized in the following scheme.

$$\begin{array}{c} \text{Strong zero-controllability} \\ \Updownarrow \\ \text{Strong controllability} \\ \Downarrow \\ \text{Controllability (Willems)} \\ \Downarrow \\ \text{Symmetric controllability} \\ \Downarrow \\ \text{Zero-controllability} \end{array}$$

Note that both strong controllability and controllability are the original definitions proposed by Willems in [25], while strong zero-controllability and symmetric controllability have been introduced by Trott in [22]. The symmetric version of zero-controllability, that could be called zeroreachability, can be introduced and connected with the other notions of controllability in an obvious way. The following proposition shows that if R is Noetherian, then symmetric controllability, controllability and strong controllability coincide. If moreover the system is strongly complete, then all the controllability notions are equivalent.

Proposition 2 *Let $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant system over a Noetherian ring R. Then Σ is symmetric controllable if and only if Σ is strongly controllable. If moreover Σ is strongly complete, then all notions of controllability are equivalent.*

Proof: Suppose that Σ is symmetric controllable. Let $\hat{\mathcal{B}}$ is the set of trajectories in $\mathcal{B}$ with finite support. Note that $\hat{\mathcal{B}}$ is a Noetherian module over $R[z, z^{-1}]$ (a module M is Noetherian if each submodule of M is finitely generated; see problem 10 pag. 85 in [2]). Let $w_1, \ldots, w_n$ a set of generators for $\hat{\mathcal{B}}$ and suppose that their supports are included in $[\Leftrightarrow N, N]$. Take now a $w \in \mathcal{B}$. We want to show that there exists $w' \in \mathcal{B}$ such that $w'_{|(-\infty,0]} = w_{|(-\infty,0]}$ and $w'_{|[2N,+\infty)} = 0$. By symmetric controllability it is easy to see that there exists $\hat{w} \in \hat{\mathcal{B}}$ and $w_1, w_2 \in \mathcal{B}$ such that $w_{1|(-\infty,0]} = 0$, $w_{2|[0,+\infty)} = 0$ and $w = \hat{w} + w_1 + w_2$. Then

$$\hat{w} = \sum_{j=1}^{n} \sum a_{ij} z^i w_j = \sum_{j=1}^{n} \sum_{|i| \leq N} a_{ij} z^i w_j + \sum_{j=1}^{n} \sum_{|i| > N} a_{ij} z^i w_j,$$

where $a_{ij} \in R$. Defining

$$\hat{w}' := \sum_{j=1}^{n} \sum_{|i| \leq N} a_{ij} z^i w_j \in \hat{\mathcal{B}},$$

$$w'_1 := w_1 + \sum_{j=1}^{n} \sum_{i > N} a_{ij} z^i w_j \in \mathcal{B}$$

and

$$w'_1 := w_1 + \sum_{j=1}^{n} \sum_{i < -N} a_{ij} z^i w_j \in \mathcal{B},$$

then we have that $w'_{1|(-\infty,0]} = 0$, $w'_{2|[0,+\infty)} = 0$ and $w = \hat{w}' + w'_1 + w'_2$. Define finally $w' := \hat{w}' + w'_2 = w \Leftrightarrow w'_1$. Then it is easy to verify that $w'_{|(-\infty,0]} = w_{|(-\infty,0]}$ and $w'_{|[2N,+\infty)} = 0$.

In order to prove the second assertion it is sufficient to show that if Σ is zero-controllable, then it is strongly controllable. Suppose that Σ is zero-controllable and let for all $n \in \mathbb{N}$

$$\mathcal{M}_n := \{w_{|[-L,0]} : w \in \mathcal{B},\ w_{|[n,+\infty)} = 0\}.$$

Then we have that $\mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \mathcal{M}_3 \subseteq \cdots \subseteq W^{L+1}$ and so, since R is Noetherian and W^{L+1} is finitely generated, there exists $N \in \mathbb{N}$ such that $\mathcal{M}_N = \mathcal{M}_{N+1} = \mathcal{M}_{N+2} = \cdots$. Take now any $w \in \mathcal{B}$. Then, since Σ is zero-controllable then $w_{|[-L,0]} \in \mathcal{M}_n$ for some n and so $w_{|[-L,0]} \in \mathcal{M}_N$. Therefore any trajectory in $\mathcal{B}$ can be controlled to zero in at most N steps and hence Σ is strongly controllable. ■

Presently it is not known if the second part of the previous proposition holds true for complete systems. In the following we give an example showing show that in general zero-controllability does not imply strong controllability.

Example Let $\Sigma = (\mathbb{Z}, \mathbb{R}, \mathcal{B})$ be a linear shift invariant system over the real field $\mathbb{R}$, where

$$\mathcal{B} := \{pw : p \in \mathbb{R}[z, z^{-1}]\}$$

and where w is any irrational trajectory in $\mathbb{R}^{\mathbb{Z}}$ (i.e. a trajectory such that pw has infinite support, for every polynomial $p \in \mathbb{R}[z, z^{-1}]$) such that $w_{|(0,+\infty)} = 0$. It is clear that Σ is a linear shift-invariant zero-controllable system. However it is not difficult to verify that it is not symmetric controllable, since the only trajectories in $\mathcal{B}$ with finite support is the zero trajectory. This implies that Σ is not strongly complete and so it is not complete either, since for linear shift invariant systems over fields completeness and strongly completeness coincide.

3.1 Image representation for controllable systems

As shown in [25], for linear shift-invariant complete and controllable systems over fields there exists a useful representation that is called image representation. More precisely the behaviour of these systems coincides with the image of a suitable linear operator that is called shift operator. We will see now that this representation holds true also for systems over Noetherian rings.

Let V and W be two modules over a ring R and let $\mathrm{Hom}(V, W)$ be the set of all the R-homomorphisms from V to W. The set $\mathrm{Hom}(V, W)[z, z^{-1}]$ of all Laurent polynomials with coefficients in $\mathrm{Hom}(V, W)$ can be defined in the usual way. This is not a ring but only an $R[z, z^{-1}]$-module. Given an $M \in \mathrm{Hom}(V, W)[z, z^{-1}]$

$$M = \sum_{i=l}^{L} M_i z^i,$$

where $M_i \in \mathrm{Hom}(V, W)$, we can associate an $R[z, z^{-1}]$-homomorphism Ψ_M from $V^{\mathbb{Z}}$ to $W^{\mathbb{Z}}$ in the following way:

If $v \in V^{\mathbb{Z}}$, then for all $t \in \mathbb{Z}$ we define

$$\Psi_M(v)(t) := \sum_{i=l}^{L} M_i v(t+i).$$

The homomorphisms defined in this way are called *shift operators* while a linear shift-invariant system admits an *image representation* if there exists a shift operator whose image coincides with the behaviour of the system.

The family of shift operators can be characterized in terms of continuity w.r. to a suitable topology defined on the signal spaces. More precisely consider the discrete topology on V and W and the product topology on the signal spaces $V^{\mathbb{Z}}$ and $W^{\mathbb{Z}}$. Such a topology is called pointwise convergence topology since a sequence $\{w_n\}_{n=1}^{\infty} \subseteq W^{\mathbb{Z}}$ converges to $w \in W^{\mathbb{Z}}$ if and only if the sequence $\{w_n(t)\}_{n=1}^{\infty} \subseteq W$ converges to $w(t) \in W$ in the discrete topology for all $t \in \mathbb{Z}$ and so if and only if $w_n(t)$ is eventually equal to $w(t)$ for all $t \in \mathbb{Z}$.

It is not difficult to prove that closed subsets of $W^{\mathbb{Z}}$ corresponds to complete behaviours. Note moreover that $W^{\mathbb{Z}}$ with this topology satisfies the first axiom of countability (see [16, pag. 92]) and therefore (see [6, pag. 218]) a subset $\mathcal{B}$ of $W^{\mathbb{Z}}$ is closed if and only if the fact that $\{w_n\}_{n=1}^{\infty} \subseteq \mathcal{B}$ converges to $w \in W^{\mathbb{Z}}$ implies that $w \in \mathcal{B}$. Moreover a map $\Phi : V^{\mathbb{Z}} \to W^{\mathbb{Z}}$ is continuous if and only if for every sequence $\{w_n\}_{n=1}^{\infty} \subseteq V^{\mathbb{Z}}$ converging to w we have that $\{\Phi(w_n)\}_{n=1}^{\infty} \subseteq W^{\mathbb{Z}}$ converges to $\Phi(w)$. It is possible to characterize the shift operators in terms of continuity w.r. to the pointwise topology.

Proposition 3 *Let V and W be modules over a ring R. Let Φ be an operator from $V^{\mathbb{Z}}$ to $W^{\mathbb{Z}}$ and consider in $V^{\mathbb{Z}}$ and $W^{\mathbb{Z}}$ the pointwise convergence topology. Then Φ is a continuous $R[z,z^{-1}]$-homomorphism from $V^{\mathbb{Z}}$ to $W^{\mathbb{Z}}$ if and only if Φ is a shift operator.*

Proof: Suppose that Φ is continuous. Consider for all $v \in V$ the signal $\delta_v \in V^{\mathbb{Z}}$ such that $\delta_v(0) = v$ and $\delta_v(t) = 0$ for all $t \neq 0$. Since the sequence $\{z^n \delta_v\}_{n=1}^{\infty}$ converges to zero, then, by continuity, the sequence $\{z^n \Phi(\delta_v)\}_{n=1}^{\infty}$ converges to zero. Analogously, since the sequence $\{z^{-n}\delta_v\}_{n=1}^{\infty}$ converges to zero, then the sequence $\{z^{-n}\Phi(\delta_v)\}_{n=1}^{\infty}$ converges to zero. We can argue that $\Phi(\delta_v)$ has finite support. Since Φ is an R-homomorphism, then for all $i \in \mathbb{Z}$ there exists $M_i \in \mathrm{Hom}(V,W)$ such that $\Phi(\delta_v)(i) = M_i v$ for all $v \in V$. Consider the polynomial $M := \sum M_i z^i$ in $\mathrm{Hom}(V,W)[z,z^{-1}]$.
We want to show that Φ coincides with the shift operator Ψ_M and consequently we have to show that for all $v \in (R^l)^{\mathbb{Z}}$ we have that $\Phi(v) = \Psi_M(v)$. If v has finite support, then this is true. For any $v \in (R^l)^{\mathbb{Z}}$ consider the

sequence $\{v_n\}_{n=1}^{\infty}$ defined as follows:

$$v_{n|[-n,n]} = v_{|[-n,n]},\ v_{n|(-\infty,-n)} = 0 \text{ and } v_{n|(n,+\infty)} = 0.$$

It is clear that $\{v_n\}$ converges to v and so, by continuity of Φ, $\{\Phi(v_n)\}$ converges to $\Phi(v)$. On the other hand, since v_n has finite support, then $\Phi(v_n) = \Psi_M(v_n)$ and so, for every $t \in \mathbb{Z}$, there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $\Psi_M(v_n)(t) = \Phi(v)(t)$. It is clear that, if n is big enough, then $\Psi_M(v_n)(t) = \Psi_M(v)(t)$.
Conversely it is easy to see that Ψ_M, where $M \in \mathrm{Hom}(V,W)[z,z^{-1}]$, is a continuous $R[z,z^{-1}]$ homomorphism from $V^{\mathbb{Z}}$ to $W^{\mathbb{Z}}$. Suppose that the sequence $\{v_n\}_{n=1}^{\infty} \subseteq (R^l)^{\mathbb{Z}}$ converges to v. Actually, we have that

$$\Psi_M(v_n)(t) = \sum_{i=l}^{L} M_i v_n(t+i)$$

and so, since there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $v_n(t+i) = v(t+i),\ \forall i = l, l+1, \ldots, L$, then for all $n > N$ we have

$$\Psi_M(v_n)(t) = \sum_{i=l}^{L} M_i v_n(t+i) = \sum_{i=l}^{L} M_i v(t+i) = \Psi_M(v)(t).$$

Consequently the sequence $\{\Psi_M(v_n)\}_{n=1}^{\infty} \subseteq (R^l)^{\mathbb{Z}}$ converges to $\Psi_M(v)$ and we have the continuity of Ψ_M. ■

The following theorem shows that complete controllable linear shift-invariant systems over Noetherian rings admit an image representation. The proof of this theorem is based on the proof of the strong controllability theorem in [12].

Theorem 1 *Let R be a Noetherian ring and let $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant complete system. Then Σ is symmetric controllable if and only if $\mathcal{B} = \mathrm{im}\ \Psi_M$ for some shift operator Ψ_M from $V^{\mathbb{Z}}$ to $W^{\mathbb{Z}}$, where V is a suitable finitely generated R-module.*

Proof: It is trivial to prove that if $\mathcal{B} = \mathrm{im}\ \Psi_M$, then Σ is symmetric controllable. Suppose conversely that Σ is symmetric controllable and so also strongly controllable. Then there exists $k \in \mathbb{N}$ such that for all $w \in \mathcal{B}$, there exists $w' \in \mathcal{B}$ such that

$$w'_{|(-\infty,0]} = w_{|(-\infty,0]},\ w'_{|[k,+\infty)} = 0.$$

Let

$$\mathcal{B}_{[0,k)} := \{w \in \mathcal{B} : w(t) = 0, \forall t \notin [0,k)\}.$$

Since R is Noetherian, then $\mathcal{B}_{[0,k)}$ is finitely generated over R. Let $u_1, \dots, u_l$ be a family of generators. Fix $V := R^l$ and let $e_1, \dots, e_l$ be the canonical basis in R^l. Let moreover M_t be the unique homomorphism in $\mathrm{Hom}(R^l, W)$ such that for all $i = 1, \dots, l$ we have $M_t(e_i) = u_i(-t)$ and define

$$M := \sum_{t=-k+1}^{0} M_t z^t$$

as an element in $\mathrm{Hom}(R^l, W)[z, z^{-1}]$. We want to show that $\mathcal{B} = \mathrm{im}\ \Psi_M$. Let $w \in \mathrm{im}\ \Psi_M$. Then $w = \Psi_M(v)$ for some $v \in (R^l)^{\mathbb{Z}}$. Let $v_h \in (R^l)^{\mathbb{Z}}$ be defined as follows:

$$v_h(t) = \begin{cases} v(t) & \text{if } |t| \leq h \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that, since the support of M is included in $(-k, 0]$, then we have that $w_h := \Psi_M(v_h)$ coincides with w in the interval $[-h + k, h]$. Since $w_h \in \mathcal{B}$ and since Σ is complete, then we have $w \in \mathcal{B}$.

Let $w \in \mathcal{B}$. First we show that $w = w_p + w_f$, where $w_p, w_f \in \mathcal{B}$ and $w_p(t) = 0$ for all $t \geq k$ and $w_f(t) = 0$ for all $t \leq 0$. Actually, by strong controllability, there exists $w_p \in \mathcal{B}$ such that $w_{p|(-\infty,0]} = w_{|(-\infty,0]}$, $w_{p|[k,+\infty)} = 0$ and so, if we define $w_f := w - w_p$ we have that w_p, w_f satisfy the conditions we required. We want to show now that $w_p, w_f \in \mathrm{im}\ \Psi_M$. Consider the sequence $w_0, w_1, w_2, \dots$ such that $w_{i|(-\infty,i]} = 0$ constructed recursively in the following way:

Let $w_0 := w_f$. If we suppose we have found w_i such that $w_{i|(-\infty,i]} = 0$, then, by strong controllability, there exists $\hat{w}_i$ such that $\hat{w}_{i|(-\infty,i+1]} = w_{i|(-\infty,i+1]}$, $\hat{w}_{i|[k+i+1,+\infty)} = 0$. Define $w_{i+1} := w_i - \hat{w}_i$. It is clear that $w_{i+1|(-\infty,i+1]} = 0$. Moreover $z^{i+1}\hat{w}_i \in \mathcal{B}_{[0,k)}$ and so

$$z^{i+1}\hat{w}_i = \sum_{j=1}^{l} a_{i+1,j} u_j$$

Consequently we have that

$$(z^{i+1}\hat{w}_i)(t) = \sum_{j=1}^{l} a_{i+1,j} u_j(t) = \sum_{j=1}^{l} a_{i+1,j} M_{-t} e_j = M_{-t} \begin{bmatrix} a_{i+1,1} \\ \vdots \\ a_{i+1,l} \end{bmatrix}$$

and so $\hat{w}_i = \Psi_M(a_{j+1})$ where $a_{j+1} \in (R^l)^{\mathbb{Z}}$ such that

$$a_{j+1}(t) := \begin{cases} \begin{bmatrix} a_{i+1,1} \\ \vdots \\ a_{i+1,l} \end{bmatrix} & \text{if } t = i+1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, if $v_i := \sum_{j=0}^{i} a_{j+1}$, then $w_f \Leftrightarrow w_i = \Psi_M(v_i)$. We have that w_i converges to zero and so $\Psi_M(v_i)$ converges to w_f. It is clear that also the sequence v_i converges to a limit signal that we call v_f, and, by continuity of Ψ_M we have that $w_f = \Psi_M(v_f)$ and so $w_f \in \text{im } \Psi_M$. In a similar way it can be shown that $w_p \in \text{im } \Psi_M$ and so also $w = w_f + w_p \in \text{im } \Psi_M$. ■

If $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ is complete and symmetric controllable, then, as shown by the previous theorem, $\mathcal{B}$ is the image of a shift operator Ψ_M from $V^{\mathbb{Z}}$ to $W^{\mathbb{Z}}$. From the point of view of convolutional codes, the behaviour $\mathcal{B}$ can be seen as the code and so the map Ψ_M can be seen as an encoder. It is clear that Ψ_M represents really an encoder only if this map is injective. Note that injective encoders correspond to the so called noncatastrophic encoders in the convolutional codes literature. It is not true in general that the behaviour of complete and symmetric controllable system coincides with the image of an injective map shift operator Ψ_M. Actually, in this case the $R[z, z^{-1}]$ module $\mathcal{B}$ would be homomorphic to $V^{\mathbb{Z}}$ and this is not always possible. Some additional requirements stronger than strong controllability are necessary in some cases(see [17, 8]). When the ring is a principal ideal domain and when the alphabet W is a finitely generated free module, then strong controllability is the necessary and sufficient condition for the existence of a injective image representation of a complete system as shown in [9].

3.2 Controllable subsystems

Given a linear shift-invariant system $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ over a Noetherian ring, it is possible to define the concept of controllable subsystem $\Sigma_c = (\mathbb{Z}, W, \mathcal{B}_c)$ as the largest linear shift-invariant controllable subsystem of Σ. More precisely $\Sigma_c = (\mathbb{Z}, W, \mathcal{B}_c)$ is the *controllable subsystem* of Σ if

1. $\mathcal{B}_c \subseteq \mathcal{B}$.
2. $\Sigma_c = (\mathbb{Z}, W, \mathcal{B}_c)$ is a linear shift-invariant symmetric controllable system.
3. For any linear shift-invariant symmetric controllable system $\Sigma' = (\mathbb{Z}, W, \mathcal{B}')$, such that $\mathcal{B}' \subseteq \mathcal{B}$, we have that $\mathcal{B}' \subseteq \mathcal{B}_c$.

The existence of such a system is ensured by the observation that, if $\Sigma_i = (\mathbb{Z}, W, \mathcal{B}_1)$, $i \in I$, is a family of linear shift-invariant symmetric controllable systems such that $\mathcal{B}_i \subseteq \mathcal{B}$, then $(\mathbb{Z}, W, \mathcal{B})$, where

$$\mathcal{B} := \sum_{i \in I} \mathcal{B}_i,$$

is a linear shift-invariant symmetric controllable system. Note that, by Proposition 2, the controllable subsystem is strongly controllable and so it is the biggest strongly controllable subsystem of Σ. Note that he existence of the biggest strongly controllable subsystem is not obvious in general.

Note moreover that if $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ is symmetric controllable, then the controllable subsystem Σ_c coincides with Σ. If Σ is not symmetric controllable, then the controllable subsystem can give an estimate of how far is Σ from being symmetric controllable and the distance can be estimated evaluating how big the module $\mathcal{B}/\mathcal{B}_c$ is. When Σ is strongly complete, then such module is easier to estimate since it is finitely generated over R, as shown in Proposition 5 below. Note moreover that for strongly complete systems controllable subsystem can be characterized in a nice way. More precisely the controllable subsystem of a linear shift-invariant strongly complete system $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ is completely determined by the set of trajectories in $\mathcal{B}$ with finite support, that is a finitely generated $R[z, z^{-1}]$-submodule. To show this we need to introduce the concept of L-completion of a behaviour. If $\mathcal{B}$ is any behaviour, then define $CP_L(\mathcal{B})$ to be the smallest L-complete behaviour containing $\mathcal{B}$. More explicitly

$$CP_L(\mathcal{B}) = \{w \in W^{\mathbb{Z}} : w_{|[t,t+L]} \in \mathcal{B}_{|[t,t+L]}\}.$$

Proposition 4 *Let $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant L-complete system over a ring R and $\Sigma_c = (\mathbb{Z}, W, \mathcal{B}_c)$ be its controllable subsystem. Then Σ_c is strongly complete and*

$$\mathcal{B}_c = CP_L(\hat{\mathcal{B}}),$$

where $\hat{\mathcal{B}} := \{\hat{w} \in \mathcal{B} : \hat{w} \text{ has finite support}\}$.

Proof: First, if we show that $CP_L(\hat{\mathcal{B}})$ is symmetric controllable, then we would argue that $\mathcal{B}_c \supseteq CP_L(\hat{\mathcal{B}})$. Actually, if $w \in CP_L(\hat{\mathcal{B}})$, then there exists $\hat{w} \in \hat{\mathcal{B}}$ such that $w_{|[-L,0]} = \hat{w}_{|[-L,0]}$. Let $\bar{w}$ be a trajectory in $W^{\mathbb{Z}}$ such that $\bar{w}_{|(-\infty,0]} = w_{|(-\infty,0]}$ and $\bar{w}_{|[-L,+\infty)} = w_{|[-L,+\infty)}$. By L-completeness of $CP_L(\hat{\mathcal{B}})$ we have that $\bar{w} \in CP_L(\hat{\mathcal{B}})$. The symmetric can be shown similarly.
Finally if we show that $\mathcal{B}_{c|[0,L]} \subseteq \hat{\mathcal{B}}_{|[0,L]}$, then we would argue that $\mathcal{B}_c \subseteq CP_L(\mathcal{B}_c) \subseteq CP_L(\hat{\mathcal{B}})$. Actually, suppose that $w \in \mathcal{B}_c$. Then there exists $w' \in \mathcal{B}_c$ such that $w'_{|(-\infty,L]} = w_{|(-\infty,L]}$ and $w'_{|[L+h,+\infty)} = 0$. Moreover there exists $w'' \in \mathcal{B}_c$ such that $w''_{|[0,+\infty)} = w'_{|[0,+\infty)}$ and $w''_{|(-\infty,-k]} = 0$. It is easy to see that $w'' \in \hat{\mathcal{B}}$ and that $w''_{|[0,L]} = w_{|[0,L]}$. ■

The next proposition shows that if a linear shift-invariant $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ is strongly complete, then $\mathcal{B}/\mathcal{B}_c$ is a finitely generated R-module, where $\mathcal{B}_c$ is the behaviour of controllable subsystem of Σ.

Proposition 5 *Let R be a Noetherian ring and $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant strongly complete system. Let moreover $\Sigma_c = (\mathbb{Z}, W, \mathcal{B}_c)$ the* *controllable subsystem of* Σ. *Then the module* $\mathcal{B}/\mathcal{B}_c$ *is finitely generated over* R.

Proof: Suppose that Σ is L-complete. Consider the projection map

$$\Phi : \mathcal{B}_{|[0,L]} \to \mathcal{B}/\mathcal{B}_c$$

defined as follows: If we take $m \in \mathcal{B}_{|[0,L]}$, then there exists $w \in \mathcal{B}$ such that $w_{|[0,L]} = m$. Define $\Phi(m) := w + \mathcal{B}_c$. This is a good definition since if $w_1, w_2 \in \mathcal{B}$ are such that $w_{1|[0,L]} = w_{2|[0,L]}$, then $w := w_1 \Leftrightarrow w_2 \in \mathcal{B}$ and $w_{|[0,L]} = 0$. By L-completeness it is easy to see that $w = w' + w''$ where $w', w'' \in \mathcal{B}$, $w'_{|(0,+\infty)} = 0$ and $w''_{|(-\infty,0]} = 0$. We want to show that $w', w'' \in \mathcal{B}_c$. Let $\mathcal{B}'$ to be the set of all trajectories w in $\mathcal{B}$ such that $w(t) = 0$ for all $t \geq N$ for some $N \in \mathbb{N}$. It is clear that $w' \in \mathcal{B}'$ and moreover $CP_L(\mathcal{B}')$ is zero controllable and so it is symmetric controllable by Proposition 2. Consequently $CP_L(\mathcal{B}') \subseteq \mathcal{B}_c$ and so $w' \in \mathcal{B}_c$. Similarly it can be seen that $w'' \in \mathcal{B}_c$. Therefore the homomorphism Φ is well defined. It is clear that it is surjective and so we have that $\mathcal{B}/\mathcal{B}_c$ is isomorphic to $\mathcal{B}_{|[0,L]}/\ker\Phi$ that is finitely generated over R. ∎

If $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ is a linear shift-invariant strongly complete system over a field F, then there exists a linear shift-invariant subsystem $\Sigma_a = (\mathbb{Z}, W, \mathcal{B}_a)$ of Σ, called autonomous subsystem, such that $\mathcal{B} = \mathcal{B}_a \oplus \mathcal{B}_c$, where $\mathcal{B}_c$ is the behaviour of the controllable subsystem. In this case the autonomous subsystem, that is not unique, has finite dimensional behaviour. Such a decomposition of a strongly complete system in the controllable part a the autonomous part is not possible in general for systems over rings. More precisely, if $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ is a linear shift-invariant strongly complete system over a Noetherian ring, then by the previous proposition there exist $w_1, \ldots, w_n \in \mathcal{B}$ such that

$$\mathcal{B} = \mathcal{B}_c + \langle w_1, \ldots, w_n \rangle,$$

where $\mathcal{B}_c$ is the behaviour of the controllable subsystem and $\langle w_1, \ldots, w_n \rangle$ is the R-module generated by $w_1, \ldots, w_n$. The previous formula can be interpreted as a decomposition of the system in the controllable subsystem and a autonomous subsystem. However the autonomous subsystem so defined, that is the system whose behaviour is $\langle w_1, \ldots, w_n \rangle$, is linear but not shift-invariant in general. When, in particular, $\mathcal{B}_c = \{0\}$, the autonomous subsystem coincides with the entire system and so it is shift-invariant. This case is studied in the following section.

4 Finitely Generated and Autonomous Systems

This section is concerned with a particular class of noncontrollable linear shift-invariant systems, i.e. the linear shift-invariant system whose controllable subsystem is zero. We will call these systems *autonomous*. It is easy to see that a linear shift-invariant system $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ is autonomous if and only if the zero trajectory is the only trajectory in $\mathcal{B}$ with finite support. The concept of autonomous system we propose here is slightly different from the one originally proposed in [25] in the general behavioural framework. As we will see these two concepts coincides for strongly complete systems. It can can be seen that autonomous systems are strictly connected with the class of *finitely generated* systems that are linear systems whose behaviour is a finitely generated R-submodule of $W^{\mathbb{Z}}$. These systems are very interesting, since they admit a nice state representation.

Suppose that $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ is a finitely generated linear shift-invariant system and let $\{w_1, \dots, w_n\}$ be a family of generators of the finitely generated R-module $\mathcal{B}$. Then for every generator w_i, since Σ is shift-invariant, we have that $zw_i \in \mathcal{B}$ and so there exist $a_{i1}, \dots, a_{in} \in R$ such that

$$zw_i = a_{i1}w_1 + \cdots + a_{in}w_n.$$

Define the matrix $A \in R^{n\times n}$ as $A := \{a_{ij}\}_{i,j=1}^{n}$. Then we have that

$$z\,[\,w_1 \quad \cdots \quad w_n\,] = [\,w_1 \quad \cdots \quad w_n\,]\,A.$$

Note that A must be invertible and so $\det A$ is a unit in R.

Let w be any signal in $\mathcal{B}$. Then

$$w = \alpha_1 w_1 + \cdots + \alpha_n w_n = [\,w_1 \quad \cdots \quad w_n\,]\,x_0,$$

where $x_0 := [\alpha_1, \dots, \alpha_n]^T \in R^n$. For all $t \in \mathbb{Z}$ we have that

$$\begin{aligned} w(t) &= (z^t w)(0) = ([\,w_1 \quad \cdots \quad w_n\,]\,A^t x_0)(0) = \\ &= [\,w_1(0) \quad \cdots \quad w_n(0)\,]\,A^t x_0 = CA^t x_0, \end{aligned}$$

where $C := [\,w_1(0) \quad \cdots \quad w_n(0)\,] \in W^{1\times n}$. Therefore $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ is a finitely generated linear shift-invariant system if and only if for some $n \in \mathbb{N}$ there exist an invertible $A \in R^{n\times n}$ and $C \in W^{1\times n}$ such that

$$\mathcal{B} = \{w \in W^{\mathbb{Z}} \ : \ \exists x_0 \in R^n, \ w(t) = CA^t x_0, \ \forall t \in \mathbb{Z}\}. \tag{1}$$

In other words, all finitely generated linear shift-invariant systems admit a state representation of the following kind:

$$\left\{ \begin{array}{rcl} x(t+1) &=& Ax(t) \\ w(t) &=& Cx(t) \end{array} \right. ,$$

where $x \in (R^n)^{\mathbb{Z}}$.

The following proposition clarifies the relation between finitely generated systems and autonomous systems.

Proposition 6 *Let R be a Noetherian ring and $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant system. Then Σ is finitely generated if and only if Σ is autonomous and strongly complete.*

Proof: Since Σ is finitely generated, it admits a representation similar to (1). We want to show now that Σ is n-complete. Let $p \in R[z, z^{-1}]$ be the characteristic polynomial of the matrix A. Then $p = p_0 + p_1 z + \cdots + p_n z^n$, with p_0, p_n invertible elements in R and by Cayley-Hamilton theorem it is easy to see that

$$\bar{\mathcal{B}} := \{w \in W^{\mathbb{Z}} : pw = 0\} \supseteq \mathcal{B}.$$

It is clear that $\bar{\Sigma} = (\mathbb{Z}, W, \bar{\mathcal{B}})$ is n-complete. Suppose that $w_{|[t,t+n]} \in \mathcal{B}_{|[t,t+n]}$ for all $t \in \mathbb{Z}$. Then $w_{|[t,t+n]} \in \bar{\mathcal{B}}_{|[t,t+n]}$ for all $t \in \mathbb{Z}$ and so $w \in \bar{\mathcal{B}}$. Since $w_{|[0,n]} \in \mathcal{B}_{|[0,n]}$, there exists $w' \in \mathcal{B}$ such that $w'_{|[0,n]} = w_{|[0,n]}$. Let $\delta := w' \Leftrightarrow w$. Then $\delta \in \bar{\mathcal{B}}$ and $\delta_{|[0,n)} = 0$. It is easy to verify that, since p_0, p_n are invertible, this implies that $\delta = 0$ and so $w = w' \in \mathcal{B}$. Suppose that $w \in \mathcal{B}$ has finite support. Then $w \in \bar{\mathcal{B}}$ and so $pw = 0$. Then it is easy to see that this implies that w must be zero and so $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ is autonomous.
The converse is a direct consequence of Proposition 5. ■

We will show now another nice property of finitely generated systems. Actually, for these systems it is possible to extend the classical Rouchaleau-Kalman-Wyman theorem (see [21]) to the behavioural approach. The proof of the proposition above, that shows this extension, is based on the proof of RouchaleauKalman-Wyman theorem given by Johnston in [5]. Before we need to give a lemma providing a characterization of finitely generated linear shift-invariant systems $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ in term of the ideal

$$Ann(\mathcal{B}) := \{p \in R[z, z^{-1}] \ : \ pw = 0, \ \forall \ w \in \mathcal{B}\},$$

of $R[z, z^{-1}]$.

Lemma 1 *Let R be a Noetherian ring and $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant system. Then Σ is finitely generated if and only if $Ann(\mathcal{B})$ contains a polynomial $p = \sum_{i=l}^{L} p_i z^i$ such that both p_l and p_L are not zero-divisors (i.e. there does not exist $a, b \in R$ different from zero such that $ap_l = bp_L = 0$).*

Proof: Suppose first that Σ is finitely generated. Then it admits a representation similar to (1). Let $p = \sum_{i=0}^{n} p_i z^i$ be the characteristic polynomial of A. Then, by the Cayley-Hamilton theorem (see [18]), $p(A) = 0$ and so it is easy to see that $p \in Ann(\mathcal{B})$. Since p is a characteristic polynomial of a matrix, then $p_n = 1$. Moreover, since A is invertible, then it is easy to see that also $p_n = \det A$ is invertible. Suppose conversely that

$$p = p_l z^l + p_{l+1} z^{l+1} + \cdots + p_{L-1} z^{L-1} + p_L z^L \in Ann(\mathcal{B})$$

where both p_l, p_l are not zero-divisors in R. Then it can be seen that $\bar{\mathcal{B}} := \{w \in W^{\mathbb{Z}} \ : \ pw = 0\}$ is a finitely generated R-module containing $\mathcal{B}$. Actually, consider the map

$$\Phi \ : \ \mathcal{B} \ \rightarrow \ W^{L-l} \ : \ w \ \mapsto \ w_{|[l,L)}.$$

It is easy to see that it is an R homomorphism and that this homomorphism is injective. Suppose that $\Phi(w) = 0$. Then, since $w \in \bar{\mathcal{B}}$, then it satisfies the difference equation

$$p_l w(t+l)+p_{l+1}w(t+l+1)+\cdots+p_{L-1}w(t+L\Leftrightarrow 1)+p_L w(t+L) = 0, \quad \forall t \in \mathbb{Z}.$$

If we apply this equation for $t = 0$ and we exploit the fact that $w(l) = w(l+1) = \cdots = w(L \Leftrightarrow 1) = 0$ and that p_l, p_l are not zero-divisors in R, then we argue that $w(L) = 0$. Repeating the same kind of argument and using induction we see that $w(t) = 0$ for all $t \geq l$. In the same way $w(t) = 0$ for all $t \leq l$ and so $w = 0$. By prop. 6.3 of [2] $\bar{\mathcal{B}}$ and so also $\mathcal{B}$ are finitely generated over R. ∎

If R is a domain, the equivalent characterization provided by the previous proposition becomes simpler since in a domain there are not nonzero-divisors and so in this case a linear shift-invariant system $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ is finitely generated if and only if $Ann(\mathcal{B}) \neq \{0\}$.

Consider the linear shift-invariant system $\Sigma = (\mathbb{Z}, R^q, \mathcal{B})$, where W is a finitely generated free R-module. Therefore, up to isomorphisms, we have that $W = R^q$ for some $q \in \mathbb{N}$. Let F be the field of fractions of R. We can define a system $\Sigma_e = (\mathbb{Z}, F^q, \mathcal{B}_e)$ as follows:

$$\mathcal{B}_e = \{\bar{a}w \in (F^q)^{\mathbb{Z}} \ : \ \bar{a} \in F, \ w \in \mathcal{B}\}.$$

We will call Σ_e the *localization* of Σ. It is clear that Σ_e is a linear shift-invariant system on the field F and so it can be studied using all the techniques that are available for these kind of dynamical systems (see [25]). Therefore it is useful to connect properties of Σ with the properties its localization. For finitely generated dynamical systems this is provided by the following proposition, that seems to be the extension of of Rouchaleau-Kalman-Wyman theorem to the behavioural approach.

Proposition 7 *Let R be a Noetherian domain, $\Sigma = (\mathbb{Z}, R^q, \mathcal{B})$ be linear shift-invariant system and $\Sigma_e = (\mathbb{Z}, F^q, \mathcal{B}_e)$ be its localization. Then Σ is finitely generated if and only if Σ_e is finitely generated.*

Proof: One way is obvious.
Suppose conversely that $\mathcal{B}_e$ is finitely generated. Then, by Lemma 1, $Ann(\mathcal{B}_e) = \{\bar{p} \in F[z, z^{-1}] : \bar{p}\bar{w} = 0, \ \forall \bar{w} \in \mathcal{B}_e\} \neq \{0\}$. Let $\bar{p}$ be a nonzero element of $Ann(\mathcal{B}_e)$. Then there exists $a \in R$ such that $p = a\bar{p} \in R[z, z^{-1}]$

and so $p \in Ann(\mathcal{B})$. Again by Lemma 1 we argue that Σ is finitely generated. ■

From the previous proposition we can argue a nice method for checking if a linear shift-invariant system is finitely generated. Let $\Sigma = (\mathbb{Z}, R^q, \mathcal{B})$ be a linear shift-invariant system and suppose that it is L-complete. Note that if Σ is not strongly complete, then it cannot be finitely generated.

Since Σ is L-complete, then it is completely described by the finite generated submodule $\mathcal{M} := \mathcal{B}_{|[0,L]}$ of $R^{(L+1)q}$. Consider the subspace of $F^{(L+1)q}$ defined in this way

$$\mathcal{V} = \{am : a \in F,\ m \in \mathcal{M}\}$$

and define the L-complete linear shift-invariant system $\bar{\Sigma} = (\mathbb{Z}, F^q, \bar{\mathcal{B}})$ where

$$\bar{\mathcal{B}} = \{\bar{w} \in (F^q)^{\mathbb{Z}} : \bar{w}_{|[t,t+L]} \in \mathcal{V},\ \forall t \in \mathbb{Z}\}.$$

It is clear that $\mathcal{V} = \mathcal{B}_{e|[0,L]}$ and so $\bar{\mathcal{B}} = CP_L(\mathcal{B}_e)$. We want to show that Σ is finitely generated if and only if $\bar{\Sigma}$ is finitely generated.

If $\bar{\Sigma}$ is finitely generated, then Σ_e is finitely generated, since $\mathcal{B}_e \subseteq \bar{\mathcal{B}}$ and so Σ is finitely generated by the previous proposition.

Suppose conversely that Σ is finitely generated. Then, by the previous proposition, Σ_e is finitely generated and so it is complete. It can be easily proved that, since Σ is L-complete, then also Σ_e is L-complete and so it coincides with $\bar{\Sigma}$ which is finitely generated.

Therefore for checking if Σ is finitely generated, one has to check if $\bar{\Sigma}$ is finitely generated. Therefore if we have a set of generators of $\mathcal{M}$, then these constitute also a set of generators of $\mathcal{V}$ and from them it is possible to compute a kernel representation of $\bar{\Sigma}$, i.e. a polynomial matrix $N \in F[z, z^{-1}]^{g \times q}$ such that

$$\bar{\mathcal{B}} = \ker N := \{w \in (F^q)^{\mathbb{Z}} : Nw = 0\}.$$

Since (see [25]) $\bar{\Sigma}$ is finitely generated if and only if N is full column rank, then this provides a test also for Σ. Actually, by Proposition 7, Σ is finitely generated if and only if N is a full column rank polynomial matrix.

5 Finitely Generated State Space Module and Realizable Systems

This last section will be devoted to the study of the state space module of a linear shift-invariant system over a Noetherian ring. The concept of state space have been introduced in the behavioural approach by Willems in [25] in its greatest generality. For systems over groups it is possible to define a canonical state space as a quotient group.

Definition 3 *Let $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant system over a ring R. Then the state space module $\mathcal{X}$ of Σ is the R-module defined as follows*

$$\mathcal{X} := \mathcal{B}/(\mathcal{B}_- + \mathcal{B}_+),$$

where $\mathcal{B}_-$ is the subset of all the trajectories w in $\mathcal{B}$ supported in $(-\infty, 0)$ and analogously $\mathcal{B}_+$ is the subset of all the trajectories w in $\mathcal{B}$ supported in $[0, +\infty)$.

As shown by Willems in [25], the system $\Sigma_s = (\mathbb{Z}, W \times \mathcal{X}, \mathcal{B}_s)$, with

$$\mathcal{B}_s = \{(w, x) \in (W \times \mathcal{X})^{\mathbb{Z}} : w \in \mathcal{B},\ x(t) := z^t w + \mathcal{B}_- + \mathcal{B}_+,\ \forall t \in \mathbb{Z}\},$$

constitutes a minimal state space representation (or state realization) of Σ in the sense that, up to isomorphisms, it is the smallest system of this form such that $\mathcal{B} = \{w : (w, x) \in \mathcal{B}_s\}$ and satisfying the axiom of state. Willems showed moreover that, when Σ is complete, Σ_s is 2-complete. In other words he showed that in this case Σ_s is completely determined by its evolution low, i.e.

$$(w, x) \in \mathcal{B}_s \ \Leftrightarrow\ (x(t), x(t+1), w(t)) \in \mathcal{M}, \quad \forall t \in \mathbb{Z}$$

where

$$\begin{aligned} \mathcal{M} &= \{(x(0), x(1), w(0)) \in \mathcal{X} \times \mathcal{X} \times W : (x, w) \in \mathcal{B}_s\} = \\ &= \{(w + \mathcal{B}_- + \mathcal{B}_+, z^{-1} w + \mathcal{B}_- + \mathcal{B}_+, w(0)) : w \in \mathcal{B}\}. \end{aligned}$$

In coding theory words, the evolution law determined by the module $\mathcal{M}$ provides the trellis diagram describing the code associated to the system Σ. Note that these considerations are really useful in practice only if the state space module $\mathcal{X}$ is finitely generated over R. In this case we say that the system Σ is *realizable*. Only when Σ is realizable, the signal alphabet $W \times \mathcal{X}$ in the state space representation Σ_s is a finitely generated R-module and so is the module $\mathcal{M}$. Consequently the evolution law can be expressed in a constructive way. More precisely, if $m_1, \ldots, m_l$ is a set of generators of $\mathcal{M}$, then the evolution law can be expressed in the following way:
$(w, x) \in \mathcal{B}_s$ if and only if the equation

$$(x(t), x(t+1), w(t)) = a_1 m_1 + \cdots + a_l m_l$$

admits solutions $a_1, \ldots, a_l \in R$ for all $t \in \mathbb{Z}$.

In the following theorem some conditions ensuring the realizability of a linear shift-invariant system over a Noetherian ring are analyzed.

Theorem 2 *Let R be a Noetherian ring and $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant system. Then the following facts hold:*

1. *If Σ is strongly complete system, then Σ is realizable.*
2. *If Σ is a symmetric controllable system, then Σ is realizable.*

Proof: 1. Let $\mathcal{X}$ be the state space module of Σ. Suppose that Σ is L-complete. Consider the the R-homomorphism

$$\Phi : \mathcal{B}_{|[0,L]} \to \mathcal{X}$$

defined as follows: if we take $m \in \mathcal{B}_{|[0,L]}$, then there exists $w \in \mathcal{B}$ such that $w_{|[0,L]} = m$. Then we define $\Phi(m) := w + \mathcal{B}_- + \mathcal{B}_+$. This is a good definition. Actually, if $w' \in \mathcal{B}$ is such that $w'_{|[0,L]} = m$, then $\delta := w \Leftrightarrow w' \in \mathcal{B}$ and $\delta_{|[0,L]} = 0$ and so, by L-completeness the signal w_1 such that $w_{1|(-\infty,0]} = \delta_{|(-\infty,0]}$ and $w_{1|(0,+\infty)} = 0$ is in $\mathcal{B}$. It is clear that $w_1 \in \mathcal{B}_-$ and $w_2 := \delta \Leftrightarrow w_1 \in \mathcal{B}_+$. Therefore $\delta \in \mathcal{B}_- + \mathcal{B}_+$. Finally, it is easy to see that the map Φ is surjective and so $\mathcal{X} \cong \mathcal{B}_{|[0,L]}/\ker\Phi$ that is finitely generated over R.
2. Suppose now that Σ is symmetric controllable. Then it is easy to see that

$$\mathcal{B} = \hat{\mathcal{B}} + \mathcal{B}_- + \mathcal{B}_+,$$

where $\hat{\mathcal{B}}$ is the set of trajectories in $\mathcal{B}$ with finite support. Consequently we have that

$$\mathcal{X} \cong \hat{\mathcal{X}},$$

where $\hat{\mathcal{X}} := \hat{\mathcal{B}}/\hat{\mathcal{B}}_- + \hat{\mathcal{B}}_+$, where $\hat{\mathcal{B}}_-$ is the set of trajectories in $\mathcal{B}_-$ with finite support and similarly $\hat{\mathcal{B}}_+$ is the set of trajectories in $\mathcal{B}_+$ with finite support. Note that $\hat{\mathcal{B}}$ is a finitely generated module over $R[z, z^{-1}]$. Let $w_1, \ldots, w_n$ a set of generators for $\hat{\mathcal{B}}$ and suppose that their support are included in $[\Leftrightarrow N, N]$. We want to show that

$$\{z^i w_j : i = \Leftrightarrow N, \Leftrightarrow N+1, \ldots, \Leftrightarrow 1, 0, 1, \ldots, N \Leftrightarrow 1, N;\ j = 1, 2, \ldots, n\}$$

constitutes a set of generators for $\hat{\mathcal{X}}$. Take $w \in \hat{\mathcal{B}}$. Then

$$w = \sum_{j=1}^{n} \sum a_{ij} z^i w_j = \sum_{j=1}^{n} \sum_{|i| \le N} a_{ij} z^i w_j + \sum_{j=1}^{n} \sum_{|i|>N} a_{ij} z^i w_j.$$

It is clear that the second summand is in $\hat{\mathcal{B}}_- + \hat{\mathcal{B}}_+$ and so we have the thesis. ∎

As shown in [17] the canonical state space of a complete linear shift-invariant system over a finite Abelian group is a finite Abelian group. This could be easily argued from the previous theorem and from Proposition 1. On the other hand a result like this does not hold for systems over Noetherian rings, i.e. it is not true in general that the canonical state space of a complete linear shift-invariant system over a Noetherian ring is a finite generated module. Actually, it can be seen that the complete linear system over $\mathbb{Z}$ given in [7] is not realizable.

The last result presented in this paper shows that there exists a strict relation between the realizability and the module $\mathcal{B}/\mathcal{B}_c$ that has been defined in the previous section.

Proposition 8 *Let R be a Noetherian ring and $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant system. Let moreover $\Sigma_c = (\mathbb{Z}, W, \mathcal{B}_c)$ be the controllable subsystem of Σ. Then Σ is realizable if and only if $\mathcal{B}/\mathcal{B}_c$ is finitely generated over R.*

Proof: ($\Leftarrow$) Let $\mathcal{X} = \mathcal{B}/\mathcal{B}_\pm$, where $\mathcal{B}_\pm := \mathcal{B}_- + \mathcal{B}_+$, be the state space module of Σ. The state space module $\mathcal{X}_c := \mathcal{B}_c/\mathcal{B}_c \cap \mathcal{B}_\pm$, where $\mathcal{B}_\pm := \mathcal{B}_- + \mathcal{B}_+$, is finitely generated over R by the previous theorem and so, since $\mathcal{B}/\mathcal{B}_c$ is finitely generated over R, $\mathcal{B}/\mathcal{B}_c \cap \mathcal{B}_\pm$ is finitely generated too. Consequently $\mathcal{X} = \mathcal{B}/\mathcal{B}_\pm$ is finitely generated over R.
($\Rightarrow$) If $\mathcal{X} = \mathcal{B}/\mathcal{B}_\pm$ is finitely generated over R, then $(\mathcal{B}_f + \mathcal{B}_p)/\mathcal{B}_\pm$ is finitely generated over R, where $\mathcal{B}_f$ denotes the submodule $\{w \in \mathcal{B} : w_{|(-\infty,h]} = 0,\ \exists h \in \mathbb{Z}\}$ and $\mathcal{B}_p$ denotes the submodule $\{w \in \mathcal{B} : w_{|[k,+\infty)} = 0,\ \exists k \in \mathbb{Z}\}$. We can argue that both submodules $\mathcal{B}_f/\mathcal{B}_f \cap \mathcal{B}_\pm$ and $\mathcal{B}_p/\mathcal{B}_p \cap \mathcal{B}_\pm$ are finitely generated over R.
Let $w_1, \ldots, w_n \in \mathcal{B}_f$ such that $w_{i|(-\infty,0)} = 0$ and $w_1(0), \ldots, w_n(0)$ is a set of generators for the R-module $\{w(0) \in W : w \in \mathcal{B},\ w_{|(-\infty,0)} = 0\}$. Then it is clear that

$$\mathcal{B}_f = \mathcal{B}(w_1) + \cdots + \mathcal{B}(w_n) + \mathcal{B}_+,$$

where $\mathcal{B}(w_i) := \{p w_i : p \in R[z, z^{-1}]\}$. Consider the following increasing sequence of modules

$$\langle w_i \rangle + \mathcal{B}_f \cap \mathcal{B}_\pm \subseteq \langle w_i, z w_i \rangle + \mathcal{B}_f \cap \mathcal{B}_\pm \subseteq \langle w_i, z w_i, z^2 w_i \rangle + \mathcal{B}_f \cap \mathcal{B}_\pm \subseteq \cdots,$$

where with $\langle w_i, z w_i, \ldots, z^n w_i \rangle$ we mean the R-module generated by w_i, $z w_i$,..., $z^n w_i$. Since $\mathcal{B}_f/\mathcal{B}_f \cap \mathcal{B}_\pm$ is finitely generated over R, then there exists $N \in \mathbb{N}$ such that

$$\langle w_i, z w_i, \ldots, z^{N-1} w_i \rangle + \mathcal{B}_f \cap \mathcal{B}_\pm = \langle w_i, z w_i, \ldots, z^N w_i \rangle + \mathcal{B}_f \cap \mathcal{B}_\pm$$

and so there exist $p_j \in R$, $j = 0, 1, \ldots, N \Leftrightarrow 1$, such that

$$z^N w_i = \sum_{j=0}^{N-1} p_j z^j w_i + w_+ + w_-,$$

where $w_- \in \mathcal{B}_-$ and $w_+ \in \mathcal{B}_+$. It is clear that $\bar{w}_i := z^{-N} w_-$ has finite support and moreover $\bar{w}_i(0) = w_i(0)$. Doing the same with every w_i, we obtain a family $\bar{w}_1, \ldots, \bar{w}_n \in \mathcal{B}$ with finite support satisfying

$$\mathcal{B}_f = \mathcal{B}(\bar{w}_1) + \cdots + \mathcal{B}(\bar{w}_n) + \mathcal{B}_+.$$

In a similar way there exists a family $\hat{w}_1, \dots, \hat{w}_m \in \mathcal{B}$ with finite support satisfying

$$\mathcal{B}_p = \mathcal{B}(\hat{w}_1) + \cdots + \mathcal{B}(\hat{w}_m) + \mathcal{B}_-.$$

We can argue that

$$\mathcal{B}_f + \mathcal{B}_p = \hat{\mathcal{B}} + \mathcal{B}_\pm, \tag{2}$$

where with $\hat{\mathcal{B}}$ we mean the set of trajectories in $\mathcal{B}$ with finite support.
If we show that $\mathcal{B}_f + \mathcal{B}_p = \hat{\mathcal{B}} + \mathcal{B}_\pm$ is symmetric controllable, then we are done, since in this case $\mathcal{B}_f + \mathcal{B}_p \subseteq \mathcal{B}_c$ and so $\mathcal{B}/\mathcal{B}_f + \mathcal{B}_p$ finitely generated over R would imply $\mathcal{B}/\mathcal{B}_c$ finitely generated over R.
Let $w \in \hat{\mathcal{B}} + \mathcal{B}_\pm$. Then $w = \hat{w} + w_+ + w_-$, where $\hat{w} \in \hat{\mathcal{B}}$, $w_- \in \mathcal{B}_-$ and $w_+ \in \mathcal{B}_+$. Consider $w_1 := w_- + \hat{w}$. Then $w_{1|(-\infty,0]} = w_{|(-\infty,0]}$ and $w_{1|[k,+\infty)} = 0$ for some $k \in \mathbb{N}$. On the other hand defining $w_2 := w_+ + \hat{w}$, we have that $w_{2|[0,+\infty)} = w_{|[0,+\infty)}$ and $w_{2|(-\infty,-h]} = 0$ for some $h \in \mathbb{N}$. This shows that $\hat{\mathcal{B}} + \mathcal{B}_\pm$ is symmetric controllable. ■

Remark Note that in the proof of the first part of Theorem 2 something weaker than strongly completeness is really needed. Actually we need only that the system has finite memory. A linear shift-invariant system $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ has finite memory if there exists $L \in \mathbb{N}$ such that if $w \in \mathcal{B}$ and $w_{|[0,L)} = 0$, then $\bar{w}$ such that $\bar{w}_{|(-\infty,0]} = 0$ and $\bar{w}_{|(0,+\infty)} = w_{|(0,+\infty)}$ is contained in $\mathcal{B}$. As shown in [25], a system is strongly complete if and only if it is complete and has finite memory.

This observation and the previous proposition imply that all the results given for strongly complete systems hold true for systems that are only finite memory. More specifically:

1. The second part of Proposition 2 holds true for finite memory systems, i.e. if a finite memory linear shift-invariant system $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ over a Noetherian ring is zero controllable, then it is strongly controllable. Actually, it is easy to verify that, if Σ is zero-controllable, then $\mathcal{B}_p + \mathcal{B}_f = \mathcal{B}$, where $\mathcal{B}_p$ and $\mathcal{B}_f$ are the submodule of $\mathcal{B}$ defined in the proof of the previous proposition. If Σ is finite memory, then it is realizable and so, as seen in the proof of the previous proposition, $\mathcal{B}_p + \mathcal{B}_f$ is symmetric controllable.

2. Proposition 5 holds true for finite memory systems, i.e. If $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ is a linear shift-invariant finite memory system and $\Sigma_c = (\mathbb{Z}, W, \mathcal{B}_c)$ is the controllable subsystem of Σ, then the module $\mathcal{B}/\mathcal{B}_c$ is finitely generated over R.

3. Proposition 6 can be easily weakened in the following way. If $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ is a linear shift-invariant system over a Noetherian ring, then the following facts are equivalent:

i) Σ is finitely generated.

ii) Σ is autonomous and finite memory.

iii) Σ is autonomous and strongly complete.

This equivalence has the surprising consequence that linear shift-invariant finite memory autonomous systems are automatically complete.

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Communicated by Anders Lindquist