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$\mathscr{H}^\infty$ sensitivity minimization for delay systems

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Abstract: This paper announces results on the problem of feedback compensator design for $\mathscr{H}^\infty$ norm weighted sensitivity minimization when the plant contains a delay in the input. A complete solution is presented for the case of one pole/zero weighting function and a single-input/single-output plant for stable minimum-phase rational part. Generalizations and proofs will be published elsewhere.

Keywords: $\mathscr{H}^\infty$ sensitivity, Delay system, Stable minimum-phase plant, Interpolation theory, Proper approximation of compensator.

1. Introduction and problem formulation

We consider the single-input/single output $\mathscr{H}^\infty$ weighted sensitivity minimization control problem formulated in Zames [18], but with transfer functions of the form

$$P(s) = e^{-s\Delta} P_0(s) \tag{1.1}$$

where $P_0(s)$ is a minimum-phase and stable rational function, and $\Delta > 0$. The block diagram in Figure 1 shows the feedback system models we are considering.

Part of the motivation of this work is that all real systems contain delays, so that it is only by examining the optimum for such systems that an understanding can be obtained of how delays limit achievable performance. It is for this reason that we introduce the factor $e^{-s\Delta}$ into the plant transfer function.

The closed-loop sensitivity $S(s)$ is the transfer function from d to y. The weighted sensitivity $X(s)$ for the weighting function $W(s)$ is given by

$$X(s) = W(s)S(s) = W(s)[1 + P(s)C(s)]^{-1}. \tag{1.2}$$

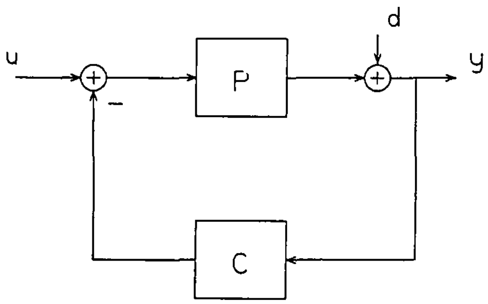

Fig. 1. Feedback system considered.

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The problem is to minimize the $\mathscr{H}^\infty$ norm of $X(s)$ over all stabilizing proper feedbacks $C(s)$, that is, to solve

$$\inf_{C(s)} \| W(s)[1+P(s)C(s)]^{-1} \|_\infty \tag{1.3}$$

where $C(s)$ ranges over all proper compensators for which the feedback system in Figure 1 is internally stable.

The criterion of minimizing the norm of (1.2) is introduced and motivated in Zames [19, pp. 585–586]. See Francis and Doyle [9] for a tutorial and bibliography on this and related problems. In this paper we also assume that the weighting function is given by

$$W(s) = \frac{s+1}{s+\beta} \tag{1.4}$$

with $0 < \beta$. We have normalized the frequency scale to put the zero of the weighting function at the point -1, and so that $\lim_{|s| \to \infty} W(s) = 1$. (1.4) is a general one pole/one zero weighting function, subject to stability and minimum-phase conditions, and the assumption that $W(s)$ is bounded away from zero at ∞.

Remark. Assuming that $\lim_{|s| \to \infty} W(s) = 0$ would avoid problems with construction and uniqueness of solutions. However, we believe that $W(\infty) = 0$ is not a reasonable choice since the resulting unweighted closed-loop sensitivities would be unbounded at ∞.

In this paper we first show how to compute the infimal sensitivity for the modified (Problem 2.2) below. When the parameter β in the weighting function satisfies $\beta < 1$ there exists a unique solution to this minimization problem. We obtain this result using the work of Sarason [13]. In the case $\beta > 1$ there is an infinity of solutions, and when $\beta = 1$ there is again a unique solution. We state a theorem due to Fagnani [1] which parametrizes all solutions of the problem for the case $\beta \geqslant 1$.

We also compute the optimal feedback compensator corresponding to the infirmal sensitivity. We show that the optimal feedback compensator is improper, unstable and infinite dimensional, and we construct proper rational approximations of the optimal feedback compensator. It is by means of these approximations that we actually solve (1.3) or (Problem 2.1) below.

In Flamm [3] we discuss various aspects of the problem when more general weighting functions and plants are considered. Here we only announce the results for the case of the weighting function as in (1.4). The detailed proofs appear in Flamm and Mitter [5,6] and Fagnani [1].

The idea of using Sarason's work for solving the sensitivity minimization problem *for delay systems* first appeared in Flamm [2]; see also [4].

(Another part of Sarason [13] was applied to the case of rational plants in Francis and Zames [10].)

Work similar to that of this paper and the more general treatment in [3] has been done independently by Foias, Tannenbaum and Zames [7,8]). Some extensions of this work to multiple delays in the plant have been achieved by G. Tadmor. See Tadmor [15,16].

2. Reformulation of the problem

We shall assume a basic knowledge on the part of the reader of the theory of $\mathscr{H}^p$ spaces, the Hardy spaces of analytic functions. We refer the reader unfamiliar with this material to the reference Hoffman [12]. We note that we shall work with functions analytic in the right half plane as in [12, Ch. 8]. We also assume knowledge of the prior work of Zames and others, as represented, for example, in Francis and Zames [10].

Following the argument which appears, for example, in [10, p. 10] we assume the problem of minimizing the weighted sensitivity (1.3) has been transformed to the form

$$\inf_{H \in \mathscr{H}^\infty} \left\| W(s) - \mathrm{e}^{-s\Delta} P_0(s) H(s) \right\|_\infty . \tag{Problem 2.1}$$

Just as in the same reference, we first restrict our attention to the modified problem

$$\inf_{H\in\mathscr{H}^\infty} \| W(s) - e^{-s\Delta}H(s) \|_\infty \qquad \text{(Problem 2.2)}$$

since the infimum in (Problem 2.1) is generally not realized when $P_0(s)$ is strictly proper. Note that the infimum in (Problem 2.2) is not greater than that in (Problem 2.1). Later a compensator arising from a solution to (Problem 2.2) is used to find a sequence of compensators for which the closed-loop weighted sensitivity approaches the infimum in (2.1). We note that (Problem 2.2) is equivalent to the Nehari problem

$$\inf_{H\in\mathscr{H}^\infty} \| e^{s\Delta}W(s) - H(s) \|_\infty$$

where $e^{s\Delta}W(s) \in L^\infty$.

3. The basic strategy for solution

Following Sarason [13] we view $[W(s) - e^{-s\Delta}H(s)]$ in (Problem 2.2) as an operator on $\mathscr{H}^2$. The compression of this operator to $K = \mathscr{H}^2 \ominus e^{-s\Delta}\mathscr{H}^2$ is equal to the compression of $W(s)$ on the same subspace. Call this latter operator $T = \Pi_K W|_K$. (We note that $\|T\| \leqslant \|W\|$.) The infimum in (Problem 2.2) cannot be less than the operator norm of T. Theorem 1 in [13] says that the desired infimum is in fact equal to $\|T\|$.

Following [13, Section 7] a way to find this supremum is to use the fact that $\|T\| = \rho(T^*T)^{1/2}$. ($\rho(\cdot)$ is the spectral radius.) If T is compact we need only find the largest eigenvalue of T^*T, but this is not the general case.

In our case we have normalized $W(s)$ with $W(\infty) = 1$. Since $W(s)$ is rational it is then equal to 1 plus a strictly proper stable rational function. Thus $T^*T - I$ is compact, and we know from Weyl's theorem that $\sigma(T^*T)$ and $\sigma(I)$ differ only by eigenvalues. [11, pp. 92&295]. ($\sigma(\cdot)$ denotes the spectrum.) Also, 1 is the only cluster point of $\sigma(T^*T)$.

The idea will be to examine the eigenvalues of T^*T for a maximum. If none exists, we will have $\|T\| = 1$.

For computational reasons we transform the operator T to the time domain. The operator T is equivalent to an operator $V: \mathscr{L}^{-1}(K) \to \mathscr{L}^{-1}(K)$ via the inverse Laplace transformation. ($\mathscr{L}(\cdot)$ is the bilateral Laplace transform operator.) $\mathscr{L}^{-1}(K)$ is a subspace of $L^2(0, \infty)$. One can think of V acting on time functions via convolution and T on transfer functions via multiplication, each followed by the corresponding projection. In particular, there is a one-to-one correspondence between eigenvectors, and the eigenvalues of T^*T and V^*V are the same. Furthermore, compactness of V is equivalent to compactness of T.

If T^*T has a largest eigenvalue, say λ^2, then $\|T\|^2 = \lambda^2$, and the corresponding eigenfunction will be a maximal vector for T. According to Proposition 5.1 in [13], in this case $T/\|T\|$ will be dilated by an inner function given by $T\hat{f}/\|T\|\hat{f}$. This is the *unique* minimal dilation. In our case, $T\hat{f}/\hat{f}$ would be the optimal sensitivity.

When T does not have a maximal vector, we still know from Theorem 1 in [13] that a minimal dilation of T exists, but we do not know that it is unique. Nevertheless using the theory of extension of Hankel operators, one can, in principle, parametrize all solutions of the problem (cf. Section 5 below).

4. Computation of the infimal weighted sensitivity for $\beta < 1$

Theorem 1. *For $\beta < 1$ the infimal sensitivity $\overline{X}(s)$ corresponding to Problem* (2.2) *is given by*

$$\overline{X}(s) = \lambda_{\max} \frac{s + 1 - e^{-s\Delta}\lambda_{\max}(s - \beta)}{\lambda_{\max}(s + \beta) - e^{-s\Delta}(s - 1)}, \tag{4.1}$$

where $\lambda_{\max} = [(\omega_0^2 + 1)/(\omega_0^2 + \beta^2)]^{1/2}$ *and* ω_0 *is the smallest positive solution of* $\cot(\omega\Delta) = (\omega^2 - \beta)/\omega(1 + \beta)$. *Furthermore,* $\bar{X}(s)$ *is a scalar multiple of an inner function, and* $\| \bar{X}(s) \| = \lambda_{\max}$.

The proof of this theorem proceeds via several propositions.

Proposition 1. *Let* $K = \mathscr{H}^2 \ominus \mathrm{e}^{-s\Delta}\mathscr{H}^2$. *Then* $K = \{ f = \mathscr{L}(f) \mid f \in L^2(0, \Delta)\}$. *Moreover*

$$T: K \to K: \quad f \mapsto \Pi_K(Wf) = \left[\left(\frac{1 - \mathrm{e}^{-s\Delta}}{s}\right) * W(s)\right] f(s),$$

where $*$ *denotes convolution*;

$$V: L^2(0, \Delta) \to L^2(0, \Delta): \quad f \mapsto (Vf)(t) = f(t) + \int_0^t w(t - \tau) f(\tau)\, \mathrm{d}\tau,$$

where $w(t) \in L^1(0, \infty)$ *is given by* $w(t) = \mathscr{L}^{-1}(W_0(s))$ *and* $W_0(s) = W(s) - 1$, *and the adjoint operator to* V *is given by*

$$V^*: L^2(0, \Delta) \to L^2(0, \Delta): \quad f \mapsto (V^* f)(t) = f(t) + \int_t^\Delta w(\tau - t) f(\tau)\, \mathrm{d}\tau.$$

To proceed further, it is convenient to obtain a state space realization of the operator V^*V.

Proposition 2. *The operator*

$$V^*V: L^2(0, \Delta) \to L^2(0, \Delta): \quad f \mapsto V^*Vf$$

is realized by the mapping $f \mapsto z$ *where*

$$\frac{\mathrm{d}}{\mathrm{d}t} x_1(t) = -\beta x_1(t) + (1 - \beta) f(t), \quad x_1(0) = 0, \qquad y(t) = x_1(t) + f(t), \tag{4.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} x_2(t) = \beta x_2(t) - (1 - \beta) y(t), \quad x_2(\Delta) = 0, \qquad z(t) = x_2(t) + y(t). \tag{4.3}$$

Remark. Although we are dealing with the case of a weighting function $W(s) = (s + 1)/(s + \beta)$, $\beta < 1$, the results derived so far are valid with obvious modifications for a more general rational proper weighting function.

The next proposition solves the eigenvalue problem $V^*Vf = \lambda^2 f$. The proof uses the state space realization (4.2)–(4.3) and also determines the maximal eigenvalue and the corresponding maximal eigenvector.

Proposition 3. *For the eigenvalue problem* $V^*Vf = \lambda^2 f$ *(with* $\beta < 1$*), a maximal eigenvalue* $\lambda^2_{\max}$ *and a corresponding maximal vector* $f_{\max}$ *exist and are given by*

$$\lambda^2_{\max} = (\omega_0^2 + 1)/(\omega_0^2 + \beta^2)$$

where ω_0 *is the smallest positive solution of* $\cot(\omega\Delta) = (\omega^2 - \beta)/\omega(1 + \beta)$, *and* $f_{\max}(t) = \cos(\omega_0 t + \varphi)$, *where* $\tan(\varphi) = -\beta/\omega_0$. *Furthermore,* $\sigma(V^*V) \subseteq (1, \lambda^2_{\max}]$.

Remark. When $\beta < 1$, $|W(\mathrm{j}\omega)|$ approaches 1 from above as $\omega \to \infty$.

Proposition 2 shows that for $\beta < 1$. $\lambda_{\max} = \| T \| > 1$, and hence according to the proof of Proposition 5.1 in [13] (cf. Section 3) $T/\| T \|$ will be interpolated by an inner function given by $T\hat{f}_{\max}/\| T \| \hat{f}_{\max}$. The minimal dilation of T will be given by $T\hat{f}_{\max}/\hat{f}_{\max}$. The explicit computation of this leads to Theorem 1.

Remark. As $\Delta \to 0$, $\lambda^2_{\max} \to 1$, and as $\Delta \to \infty$, $\lambda^2_{\max} \to 1/\beta$.

5. Computation of the ideal compensator

Given the infimal sensitivity $\bar{X}(s)$ in Theorem 1 which is the solution to the (Problem 2.2), it is easy to calculate a formula for the corresponding feedback compensator using $\bar{C}(s) = (W(s) - \bar{X}(s))/P(s)\bar{X}(s)$. For the case $W(s) = (s+1)/(s+\beta)$ with $\beta < 1$ we get

$$\bar{C}(s) = \zeta P_0(s)^{-1} \frac{s^2 + \omega_0^2}{s^2 + (\beta+1)s + \beta} \frac{1}{1 + \mathrm{e}^{-s\Delta}\lambda(\beta - s)/(s+1)},$$

where $\zeta = (\lambda^2_{\max} - 1)/\lambda_{\max}$, $\lambda_{\max}$ and ω_0 are as given in Theorem 1, and $P_0(s)$ is the outer factor of the plant.

Since $P_0(s)$ will generally be strictly proper, $\bar{C}(s)$ will be improper. $\bar{C}(s)$ also contains a pure delay term, and is therefore an infinite-dimensional system. In the next section we show how to construct a sequence $\{C_n(s)\}$ of proper rational transfer functions, such that closed-loop stability is maintained and the closed-loop weighted sensitivities approach the infimal value $\| \bar{X}(s) \|_\infty$.

Remark 1. In addition to being improper, $\bar{C}(s)$ is unstable. (In fact, we can show $\bar{C}(s)$ has infinitely many right half plane poles.) This can be demonstrated following an argument in [13, p. 194] which shows that $\bar{X}(s)$ is a Blaschke product.

Remark 2. One may consider the question of 'delay margin' when the true delay differs from the value of delay used to compute $\bar{C}(s)$. Initial investigation indicates that $\mathscr{H}^\infty$ minimal weighted sensitivity designs are characterized by zero delay margin. A feedback system using the approximations given in the next section would be stable in spite of small errors in the delay because the loop transfer function would be strictly proper.

6. Computation of the infimal sensitivity for $\beta > 1$

Theorem 2. *For $\beta \geqslant 1$, all infimal sensitivities $X(s)$ corresponding to* (Problem 2.2) *are given by*

$$X(s) = \frac{s + 1 - \mathrm{e}^{-s\Delta}\varphi(s)(s-\beta)}{s + \beta - \mathrm{e}^{-s\Delta}\varphi(s)(s-1)} \tag{6.1}$$

as $\varphi(s)$ ranges over $B(\mathscr{H}^\infty)$, the unit ball in $\mathscr{H}^\infty$. Moreover, $\| X(s) \|_\infty = 1$, and for $\beta > 1$, $X(s)$ is inner $\Leftrightarrow$ φ is inner.

Remark. The proof of this theorem, first presented in [1], is based on Sarason [14] and will appear elsewhere.

We give two examples of infimal sensitivities corresponding to $\varphi(s) = 0$ and $\varphi(s) = 1$. Note that $\beta = 1$ gives the unique solution $X(s) = 1$.

First, we note that $\varphi(s) = 0$ gives $X(s) = W(s)$ as an infimal sensitivity when $\beta > 1$. In this case all eigenvalues are on the interval $[\lambda_{\min}, 1)$, clustering at 1. We then can conclude that $\| V \| = 1$, and therefore $W(s)$ itself is an infimal sensitivity. The corresponding compensator is $C(s) = 0$.

For $\beta > 1$ the choice of $\varphi(s) = 1$ in equation (6.1) gives an infimal sensitivity

$$X(s) = \frac{s + 1 - \mathrm{e}^{-s\Delta}(s-\beta)}{s + \beta - \mathrm{e}^{-s\Delta}(s-1)},$$

and a corresponding compensator

$$C(s)=\frac{W(s)-X(s)}{P(s)X(s)}=P_0(s)^{-1}\frac{1-\beta^2}{s^2+(\beta+1)s+\beta}\frac{1}{1+e^{-s\Delta}(\beta-s)/(s+1)}.$$

7. Approximation by proper finite-dimensional compensators

Theorem 3. *There exists a sequence of rational proper feedback compensators which result in weighted sensitivities of norm approaching the optimal value $\lambda_{\max}$ of Theorem* 1.

Remark. Theorem 3 solves (Problem 2.1) for $\beta<1$. A similar result holds for $\beta \geqslant 1$. Since the infimal weighted sensitivity is unique (when it has norm greater than 1), and the corresponding compensator is improper when the plant is strictly proper, there can be no proper compensator which achieves the infimum.

There are procedures in the literature to compute proper compensators for the case of a purely rational plant [19, p. 591] and [17, p. 178]. The procedure in [19] requires the evaluation of the term '$B_z(\infty)$', where $B_z(s)$ is the Blaschke product formed from plant zeros. In our case there is no apparent way to interpret '$\lim_{|s|\to\infty} e^{-s\Delta}$'. The procedure in [17] does not work for our case either. The essence of the difficulty is the same as in the Zames–Francis procedure – the inner factor of the plant is not continuous at infinity.

The proof proceeds by developing approximations to $\overline{C}(s)$, first to make $C(s)P(s)$ strictly proper, next to make $C(s)$ proper, and finally to make $C(s)$ finite dimensional. We emphasize that we are not necessarily approximating the optimal compensator, but rather describing a sequence of compensators for which the weighted $\mathscr{H}^\infty$ norm of the closed-loop sensitivity approaches the infimal value.

Proposition 4. *Let $\overline{X}(s)$ be the infimal sensitivity given in* (4.1), *with norm $\|\overline{X}(s)\|_\infty=\lambda_{\max}\geqslant 1$. Let $h_n(s)=[n/(s+n)]^{1/n}$. Then as $n\to\infty$, the compensator given by*

$$C_n'(s)=\frac{h_n(s)[W(s)-\overline{X}(s)]}{P(s)[(1-h_n(s))W(s)+h_n(s)\overline{X}(s)]}$$

results in a stable closed loop for which the sensitivity approaches $\lambda_{\max}$. Furthermore, the loop gain $|P(j\omega)C_n'(j\omega)|\to 0$ as $|\omega|\to\infty$.

The reason for the given form of the 'roll-off' function $h_n(s)$ is that it is necessary to control the phase of $h_n(s)$ until the loop gain has decreased sufficiently. Otherwise the sensitivity can be bounded away from the infimal value as $n\to\infty$, even if the roll-off does not start until high frequency.

Having shown how to approach the optimal sensitivity with a compensator such that the loop gain goes to zero, it is simple to apply the standard approach to modify the compensator to make it proper, and at the same time introduce a rational approximation for $h_n(s)$. It is sufficient to delay the faster roll-off which makes the compensator proper until the frequency ω_n in the following proposition.

Proposition 5. *Let $h_n(s)$ be as in Proposition* 4. *For each n take ω_n to satisfy $\omega_n\geqslant n[4(\lambda_{\max}+1)/(\lambda_{\max}-1)]^n$. Take ω_r to be the least frequency above ω_n at which $|f_n(j\omega)|\leqslant|h_n(j\omega)|$ for $|\omega|\geqslant\omega_r$. Let $\tilde{h}_n(s)$ be any stable rational minimum-phase function which satisfies $\|\tilde{h}_n\|_\infty\leqslant 1$ and $|h_n(j\omega)-\tilde{h}_n(j\omega)|<1/n^2$ for $|\omega|\leqslant\omega_r$. Take $f_n(s)$ to be a stable minimum-phase rational function of s which satisfies $\|f_n\|_\infty\leqslant 1$ and $|1-f_n(j\omega)|<1/n^2$ for $\omega\leqslant\omega_n$ and which eventually rolls off at least as fast as $P(s)$. Define $g_n(s)=f_n(s)\cdot\tilde{h}_n(s)$. Define*

$$C_n(s)=\frac{g_n(s)[W(s)-\overline{X}(s)]}{P(s)[(1-g_n(s))W(s)+g_n(s)\overline{X}(s)]}.$$

Then the closed-loop feedback system using $C_n(s)$ as the compensator is stable, and the closed-loop weighted sensitivity $X_n(s) = W(s)[1 + P(s)C_n(s)]^{-1}$ has ∞-norm which approaches the infimal value $\lambda_{\max}$ as $n \to \infty$. Furthermore $C_n(s)$ is a proper function.

From the formulas in Propositions 4 and 5, C_n will in general contain a pure delay. Even though this rolled-off compensator is physically more realistic than the ideal compensator, we next show how to further approximate the ideal compensator with one which is finite dimensional. In approximating $e^{-s\Delta}$ with a rational function, we must be concerned with two things: First, we must preserve the stability of the closed loop, and second, we must preserve the approximation of the closed-loop sensitivity to the optimal sensitivity. The restrictions these impose on rational approximation of the delay amount to (1) the delay must be approximated closely enough until h_n is sufficiently small, and (2) after that the delay approximation must not exceed 1 in magnitude.

We approximate the delay by replacing $e^{-s\Delta}$ with the rational function $\rho(s)$. The following gives one set of criteria for selecting $\rho(s)$.

Proposition 6. *Let ω_r and $g_n(j\omega)$ be as in Proposition 5. Take $\omega_c \geqslant \omega_r$ such that, if $\mathrm{Re}(s) \geqslant 0$ and $|s| \geqslant \omega_c$, then $|W(s)| < \lambda_{\max}$, and if $|\omega| > \omega_c$, then $|g_n(j\omega)| < (\lambda_{\max} - |W(j\omega_c)|)/(\lambda_{\max} + |W(j\omega_c)|)$. Let*

$$\gamma = \inf_{\mathrm{Re}(s) \geqslant 0} \left| \lambda_{\max} + e^{-s\Delta} \frac{1-s}{s+\beta} \right|.$$

Let $\rho(s)$ be a stable rational approximation to $e^{-s\Delta}$ such that $\|\rho\| \leqslant 1$, and for $\mathrm{Re}(s) \geqslant 0$ and $|s| < \omega_c$, $|\rho(s) - e^{-s\Delta}| < \varepsilon < \gamma/2\|W\|_\infty$. Define $\tilde{C}_n(s)$ as $C_n(s)$ with $\rho(s)$ substituted for $e^{-s\Delta}$. Under these conditions, as $n \to \infty$ and $\varepsilon \to 0$, the closed-loop system with compensator $\tilde{C}_n(s)$ and weighted sensitivity $\tilde{X}_n(s)$ is stable and satisfies $\|\tilde{X}_n(s)\| \to \lambda_{\max}$.

This proposition completes the proof of Theorem 3.

8. Conclusion

We have presented the solution to the simplest meaningful $\mathscr{H}^\infty$ minimal weighted sensitivity problem for the case of a plant having a delay in the input. This includes the analysis of the resulting improper and infinite-dimensional compensator and its approximation by a finite-dimensional proper compensator.

Results parallel to those here are detailed in Flamm [3] for more general rational weighting functions, and for plants having right half plane poles and zeros in addition to the input delay. The basic techniques are essentially generalizations of those presented here with modifications made for right half plane poles and zeros in the plant. However, in the most general case of rational, stable, proper and minimum-phase $W(s)$ we are not necessarily able to construct an optimal sensitivity, although we know from the theory of Sarason that one exists. We are able to 'slightly' modify any given $W(s)$ so as to allow us to construct a solution to the changed problem. See [3] for details.

Areas for future work include completing the theoretical picture for the case of general rational $W(s)$, computational issues for the case of general weighting functions, and extensions to plants with other non-rational transfer functions.

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