

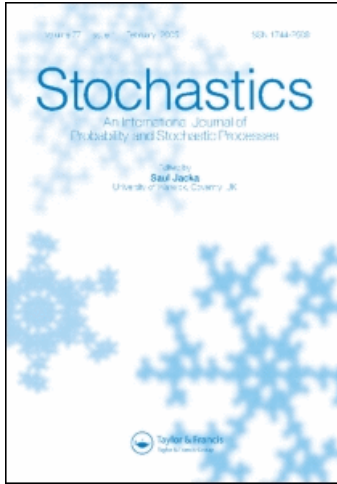
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Wendell H. Fleming<sup>a</sup>; Sanjoy K. Mitter<sup>b</sup>

<sup>a</sup> Lefschetz Center for Dynamical Systems, Division of Applied Mathematics, Brown University, Providence, Rhode Island, USA <sup>b</sup> Department of Electrical Engineering and Computer Science, and Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA

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# Optimal Control and Nonlinear Filtering for Nondegenerate Diffusion Processes

WENDELL H. FLEMING†

*Lefschetz Center for Dynamical Systems, Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912, USA*

and

SANJOY K. MITTER‡

*Department of Electrical Engineering and Computer Science, and Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*

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A linear parabolic partial differential equation describing the pathwise filter for a nondegenerate diffusion is changed, by an exponential substitution, into the dynamic programming equation of an optimal stochastic control problem. This substitution is applied to obtain results about the rate of decay as  $|x| \rightarrow \infty$  of solutions  $p(x, t)$  to the pathwise filter equation, and for solutions of the corresponding Zakai equation.

## 1. INTRODUCTION

We consider an  $n$ -dimensional signal process  $x(t) = (x_1(t), \dots, x_n(t))$  and a one-dimensional observation process  $y(t)$ , obeying the stochastic differential equations

$$dx = b[x(t)] dt + \sigma[x(t)] dw \quad (1.1)$$

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$$dy = h[x(t)] dt + d\tilde{w}, \quad y(0) = 0, \quad (1.2)$$

with  $w, \tilde{w}$  standard brownian motions of respective dimensions  $n, 1$ . (The extensions to vector-valued  $y(t)$  need only minor modifications.) The Zakai equation for the unnormalized conditional density  $q(x, t)$  is

$$dq = A^*q dt + hq dy, \quad t \geq 0. \quad (1.3)$$

where  $A$  is the generator of the signal process  $x(t)$ . See Davis and Marcus [3] for example. By formally substituting

$$q(x, t) = \exp[y(t)h(x)]p(x, t), \quad (1.4)$$

one gets instead of the stochastic partial differential equation (1.3) a linear partial differential equation of the form

$$p_t = \frac{1}{2} \text{tr } a(x)p_{xx} + g^y(x, t) \cdot p_x + V^y(x, t)p, \quad t \geq 0. \quad (1.5)$$

with  $p(x, 0) = p^0(x)$  the density of  $x(0)$ . Here

$$a(x) = \sigma(x)\sigma(x)', \quad p_x = (p_{x_1}, \dots, p_{x_n})$$

$$\text{tr } a(x)p_{xx} = \sum_{i,j=1}^n a_{ij}(x)p_{x_i x_j}$$

Explicit formulas for  $g^y, V^y$  are given in Section 6. Equation (1.5) is the basic equation of the pathwise theory of nonlinear filtering. See Davis [2] or Mitter [10]. The superscript  $y$  indicates dependence on the observation trajectory  $y = y(\cdot)$ . Of course, the solution  $p = p^y$  also depends on  $y$ .

We shall impose in (1.1) the nondegeneracy condition that the  $n \times n$  matrix  $\sigma(x)$  has a bounded inverse  $\sigma^{-1}(x)$ . Other assumptions on  $b, \sigma, h, p^0$  will be stated later. Certain unbounded functions  $h$  are allowed in the observation equation (1.2). For example,  $h$  can be a polynomial in  $x = (x_1, \dots, x_n)$  such that  $|h(x)| \rightarrow \infty$  as  $|x| \rightarrow \infty$ . The connection between filtering and control is made by considering the function  $S = -\log p$ . This logarithmic transformation changes (1.5) into a nonlinear partial differential equation for  $S(x, t)$ , of the form (2.2) below. We introduce a certain optimal stochastic control problem for which (2.2) is the dynamic programming equation.

In Section 3 upper estimates for  $S(x, t)$  as  $|x| \rightarrow \infty$  are obtained, by using an easy Verification Theorem and suitably chosen comparison controls. Note that an upper estimate for  $S$  gives a lower estimate for  $p = -\log S$ . A

lower estimate for  $S(x, t)$  as  $|x| \rightarrow \infty$  is obtained in Section 5 by another method from a corresponding upper estimate for  $p(x, t)$ . These results are applied to the pathwise nonlinear filter equation in Section 6.

Related results have been obtained using other methods by Baras, Blankenship and Hopkins [1] and by Sussmann [12]. A connection between control and nonlinear filtering was also made by Hijab [8], in a somewhat different context.

## 2. THE LOGARITHMIC TRANSFORMATION

Let us consider a linear parabolic partial differential equation of the form

$$p_t = \frac{1}{2} \operatorname{tr} a(x) p_{xx} + g(x, t) \cdot p_x + V(x, t) p, \quad t \geq 0, \quad (2.1)$$

$$p(x, 0) = p^0(x).$$

When  $g = g^v$ ,  $V = V^v$  this becomes the pathwise filter equation (1.5), to which we return in Section 6. By solution  $p(x, t)$  to (2.1) we mean a "classical" solution  $p \in C^{2,1}$ , i.e. with  $p_{x_i}, p_{x_i x_j}, p_t$  continuous,  $i, j = 1, \dots, n$ .

If  $p$  is a positive solution to (2.1), then  $S = -\log p$  satisfies the nonlinear parabolic equation

$$S_t = \frac{1}{2} \operatorname{tr} a(x) S_{xx} + H(x, t, S_x), \quad t \geq 0$$

$$S(x, 0) = S^0(x) = -\log p^0(x), \quad (2.2)$$

$$H(x, t, S_x) = g(x, t) \cdot S_x - \frac{1}{2} S'_x a(x) S_x - V(x, t).$$

Conversely, if  $S(x, t)$  is a solution to (2.2), then  $p = \exp(-S)$  is a solution to (2.1).

This logarithmic transformation is well known. For example, if  $g = V = 0$ , then it changes the heat equation into Burgers' equation (Hopf [9]).

We consider  $0 \leq t \leq t_1$ , with  $t_1$  fixed but arbitrary. Let  $Q = \mathbb{R}^n \times [0, t_1]$ . We say that a function  $\phi$  with domain  $Q$  is of class  $\mathcal{L}$  if  $\phi$  is continuous and, for every compact  $K \in \mathbb{R}^n$ ,  $\phi(\cdot, t)$  satisfies a uniform Lipschitz condition on  $K$  for  $0 \leq t \leq t_1$ . We say that  $\phi$  satisfies a polynomial growth condition of degree  $r$ , and write  $\phi \in \mathcal{P}_r$ , if there exists  $M$  such that

$$|\phi(x, t)| \leq M(1 + |x|^r), \quad \text{all } (x, t) \in Q.$$

Throughout this section and Section 3 the following assumptions are

made. Somewhat different assumptions are made in Sections 4 and 5 as needed. We assume:

$$\sigma, \sigma^{-1} \text{ are bounded, Lipschitz functions on } R^n. \quad (2.3)$$

For some  $m \geq 1$

$$g \in \mathcal{L} \cap \mathcal{P}_m, \quad V \in \mathcal{L} \cap \mathcal{P}_{2m}. \quad (2.4)$$

For some  $l \geq 0$

$$S^0 \in C^2 \cap \mathcal{P}_l. \quad (2.5)$$

For some  $M_1$ ,

$$V(x, t) \leq M_1, \quad S^0(x) \geq -M_1. \quad (2.6)$$

We introduce the following stochastic control problem, for which (2.2) is the dynamic programming equation. The process  $\zeta(t)$  being controlled is  $n$ -dimensional and satisfies

$$d\zeta = \mathbf{u}(\zeta(\tau), \tau) d\tau + \sigma[\zeta(\tau)] dw, \quad 0 \leq \tau \leq t, \quad (2.7)$$

$$\zeta(0) = x.$$

The control is feedback,  $R^n$ -valued:

$$u(\tau) = \mathbf{u}(\zeta(\tau), \tau), \quad (2.8)$$

Thus, the control  $\mathbf{u}$  is just the drift coefficient in (2.7). We admit any  $\mathbf{u}$  of class  $\mathcal{L} \cap \mathcal{P}_1$ . Note that  $\mathbf{u} \in \mathcal{P}_1$  implies at most linear growth of  $|\mathbf{u}(x, t)|$  as  $|x| \rightarrow \infty$ . For every admissible  $\mathbf{u}$ , Eq. (2.7) has a pathwise unique solution  $\zeta$  such that  $E\|\zeta\|_t^r < \infty$  for every  $r > 0$ . Here  $\|\cdot\|_t$  is the sup norm on  $[0, t]$ .

Let

$$L(x, t, u) = \frac{1}{2}(u - g(x, t))' a^{-1}(x)(u - g(x, t)) - V(x, t). \quad (2.9)$$

For  $(x, t) \in Q$  and  $\mathbf{u}$  admissible, let

$$J(x, t, \mathbf{u}) = E_x \left\{ \int_0^t L[\zeta(\tau), t - \tau, u(\tau)] d\tau + S^0[\zeta(t)] \right\}. \quad (2.10)$$

The polynomial growth conditions in (2.4), (2.5) imply finiteness of  $J$ . The stochastic control problem is to find  $\mathbf{u}^{\text{op}}$  minimizing  $J(x, t, \mathbf{u})$ . Under the above assumptions, we cannot claim that an admissible  $\mathbf{u}^{\text{op}}$  exists minimizing  $J(x, t, \mathbf{u})$ . However, we recall from Fleming Rishel [7], Thm. VI 4.1, the following result, which is a rather easy consequence of the Itô differential rule.

### Verification Theorem

Let  $S$  be a solution to (2.2) of class  $C^{2,1} \cap \mathcal{P}_r$ , with  $S(x, 0) = S^0(x)$ . Then

- $S(x, t) \leq J(x, t; \mathbf{u})$  for all admissible  $\mathbf{u}$ .
- If  $\mathbf{u}^{\text{op}} = g - aS_x$  is admissible, then  $S(x, t) = J(x, t; \mathbf{u}^{\text{op}})$ .

In Section 3 we use (a) to get upper estimates for  $S(x, t)$ , by choosing judiciously comparison controls. For  $\mathbf{u}^{\text{op}}$  to be admissible, in the sense we have defined admissibility,  $|S_x|$  can grow at most linearly with  $|x|$ ; hence  $S(x, t)$  can grow at most quadratically. By enlarging the class of admissible controls to include certain  $\mathbf{u}$  with faster growth as  $|x| \rightarrow \infty$  one could generalize (b). However, we shall not do so here, since only part (a) will be used in Section 3 to get an estimate for  $S$ .

In Section 4 we consider the existence of a solution  $S$  with the polynomial growth condition required in the Verification Theorem.

As in Fleming [6] we call a control problem with dynamics (2.7) a problem of stochastic calculus of variations. The control  $\mathbf{u}(\xi(\tau), \tau)$  is a kind of "average" time-derivative of  $\xi(\tau)$ , replacing the nonexistent derivative  $\dot{\xi}(\tau)$  which would appear in the corresponding calculus of variations problem with  $\sigma = 0$ .

### Other control problems

There are other stochastic control problems for which (2.2) is also the dynamic programming equation. One choice, which is appealing conceptually, is to require instead of (2.7) that  $\xi(\tau)$  satisfy

$$d\xi = \{g[\xi(\tau), \tau] + \mathbf{u}[\xi(\tau), \tau]\} d\tau + \sigma[\xi(\tau)] dw \quad (2.11)$$

with  $\xi(0) = x$ . We then take

$$L(x, t, u) = \frac{1}{2} u' a^{-1}(x) u - V(x, t). \quad (2.12)$$

The feedback control  $\mathbf{u}$  changes the drift in (2.11) from  $g$  to  $g + \mathbf{u}$ . When

$a = \text{identity}$ ,  $L = \frac{1}{2}|u|^2 - V(x, t)$  corresponds to an action integral in classical mechanics with time-dependent potential  $V(x, t)$ .

### 3. UPPER ESTIMATES FOR $S(x, t)$

In this section we obtain the following upper estimates for the growth of  $S(x, t)$  as  $|x| \rightarrow \infty$  in terms of the constants  $m \geq 1$ ,  $l \geq 0$  in (2.4), (2.5).

**THEOREM 3.1** *Let  $S$  be a solution of (2.2) of class  $C^{2,1} \cap \mathcal{P}_r$ , with  $S(x, 0) = S^0(x)$ . Then there exist positive  $M_1, M_2$  such that:*

- i) For  $(x, t) \in Q$ ,  $S(x, t) \leq M_1(1 + |x|^\rho)$  with  $\rho = \max(m + 1, l)$ ,
- ii) Let  $0 < t_0 < t_1$ ,  $m > 1$ . For  $(x, t) \in R^n \times [t_0, t_1]$ ,  $S(x, t) \leq M_2(1 + |x|^{m+1})$ .

The constant  $M_1$  depends on  $t_1$ , and  $M_2$  depends on both  $t_0$  and  $t_1$ . In the hypotheses of this theorem,  $S(x, t)$  is assumed to have polynomial growth as  $|x| \rightarrow \infty$  with some degree  $r$ . The theorem states that  $r$  can be replaced by  $\rho$ , or indeed by  $m + 1$  provided  $t \geq t_0 > 0$ . Purely formal arguments suggest that  $m + 1$  is best possible, and this is confirmed by the lower estimate for  $S(x, t)$  made in Section 5.

*Proof of Theorem 3.1* We first consider  $m > 1$ . By (2.3)–(2.6) and (2.9),

$$\begin{aligned} L(x, t, u) &\leq B_1(1 + |x|^{2m} + |u|^2) \\ S^0(x) &\leq B_1(1 + |x|^l) \end{aligned} \tag{3.1}$$

for some  $B_1$ . Given  $x \in R^n$  we choose the following open loop control  $u(\tau)$ ,  $0 \leq \tau \leq t$ . Let  $u(\tau) = \dot{\eta}(\tau)$ , where the components  $\eta_i(\tau)$ , satisfy the differential equation

$$\dot{\eta}_i = -(\text{sgn } x_i)|\eta_i|^m \quad i = 1, \dots, n, \tag{3.2}$$

with  $\eta(0) = x$ . From (2.7)

$$\dot{\zeta}(\tau) = \eta(\tau) + \zeta(\tau), \quad 0 \leq \tau \leq t,$$

$$\zeta(\tau) = \int_0^\tau \sigma[\zeta(\theta)] d\omega(\theta).$$

Since  $\sigma$  is bounded,  $E\|\zeta\|_t^r < \infty$  for each  $r$ . By explicitly integrating (3.2) we

find, since  $m > 1$ , that

$$\int_0^t \eta_i(\tau)^{2m} d\tau \leq \frac{1}{m+1} |x_i|^{m+1} \leq \frac{1}{m+1} |x|^{m+1},$$

$$E \int_0^t |\xi(\tau)|^{2m} d\tau \leq 2^{2m} \left[ \int_0^t |\eta(\tau)|^{2m} d\tau + E \int_0^t |\zeta(\tau)|^{2m} d\tau \right] \leq M_3(1 + |x|^{m+1})$$

for some  $M_3$ .

Since  $u_i^2 = \dot{\eta}_i^2 = \eta_i^{2m}$ ,

$$\int_0^t |u(\tau)|^2 d\tau \leq \frac{n}{m+1} |x|^{m+1}.$$

Since  $|\eta(t)| \leq |x|$ ,

$$E|\xi(t)|^l \leq E(|x| + |\zeta(t)|)^l \leq \kappa(1 + |x|^l)$$

for some  $\kappa$ . From (2.10), (3.1) we get

$$J(x, t, u) \leq M_1(1 + |x|^\rho), \quad \rho = \max(m+1, l)$$

for some  $M_1$ . By part (a) of the Verification Theorem,  $S(x, t) \leq J(x, t, u)$ , which implies (i) when  $m > 1$ .

For  $t > t_0 > 0$ ,  $|\eta(t)|$  is bounded by a constant not depending on  $x = \eta(0)$ . Since  $\xi(t) = \eta(t) + \zeta(t)$ , and  $E_x |\zeta(t)|^l$  is bounded, this bounds  $E_x S^0[\xi(t)]$  by a constant not depending on  $x$ . The estimates above and part (a) of the Verification Theorem then give (ii).

It remains to prove (i) when  $m=1$ . Consider the "trivial" control  $u(\tau) \equiv 0$ . When  $m=1$ ,  $g$  grows at most linearly and  $V$  at most quadratically as  $|x| \rightarrow \infty$ . Moreover,  $E\|\xi\|_t^2 \leq K(1 + |x|^2)$  for some  $K$ . Using again (a) of the Verification Theorem, we get again (i) with  $\rho = \max(2, l)$ . [When  $m=1$ , this is a known result, obtained without using stochastic control arguments.]

#### 4. AN EXISTENCE THEOREM

In this section we give a stochastic control proof of a theorem asserting that the dynamic programming equation (2.2) with the initial data  $S^0$  has a solution  $S$ . The argument is essentially taken from Fleming [4, p. 222 and top p. 223]. Since (2.2) is equivalent to the linear equation (2.1), with



positive initial data  $p^0$ , one could get existence of  $S$  from other results which give existence of positive solutions to (2.1), see Sheu [11]. However, the stochastic control proof gives a polynomial growth condition on  $S$  used in the Verification Theorem (Section 2).

Let  $0 < \alpha \leq 1$ . We say that a function  $\phi$  with domain  $Q$  is of class  $C_\alpha$  if the following holds. For any compact  $\Gamma \subset Q$ , there exists  $M$  such that  $(x, t), (x', t') \in \Gamma$  imply

$$|\phi(x', t') - \phi(x, t)| \leq M[|t' - t|^{\alpha/2} + |x' - x|^\alpha]. \quad (4.1)$$

We say that  $\phi$  is of class  $C_\alpha^{2,1}$  if  $\phi, \phi_{x_i}, \phi_{x_i x_j}, \phi_t$  are of class  $C_\alpha$ ,  $i, j = 1, \dots, n$ .

In this section the following assumptions are made. The matrix  $\sigma(x)$  is assumed constant. By a change of variables in  $R^n$  we may take

$$\sigma = \text{identity}. \quad (4.2)$$

For fixed  $t$ ,  $g(\cdot, t), V(\cdot, t)$  are of class  $C^1$  on  $R^n$ , and  $g, g_{x_i}, V, V_{x_i}, i = 1, \dots, n$ , are of class  $C_\alpha$  for some  $\alpha \in (0, 1]$ . Moreover,

$$|g(x, t)| \leq \gamma_1 + \gamma_2 |x|^m, \quad m \geq 1. \quad (4.3)$$

with  $\gamma_2$  small enough that (4.8) below holds. (If  $g \in \mathcal{P}_\mu$  with  $\mu < m$ , then we can take  $\gamma_2$  arbitrarily small.) We assume that

$$a_1 |x|^{2m} - a_2 \leq -V(x, t) \leq A(1 + |x|^{2m}) \quad (4.4)$$

for some positive  $a_1, a_2, A$  and that

$$g_x \in \mathcal{P}_m, \quad V_x \in \mathcal{P}_{2m}. \quad (4.5)$$

We assume that  $S^0 \in C^3 \cap \mathcal{P}_l$  for some  $l \geq 0$ , and

$$\lim_{|x| \rightarrow \infty} S^0(x) = +\infty \quad (4.6)$$

$$|S_x^0| \leq C_1 S^0 + C_2 \quad (4.7)$$

for some positive  $C_1, C_2$ .

*Example* Suppose that  $V(x, t) = -kV_0(x) + V_1(x, t)$  with  $V_0(x)$  a positive, homogeneous polynomial of degree  $2m$ ,  $k > 0$ , and  $V_1(x, t)$  a polynomial in

$x$  of degree  $\leq 2m-1$  with coefficients Hölder continuous functions of  $t$ . Suppose that  $g(x, t)$  is a polynomial of degree  $\leq m-1$  in  $x$ , with coefficients Hölder continuous in  $t$ , and  $S^0(x)$  is a polynomial of degree  $l$  satisfying (4.6). Then all of the above assumptions hold.

From (2.9), (4.2),  $L = \frac{1}{2}|u-g|^2 - V$ . If  $\gamma_2$  in (4.3) is small enough, then

$$\beta_1(|u|^2 + |x|^{2m}) - \beta_2 \leq L(x, t, u) \leq B(1 + |u|^2 + |x|^{2m}) \quad (4.8)$$

for suitable positive  $\beta_1, \beta_2, B$ . Moreover,

$$\begin{aligned} L_x &= -g_x(u-g) - V_x, \\ |L_x| &\leq \frac{1}{2}|u|^2 + |g_x|^2 + \frac{1}{2}|g|^2 + |V_x|, \end{aligned}$$

where  $|g_x|$  denotes the operator norm of  $g_x$  regarded as a linear transformation on  $R^n$ . From (4.3), (4.5), (4.8)

$$|L_x| \leq C_1 L + C_2 \quad (4.9)$$

for some positive  $C_1, C_2$  (which we may take the same as in (4.7).)

**THEOREM 4.1** *Let  $r = \max(2m, l)$ . Then Eq. (2.2) with initial data  $S(x, 0) = S^0(x)$  has a unique solution  $S(x, t)$  of class  $C_\alpha^{2,1} \cap \mathcal{P}_r$ , such that  $S(x, t) \rightarrow \infty$  as  $|x| \rightarrow \infty$  uniformly for  $0 \leq t \leq t_1$ .*

*Proof* We follow Fleming [4, Section 5]. For  $k=1, 2, \dots$ , let us impose the constraint  $|u| \leq k$  on the feedback controls admitted as drifts in (2.7). Let

$$S_k(x, t) = \min_{|u| \leq k} J(x, t; \mathbf{u}). \quad (4.10)$$

Then  $S_k$  is a  $C_\alpha^{2,1}$  solution to the corresponding dynamic programming equation

$$(S_k)_t = \frac{1}{2} \Delta S_k + H_k(x, t, (S_k)_x), \quad (4.11)$$

$$H_k(x, t, (S_k)_x) = \min_{|u| \leq k} [L(x, t, u) + (S_k)_x u].$$

The initial data are again  $S_k(x, 0) = S^0(x)$ . The minimum in (4.10) is attained by an admissible  $\mathbf{u}_k^p$ . See Fleming and Rishel [7, p. 172].

Now  $S_1 \geq S_2 \geq \dots$ ; and  $S_k$  is bounded below since  $L$  and  $S^0$  are bounded

below by (4.6), (4.9). Let  $S = \lim_{k \rightarrow \infty} S_k$ . Let us show that  $(S_k)_x$  is bounded independent of  $k$  uniformly for  $(x, t)$  in any compact set. Once this is established standard arguments in the theory of parabolic partial differential equations imply that  $S \in C_{\alpha}^{2,1}$  and  $S$  satisfies (2.2). For  $(S_k)_x$  there is the probabilistic representation

$$(S_k)_x(x, t) = E_x \left\{ \int_0^t L_x[\xi_k(\tau), t - \tau, u_k(\tau)] d\tau + S_x^0[\xi_k(t)] \right\}, \quad (4.12)$$

where  $\xi_k$  is the solution to (2.7) with  $u = \mathbf{u}_k^{\text{op}}$ ,  $\xi_k(0) = x$ , and

$$u_k(\tau) = \mathbf{u}_k^{\text{op}}(\xi_k(\tau), \tau).$$

This can be proved exactly as in Fleming [4, Lemma 3]. Another proof, based on differentiating (4.10) with respect to  $x_i$ ,  $i = 1, \dots, n$ , is given in Fleming [5, Lemma 5.3]. From (4.7), (4.9), (4.12)

$$|(S_k)_x(x, t)| \leq C_1 E_x \left\{ \int_0^t L_x[\xi_k(\tau), t - \tau, u_k(\tau)] d\tau + S_x^0[\xi_k(t)] \right\} + C_2(t + 1),$$

or since  $\mathbf{u}_k^{\text{op}}$  is optimal

$$|(S_k)_x(x, t)| \leq C_1 S_k(x, t) + C_2(t + 1). \quad (4.13)$$

Since  $S_k(x, t)$  is bounded uniformly on compact sets, (4.12) gives the required bound for  $|(S_k)_x|$  uniformly on compact sets.

For the "trivial" control 0, we have by (4.8) and  $S^0 \in \mathcal{P}_l$

$$J(x, t, 0) \leq B_1(1 + E_x \|\tilde{\xi}\|_t^r), \quad r = \max(2m, l)$$

for suitable  $B_1$ . When  $u(\tau) \equiv 0$ ,  $\sigma = I$ , we have  $\tilde{\xi}(\tau) = x + w(\tau)$ . For suitable  $M$  we have

$$S_k(x, t) \leq J(x, t, 0) \leq M(1 + |x|^r), \quad k = 1, 2, \dots$$

Hence  $S(x, t)$  satisfies the same inequality. Since  $S$  is bounded below, this implies  $S \in \mathcal{P}_r$ .

Let us show that  $S(x, t) \rightarrow \infty$  as  $|x| \rightarrow \infty$ , uniformly for  $0 \leq t \leq t_1$ . Since  $S_k(x, t) = J(x, t; \mathbf{u}_k^{\text{op}})$ , (4.8) implies

$$S_k(x, t) \geq \beta_1 E_x \int_0^t [ |u_k(\tau)|^2 + |\xi_k(\tau)|^{2m} ] d\tau - \beta_2 t + E S^0[\xi_k(t)].$$

Given  $\lambda > 0$  there exists  $R_1$  such that  $|x| \geq R_1$  implies  $S^0(x) \geq \lambda$ , by (4.6). Let  $R_2 > R_1$  and consider the events

$$A_1 = \{ \|\xi_k - x\|_t \leq R_2 - R_1 \}$$

$$A_2 = \{ \|v_k\|_t \geq \frac{1}{2}(R_2 - R_1) \}, \quad v_k(\tau) = \int_0^\tau u_k(\theta) d\theta$$

$$A_3 = \{ \|w\|_t \geq \frac{1}{2}(R_2 - R_1) \}$$

with  $\|\cdot\|_t$  the sup norm on  $[0, t]$ . Since

$$\xi_k(\tau) - x = v_k(\tau) + w(\tau), \quad 0 \leq \tau \leq t,$$

$A_1^c \subset A_2 \cup A_3$ . For  $R_2 - R_1$  large enough,  $P(A_3) < \frac{1}{4}$  and hence  $P(A_1) + P(A_2) \geq \frac{3}{4}$ . From Cauchy-Schwarz

$$\frac{1}{4}(R_2 - R_1)^2 P(A_2) \leq t \dot{L}_x \int_0^t |u_k(\theta)|^2 d\theta.$$

Let  $|x| \geq R_2$ . On  $A_1$ ,  $|\xi_k(t)| \geq R_1$  and hence  $S^0[\xi_k(t)] \geq \lambda$ . For  $|x| \geq R_2$

$$S_k(x, t) \geq \frac{\beta_1}{4t} (R_2 - R_1)^2 P(A_2) + \lambda P(A_1) - (\beta_2 t + \beta_3)$$

with  $\beta_3$  a lower bound for  $S^0(x)$  on  $R^n$ . Since the right side does not depend on  $k$ ,  $S$  satisfies the same inequality. This implies that  $S(x, t) \rightarrow \infty$  as  $|x| \rightarrow \infty$ , uniformly for  $0 \leq t \leq t_1$ .

To obtain uniqueness,  $p = \exp(-S)$  is a  $C_x^{2,1}$  solution of (2.1), with  $p(x, t) \rightarrow 0$  as  $|x| \rightarrow \infty$  uniformly for  $0 \leq t \leq t_1$ . Since  $V(x, t)$  is bounded above, the maximum principle for linear parabolic equations implies that  $p(x, t)$  is unique among solutions to (2.1) with these properties, and with initial data  $p(x, 0) = p^0(x) = \exp[-S^0(x)]$ . Hence,  $S$  is also unique, proving Theorem 4.1.

It would be interesting to remove the restriction that  $\sigma = \text{constant}$  made in this section.

## 5. A LOWER ESTIMATE FOR $S(x, t)$

To complement the upper estimates in Theorem 3.1, let us give conditions under which  $S(x, t) \rightarrow +\infty$  as  $|x| \rightarrow \infty$  at least as fast as  $|x|^{m+1}$ ,  $m \geq 1$ . This is done by establishing a corresponding exponential rate of decay to 0 for

$p(x, t)$ . In this section we make the following assumptions. We take  $\sigma \in C^2$  with

$$\sigma, \sigma^{-1}, \sigma_{x_i} \text{ bounded, } \sigma_{x_i x_j} \in \mathcal{P}_r, \quad i, j = 1, \dots, n, \quad (5.1)$$

for some  $r > 0$ . For each  $t$ ,  $g(\cdot, t) \in C^2$ . Moreover,

$$g \in \mathcal{P}_\mu, \quad \mu < m, \quad g_{x_i} \in \mathcal{P}_r, \quad g_{x_i x_j} \in \mathcal{P}_r, \quad (5.2)$$

and  $g, g_{x_i}, g_{x_i x_j}$  are continuous on  $Q$ . For each  $t$ ,  $V(\cdot, t) \in C^2$ . Moreover,  $V$  satisfies (4.4),

$$V_{x_i} \in \mathcal{P}_r, \quad V_{x_i x_j} \in \mathcal{P}_r, \quad (5.3)$$

and  $V, V_{x_i}, V_{x_i x_j}$  are continuous on  $Q$ . We assume that  $p^0 \in C^2$  and that there exist positive  $\beta, M$  such that

$$\exp[\beta|x|^{m+1}][|p^0(x)| + |p_{x_i}^0(x)| + |p_{x_i x_j}^0(x)|] \leq M. \quad (5.4)$$

**THEOREM 5.1** *Let  $p(x, t)$  be a  $C^{2,1}$  solution to (2.1) such that  $p(x, t) \rightarrow 0$  as  $|x| \rightarrow \infty$ , uniformly for  $0 \leq t \leq t_1$ . Then there exists  $\delta > 0$  such that  $\exp[\delta|x|^{m+1}]p(x, t)$  is bounded on  $Q$ .*

*Proof* Let

$$\psi(x) = (1 + |x|^2)^{\frac{m+1}{2}}, \quad \pi(x, t) = \exp[\delta\psi(x)]p(x, t).$$

Then  $\pi$  is a solution to

$$\pi_t = \frac{1}{2} \text{tr } a \pi_{xx} + \bar{g} \cdot \pi_x + \bar{V} \pi, \quad (5.5)$$

$$\bar{g} = g - \delta a \psi_x$$

$$\bar{V} = V - \delta g \cdot \psi_x + \frac{1}{2}(\delta^2 a \psi_x \cdot \psi_x - \delta \text{tr } a \psi_{xx}).$$

By Sheu [11, Theorem 1], Eq. (5.5) with initial data  $\pi^0 = \exp(\delta\psi)p^0$  has for small enough  $\delta > 0$  the probabilistic solution

$$\tilde{\pi}(x, t) = E_x \left\{ \pi^0[X(t)] \exp \int_0^t [\sigma^{-1} \bar{g} \cdot dw - \frac{1}{2} |\sigma^{-1} \bar{g}|^2 d\tau + \bar{V} d\tau] \right\}, \quad (5.6)$$

where  $X(t)$  satisfies

$$dX = \sigma[X(\tau)] dw, \quad \tau \geq 0, \quad (5.7)$$

with  $X(0)=x$ . In the integrands  $\sigma^{-1}\bar{g}$  and  $\bar{V}$  are evaluated at  $(X(\tau), \tau)$ . The proof in Sheu [11] that  $\tilde{\pi}$  satisfies (5.5) is done by approximating  $\bar{g}, \bar{V}$  by functions  $\bar{g}_n, \bar{V}_n$ , for which the corresponding  $\tilde{\pi}_n$  tend to  $\tilde{\pi}$  boundedly and pointwise. By standard estimates for partial derivatives to solutions of linear parabolic partial differential equations,  $\tilde{\pi}$  is  $C^{2,1}$  and satisfies (5.5).

Then  $\tilde{p} - \exp(-\delta\psi)\tilde{\pi}$  is a  $C^{2,1}$  solution to (2.1), with initial data  $p^0$ , and with  $\tilde{p}(x, t)$  tending to 0 as  $|x| \rightarrow \infty$  uniformly for  $0 \leq t \leq t_1$ . By the maximum principle,  $\tilde{p} = p$  which implies that  $\exp[\delta|x|^{m+1}]p$  is bounded on  $Q$ . This proves Theorem 5.1.

Since  $S = -\log p$ , we get by taking logarithms:

COROLLARY For some positive  $\delta, \delta_1$

$$S(x, t) \geq \delta|x|^{m+1} - \delta_1. \quad (5.8)$$

## 6. CONNECTION WITH THE PATHWISE FILTER EQUATION

The generator  $A$  of the signal process in (1.1) satisfies for  $\phi \in C^2$

$$A\phi = \frac{1}{2} \text{tr } a(x)\phi_{xx} + b(x) \cdot \phi_x.$$

The pathwise filter equation (1.5) for  $p = p^y$  is

$$p_t = (A^y)^* p + \tilde{V}^y p,$$

where

$$A^y \phi = A\phi - y(t)a(x)h_x(x) \cdot \phi_x$$

$$\tilde{V}^y(x, t) = \frac{-1}{2} h(x)^2 - y(t)Ah(x) + \frac{1}{2}y(t)^2 h_x(x)' a(x) h_x(x). \quad (6.1)$$

Hence, in (1.5) we should take

$$g^y = -b + y(t)ah_x + \gamma, \quad \gamma_j = \sum_{i=1}^n \frac{\partial a_{ij}}{\partial x_j}, \quad j = 1, \dots, n, \quad (6.2)$$

$$V^y = \tilde{V}^y - \text{div}(b - y(t)ah_x) + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 a_{ij}}{\partial x_i \partial x_j}. \quad (6.3)$$

To satisfy the various assumptions about  $g = g^y$ ,  $V = V^y$  made above,

suitable conditions on  $\sigma$ ,  $b$ , and  $h$  must be imposed. To obtain the local Hölder conditions needed in Section 4 we assume that  $y(\cdot)$  is Hölder continuous on  $[0, t]$ . This is no real restriction, since almost all observation trajectories  $y(\cdot)$  are Hölder continuous.

To avoid unduly complicating the exposition let us consider only the following special case. We take  $\sigma = \text{identity}$ , an assumption already made for the existence theorem in Section 4. We assume that  $b \in C^3$  with  $b, b_x$  bounded, and all second, third order partial derivatives of  $b$  of-class  $\mathcal{P}_r$  for some  $r$ . Let  $h$  be a polynomial of degree  $m$  and  $S^0$  a polynomial of degree  $l$ , such that  $h = h_1 + h_2$ ,  $S^0 = S_1^0 + S_2^0$  where  $h_1, S_1^0$  are homogeneous polynomials of degrees  $m, l$ ,

$$\lim_{|x| \rightarrow \infty} |h_1(x)| = \infty, \quad \lim_{|x| \rightarrow \infty} S_1^0 = +\infty, \quad (6.4)$$

$h_2$  is of degree  $< m$  and  $S_2^0$  of degree  $< l$

Then all of the hypotheses in Sections 2–4 hold. In (6.2),  $g^y$  has polynomial growth of degree  $m-1$  as  $|x| \rightarrow \infty$ , while in (6.3)  $V^y$  is the sum of the degree  $2m$  polynomial  $-\frac{1}{2}h^2(x)$  and terms with polynomial growth of degree  $< 2m$ .

Let  $S^y = -\log p^y$ . From Theorem 3.1 we get the upper bounds

- i)  $S^y(x, t) \leq M_1(1 + |x|^\rho)$ ,  $0 \leq t \leq t_1$ ,  $\rho = \max(m+1, l)$ ,
- ii)  $S^y(x, t) \leq M_2(1 + |x|^{m+1})$ ,  $0 < t_0 \leq t \leq t_1$ ,  $m > 1$ ,

where  $M_1, M_2$  depend on  $y$ . For  $p^0 = \exp(-S^0)$  to satisfy (5.4) we need  $l \geq m+1$ . The corollary to Theorem 5.1 then gives the lower bound

$$S^y(x, t) \geq \delta |x|^{m+1} - \delta_1, \quad 0 \leq t \leq t_1. \quad (6.6)$$

From (6.5) (ii) and (6.6) we see that  $S^y(x, t)$  increases to  $+\infty$  like  $|x|^{m+1}$ , at least for  $m > 1$  and  $t$  bounded away from 0, and for  $0 \leq t \leq t_1$ , in case  $l = m+1$ .

Finally,  $q = \exp(y(t)h)p$  is a solution to the Zakai equation. For any  $\phi \in C_b$  (i.e.,  $\phi$  continuous and bounded on  $R^n$ ) let

$$\tilde{\Lambda}_t(\phi) = \int_{R^n} \phi(x) q(x, t) dx,$$

$$\Lambda_t(\phi) = \hat{E} \left\{ \phi[x(t)] \exp \int_0^t (h[x(\tau)] dy - \frac{1}{2} h[x(\tau)]^2 d\tau) \middle| \mathcal{F}_t(y) \right\},$$

where  $\hat{E}$  denotes expectation with respect to the probability measure  $\hat{P}$  obtained by eliminating the drift term in (1.2) by a Girsanov

transformation. The measure  $\Lambda_t$  is the unnormalized conditional distribution of  $x(t)$ . Then  $\Lambda_t$  is also a (weak) solution of the Zakai equation, with  $\dot{E}\Lambda_t(1) = 1$ . By a result of Sheu [11, Theorem 4],  $\tilde{\Lambda}_t = \Lambda_t$ .

## References

- [1] J. Baras, G. L. Blankenship and W. E. Hopkins, *Existence, Uniqueness, and Asymptotic Behavior of Solutions to a Class of Zakai Equations with Unbounded Coefficients*, Elec. Eng. Dept., University of Maryland, preprint.
- [2] M. H. A. Davis, Pathwise nonlinear filtering, in: *Stochastic Systems* (ed. M. Hazewinkel and J. C. Willems), NATO Advanced Study Institute Series, Reidel, Dordrecht, June-July 1980.
- [3] M. H. A. Davis and S. I. Marcus, An introduction to nonlinear filtering, *ibid*.
- [4] W. H. Fleming, Controlled diffusions under polynomial growth conditions, in: *Control Theory and the Calculus of Variations* (ed. A. V. Balakrishnan), Academic Press, 1969, pp. 209-234.
- [5] W. H. Fleming, Stochastic control for small noise intensities, *SIAM J. on Control* **9** (1971), 473-517.
- [6] W. H. Fleming, Stochastic calculus of variations and mechanics, *J. Optimiz. Th Applic* (to appear).
- [7] W. H. Fleming and R. W. Rishel, *Deterministic and Stochastic Optimal Control*, Springer-Verlag, 1975.
- [8] O. Hijab, Minimum energy estimation, Ph.D. Thesis, University of California, Berkeley, 1980.
- [9] E. Hopf, The partial differential equation  $u_t + uu_x = \mu u_{xx}$ , *Communication Pure Appl. Math.* **3** (1950), 201-230.
- [10] S. K. Mitter, On the analogy between mathematical problems of nonlinear filtering and quantum physics, *Ricerca di Automatica* **10** (1979), 163-216.
- [11] S. J. Sheu, Solution of certain parabolic equations with unbounded coefficients and its application to nonlinear filtering, submitted to *Stochastics*.
- [12] H. J. Sussmann, Rigorous results on the cubic sensor problem, in: *Stochastic Systems* (ed. M. Hazewinkel and J. C. Willems), NATO Advanced Study Institute Series, Reidel, Dordrecht, June-July 1980.