OPTIMAL CONTROL OF LINEAR INTEGRAL EQUATIONS WITH A QUADRATIC COST FUNCTION: THE FINITE TIME INTERVAL PROBLEM.

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1. Preliminaries.

This chapter contains a generalisation to infinite dimensional systems of the classical optimal control theory of linear time - dependent systems with a linear - quadratic cost function. The structure of the problem and the results are formally identical to what is already known in the finite - dimensional case. For the infinite - dimensional case, the initial hypotheses and the interpretation of the results are both very important. A slightly too strong hypothesis can rule out a complete family of infinite - dimensional systems and reduce the applicability of the whole theory solely to finite - dimensional systems.

Important classes of systems governed by linear partial differential equations or differential delay equations admit an internal representation in terms of a certain evolution operator which characterizes the system. In this chapter we shall assume that such an evolution operator exists and that the evolution of the state of the controlled system can be described by an integral equation. We shall present a general theory of the optimal control problem for such a linear system defined in a finite time interval when the cost function to be minimized is linear-quadratic. Optimality conditions will be obtained and the feedback operator and the reference function will be characterized by integral equations. Finally in the last section we shall show how differential equations can be directly obtained from the integral equations when the evolution operator is generated by some linear operational differential equation.

Notation and Terminology.

Let $\mathbb{R}$ be the field of all reals. Let X and Y be two Banach spaces on $\mathbb{R}$ with norms $|\ |_X$ and $|\ |_Y$. We denote by $\mathcal{L}(X,Y)$ the real Banach space of all continuous linear maps $T:X \to Y$ endowed with the natural norm

$$(1.1) \qquad |T| = \sup \{ |Tx|_Y : |x|_X = 1\} ;$$

when $\mathcal{L}(X,Y)$ is endowed with the strong (resp. weak) operator topology, it will be written $\mathcal{L}_s(X,Y)$ (resp. $\mathcal{L}_w(X,Y)$). When X = Y we shall abbreviate and use the notation $\mathcal{L}(X)$, $\mathcal{L}_s(X)$ and $\mathcal{L}_w(X)$. The adjoint operator of T in $\mathcal{L}(X,Y)$ is an element of $\mathcal{L}(Y^*,X^*)$ which will be denoted T^*. When X and Y are Hilbert spaces on $\mathbb{R}$, it will be understood that the adjoint operator T^* of T in $\mathcal{L}(X,Y)$ belongs to $\mathcal{L}(Y,X)$.

Let X be a Hilbert space on $\mathbb{R}$ with inner product $(\cdot,\cdot)_X$. T in $\mathcal{L}(X)$ is said to be self-adjoint if $T^* = T$. A self-adjoint element T of $\mathcal{L}(X)$ is said to be positive, $T \geq 0$ (resp. positive-definite, $T > 0$), if for all x in X, $(x,Tx)_X \geq 0$ (resp. for all $x \neq 0$ in X, $(x,Tx)_X > 0$).

Given $-\infty \leq a < b \leq +\infty$, we define

$$(1.2) \qquad I(a,b) = [a,b] \cap \mathbb{R}$$

$$(1.3) \qquad P(a,b) = \{(t,s) \in \mathbb{R} \times \mathbb{R} : s \in I(a,b),\ t \in I(s,b)\}.$$

Let X and Y be a Banach spaces on $\mathbb{R}$. $C(a,b;X)$ will denote the Banach space of all bounded continuous maps $I(a,b) \to X$ endowed with the sup norm, $L^p(a,b;X)$ the Banach space of all Lebesgue measurable maps $I(a,b) \to X$ which are p-integrable ($1 \le p < \infty$) or essentially bounded ($p = \infty$). The space of all maps $G:I(a,b) \to \mathcal{L}(X,Y)$ such that for all x in X the map $t \mapsto G(t)x$ belongs to $L^p(a,b;Y)$ (resp. $C(a,b;Y)$) will be denoted by $L^p(a,b;\mathcal{L}_s(X,Y))$ ($C(a,b;\mathcal{L}_s(X,Y))$). Given a in $\mathbb{R}$, $L^p_{loc}(a,\infty;X)$ (resp. $L^p_{loc}(a,\infty;\mathcal{L}_s(X,Y))$) will denote the Fréchet space of all measurable maps $[a,\infty) \to X$ (resp. $\mathcal{L}_s(X,Y)$) the restriction of which to $[a,T]$ belongs to $L^p(a,T;X)$ (resp. $L^p(a,T;\mathcal{L}_s(X,Y))$) for all T, $a < T < \infty$.

When X is a Hilbert space on $\mathbb{R}$, $H^1(a,b;X)$ will denote the Sobolev space of all maps x in $L^2(a,b;X)$ with a distributional derivative Dx in $L^2(a,b;X)$; $C(a,b;X_w)$ will denote the space of all maps $g:I(a,b) \to X$ such that for all x in X the map $t \mapsto (g(t),x)_X$ belongs to $C(a,b;\mathbb{R})$. Finally $C(a,b;\mathcal{L}_w(X,Y))$ will denote the space of all maps $G:I(a,b) \to \mathcal{L}(X,Y)$ such that for all x in X the map $t \mapsto G(t)x$ belongs to $C(a,b;Y_w)$.

2. System description, adjoint system.

In this section we introduce all the elements which will allow us to start with an internal representation of a linear controlled system (S).

Definition 2.1. (i) Let G belong to $L^{\infty}_{loc}(0,\infty;\mathcal{L}_s(U,X))$, let f and g belong to $L^2_{loc}(0,\infty;X)$ and let the map $\Lambda:P(0,\infty)\to\mathcal{L}(X)$ define an evolution operator with the following properties:

a) $\Lambda(t,r) = \Lambda(t,s)\,\Lambda(s,r),\ t \ge s \ge r \ge 0,$

b) $\Lambda(t,t) = I$ (identity in $\mathcal{L}(X)$), $t \ge 0$.

c) $\forall\, x \in X,\ (t,s) \mapsto \Lambda(t,s)x:\ P(0,\infty) \to X$ is continuous.

(ii) The state $\xi(t;s,x_0,v)$ of system (S) at time $t \ge s$ with initial datum x_0 at time $s \ge 0$ and control function v in $L^2_{loc}(s,\infty;U)$ is defined as follows:

$$(2.1)\qquad \xi(t;s,x_0,v) = \Lambda(t,s)x_0 + \int_s^t \Lambda(t,r)[G(r)v(r) + f(r)]dr.$$

(iii) Given T, $0 < T < \infty$, the state $\pi(t;T,x_T)$ of the adjoint system (S*) at time t, $0 \le t \le T$, with final datum x_T at time T is defined as follows:

$$(2.2)\qquad \pi(t;T,x_T) = \Lambda(T,t)^*x_T + \int_t^T \Lambda(r,t)^*g(r)dr.\square$$

Lemma 2.2. Let X and Y be two Hilbert spaces. (i) Let the map $A:P(0,\infty)\to\mathcal{L}(X,Y)$ be weakly continuous, then the map

$$(t,s) \mapsto |A(t,s)| : \mathcal{P}(0,\infty) \to \mathbb{R}$$

is bounded and measurable on every compact subsets of $\mathcal{P}(0,\infty)$.

(ii) The map $A:\mathcal{P}(0,\infty) \to \mathcal{L}(X,Y)$ is strongly measurable if and only if it is weakly measurable.

Proof. (i) We first prove that $|A(t,s)|$ is bounded on all sets $\mathcal{P}(0,T)$. Consider the family $\{A(t,s) | (t,s) \in \mathcal{P}(0,T)\}$ in $\mathcal{L}(X,Y)$. By continuity for each x and y in X, $(y,A(t,s)x)$ is bounded in $\mathcal{P}(0,T)$. By the Uniform Boundedness Principle $|A(t,s)|$ is bounded in $\mathcal{P}(0,T)$. To show that A is also measurable we introduce for an arbitrary $\delta > 0$ the set

$$V = \{(t,s) \in \mathcal{P}(0,\infty) \mid |A(t,s)| > \delta\}.$$

If $(t_0,s_0) \in V$ we can find x and y with $|x|_X = 1$ and $|y|_Y = 1$ such that $|(y,A(t,s)x)| > \delta$ for every (t,s) in an open ball containing (t_0,s_0). Thus V is open in $\mathcal{P}(0,\infty)$ and the map $(t,s) \mapsto |A(t,s)|:\mathcal{P}(0,\infty) \to \mathbb{R}$ is measurable.

(ii) Cf. HILLE-PHILLIPS p. 72-73.□

Corollary 2.3. Given $A:\mathcal{P}(0,\infty) \to \mathcal{L}(X,Y)$ strongly measurable and bounded, the map $(t,s) \mapsto A(t,s)^*:\mathcal{P}(0,\infty) \to \mathcal{L}(Y,X)$ is also strongly measurable and bounded.□

Corollary 2.4. The conclusions of Lemma 2.2 and Corollary 2.3 remain true for a map A defined on $[0,\infty)$ rather than $\mathcal{P}(0,\infty)$.□

Proposition 2.5. (i) The map $(t,s) \mapsto |A(t,s)| : \mathcal{P}(0,\infty) \to \mathbb{R}$ is measurable and for all T, $0 < T < \infty$, and there exists a constant $c(T) > 0$ such that

$$\sup \{|A(t,s)| \mid (t,s) \in \mathcal{P}(0,T)\} \le c(T). \tag{2.3}$$

(ii) The map

$$(2.4) \qquad (x_0,v) \mapsto \xi(\cdot;s,x_0,v) : X \times L^2(s,T;U) \to C(s,T;X)$$

(resp. $(x_T,g) \mapsto \pi(\cdot;T,x_T) : X \times L^2(s,T;X) \to C(s,T;X_w)$) is affine (resp. linear) and continuous.

Proof. (i) cf. Lemma 2.2. (ii) From definitions. □

3. Formulation of the Problem.

Consider the controlled system (S) defined in a fixed time interval $[0,T]$, $0 < T < \infty$. The function $t \mapsto \xi(t;0,x_0,v)$ defined in $[0,T]$ will be denoted by x. We associate with x_0 and v the cost function

$$(3.1)\quad \begin{aligned} J(v,x_0) &= (x(T),Lx(T)) + \int_0^T [(x(t),Q(t)x(t)) + (v(t),N(t)v(t))_U]dt \\ &+ 2(x(T),\ell) + 2\int_0^T [(x(t),q(t)) + (v(t),n(t))_U]dt, \end{aligned}$$

where $\ell \in X$, $q \in L^2(0,T;X)$, $n \in L^2(0,T;U)$, $L \in \mathcal{L}(X)$, and $Q:[0,T] \to \mathcal{L}(X)$ and $N:[0,T] \to \mathcal{L}(U)$ are strongly measurable and bounded. Moreover

$$(3.2)\qquad L^* = L \geq 0,\ Q(t)^* = Q(t) \geq 0,\ N(t)^* = N(t)$$

and there exists $\nu > 0$ such that

$$(3.3)\qquad (u,N(t)u)_U \geq \nu|u|_U^2,\quad \forall\ u \text{ in } U.$$

Our objective is to show that for each x_0 there exists a unique control function u which minimizes the cost function $J(v,x_0)$ over all control functions v in a closed convex subset U_{ad} of $L^2(0,T;U)$. When U_{ad} is all of $L^2(0,T;U)$ the minimizing control function u will be completely characterized in terms of the "adjoint solution". We shall also show that the control u(t) can be synthesized using a feedback law and that the minimum of the cost function can be expressed in terms of the initial datum x_0. We shall also study the feedback operator P and the reference function r.

4. Examples.

We shall give in this section a few examples for which the general theory developped later may be applied. Our main example, the hereditary systems will however be developped in a separate chapter. Here we shall present other possibilities with less details, since they are more classical and easily available in the litterature.

4.1. Second order parabolic systems.

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$. Let $P(\mathcal{O})$ be the set of infinitely differentiable functions with compact support in $\mathcal{O}$ with the Schwartz' topology. Let $H^1(\mathcal{O})$ be the Sobolev space of order 1 on $\mathcal{O}$, and $H_0^1(\mathcal{O})$ the closure of $P(\mathcal{O})$ in $H^1(\mathcal{O})$. We denote by z the space variable ($z \in \mathcal{O}$).

We take as state space $X = L^2(\mathcal{O})$. We next consider the family of unbounded operators $F(t)$ on X defined by

$$F(t) = a(t)\Delta, \tag{4.1}$$

where Δ is the Laplacian and

$$M \geq a(t) \geq \alpha > 0, \qquad \forall\, t \geq 0. \tag{4.2}$$

The domain of $F(t)$ in X is defined by

$$D(F(t)) = \{x(z) \in H_0^1(\mathcal{O}),\ \Delta x \in L^2(\mathcal{O})\} \tag{4.3}$$

which does not depend on t.

Let us consider the solution $x(t;z)$ of the heat equation

$$\begin{cases} \dfrac{\partial x}{\partial t} - a(t)\Delta x = 0 & t > s \\ x(t)\big|_{\partial\mathcal{O}} = 0 & \\ x(s,z) = x_0(z), & \end{cases} \tag{4.4}$$

where $x_0 \in L^2(\mathcal{O})$. We define the operator $\Lambda(t,x) \in \mathcal{L}(X)$ by

$$x(t) = \Lambda(t,s)x_0, \tag{4.5}$$

then $\Lambda(t,s)$ satisfies properties a), b), c). We can consider the following control problem: choose $v \in L^2(Q)$ where $Q = \mathcal{O} \times (0,T)$ to minimize

$$\begin{aligned} J(v) &= \int_0^T \int_{\mathcal{O}} (x(t,z)-x_d(t;z))^2 dtdz \\ &+ \int_0^T \int_{\mathcal{O}} v^2(t,z)dtdz, \end{aligned} \tag{4.6}$$

where $x(t;z)$ to solution of

$$\begin{cases} \frac{\partial x}{\partial t} - a(t)\Delta x = v \\ x|_{\partial\mathcal{O}} = 0 \\ x(0) = x_0. \end{cases} \tag{4.7}$$

Remark 4.1: Problems of the form (4.6), (4.7) have been studied in full detail in the book of J.L. LIONS [1]. The reader will find in this reference numerous other examples.

4.2. First order hyperbolic systems.

Let $\mathcal{O}$ be an open domain of $\mathbb{R}^n$ with smooth boundary $\partial\mathcal{O}$. Let $a_i(z)$, $z \in \bar{\mathcal{O}}$, $i = 1,\dots,n$, be functions which are continuously differentiable with bounded derivatives. Let

$$\Gamma_- = \{z | z \in \Gamma = \partial\mathcal{O},\ \sum_{i=1}^n a_i(x)\nu_i \leq 0\}, \tag{4.8}$$

where $\nu = \{\nu_i\}$ denotes the outward normal in $z \in \partial\mathcal{O}$. Let again $X = L^2(\mathcal{O})$. We consider in X the linear unbounded operator

$$F = \sum_{i=1}^n a_i \frac{\partial x}{\partial z_i},$$

where

$$D(F) = \{x \in L^2(\mathcal{O}) \mid \sum_i a_i \frac{\partial x}{\partial z_i} \in L^2(\mathcal{O}),\ x|_{\Gamma^-} = 0\}. \tag{4.9}$$

We know (cf. N. BARDOS [1], J.L. LIONS [1]) that the problem

$$\begin{cases} \lambda x + \sum_{i=1}^{n} a_i \frac{\partial x}{\partial z_i} = f \\ x|_{\Gamma^-} = 0 \end{cases} \tag{4.10}$$

has one and only one solution in D(F), provided that

$$\lambda \geq \omega = \sup \tfrac{1}{2} \sum_{i=1}^{n} \frac{\partial a_i}{\partial z_i} . \tag{4.11}$$

The operator F is clearly a closed operator with dense domain in X. Let us check that it generates a semigroup. This follows from applying the Hille-Yosida-Phillips Criterion. Indeed, from (4.10) it follows

$$(\lambda-\omega)x + \omega x + \sum_{i=1}^{n} a_i \frac{\partial x}{\partial z_i} = f.$$

Multiply by x and integrating over $\mathcal{O}$ yields

$$(\lambda-\omega) \int_{\mathcal{O}} x^2 dz + \omega \int_{\mathcal{O}} x^2 dz + \sum_i \int_{\mathcal{O}} a_i \tfrac{1}{2} \frac{\partial x^2}{\partial z_i} = \int_{\mathcal{O}} fx dz. \tag{4.12}$$

By Green's formula we have

$$\int_{\mathcal{O}} a_i \frac{\partial x^2}{\partial z_i} = \int_{\Gamma} a_i \nu_i x^2 - \int_{\mathcal{O}} x^2 \frac{\partial a_i}{\partial z_i} .$$

By virtue of (4.11) and using the boundary condition (4.10) we get

$$(\lambda-\omega) \int_{\mathcal{O}} x^2 dz \leq \int_{\mathcal{O}} fx dz$$

from which it follows that

$$|x| \leq \frac{1}{\lambda-\omega} |f|$$

i.e.

$$(\lambda I-F)^{-1} \leq (\lambda-\omega)^{-1} \quad \text{for} \quad \lambda > \omega$$

which is the Hille Yosida Phillips criterion.

We can therefore give a meaning to the following control problem. The system is described by

$$\text{(4.13)} \qquad \begin{cases} \frac{\partial x}{\partial t} = \sum_i a_i \frac{\partial x}{\partial z} + v \\ x|_{\Gamma^-} = 0 \\ x(0,z) = x_0(z) \in L^2 \end{cases}$$

and the payoff is given by (4.6).

4.3. Boundary control.

In many distributed parameter systems, the control is exerted on the boundary. We shall describe here an approach due to A.V. BALAKRISHNAN [2]. We shall proceed formally.

We want to consider the system

$$\text{(4.14)} \qquad \begin{cases} \frac{\partial y}{\partial t} = \Delta y \\ y|_\Gamma = v \\ y(0) = 0, \end{cases}$$

where $v(\cdot) \in L^2(0,T;L^2(\Gamma))$. We note by $Gv = \xi$ the solution of

$$\text{(4.15)} \qquad \begin{cases} \Delta\xi = 0 \quad \text{in} \quad \mathcal{O} \\ \xi|_\Gamma = v, \end{cases}$$

when v is a continuous function on the boundary. We assume that

(4.16)
$$|Gv|_{L^2(\mathcal{O})} \leq |v|_{L^2(\Gamma)}.$$

Defining

$$x = -\xi + y$$

we see that x satisfies (assuming now that v is differentiable in t)

$$\frac{\partial x}{\partial t} = -\frac{\partial \xi}{\partial t} + \Delta x = -G\mathring{v} + \Delta x$$

$$x|_{\Gamma} = 0$$

$$x(0) = -Gv(0).$$

Let now F be the Laplace operator with zero boundary condition. It generates a semigroup $\Lambda(t)$. Hence we can express x(t) as follows

$$x(t) = -\Lambda(t)Gv(0) - \int_0^t \Lambda(t-s)G\mathring{v}(s)ds$$

$$x(t) = -\Lambda(t)Gv(0) - Gv(t) + \Lambda(t)Gv(0)$$
$$- \int_0^t F\Lambda(t-s)Gv(s)ds,$$

i.e.

(4.17)
$$y(t) = -\int_0^t F\Lambda(t-s)Gv(s)ds.$$

In A.V. BALAKRISHNAN [2], it is shown that (4.17) is well defined for a.e. t and that the mapping

$$v(\cdot) \mapsto y(\cdot) \text{ belongs to } \mathcal{L}\ (L^2(0,T;L^2(\Gamma));\ L^2(0,T;L^2(\mathcal{O}))).$$

To some extent x(t) appears as a state variable and y(t) as an output. Although our theory does not apply directly to these systems, similar methods can be developped (see A.V. BALAKRISHNAN [2], [3]).

5. Necessary and sufficient conditions for optimality.

We now put our original control problem in an abstract form which has a convenient structure for analysis. This is J.L. LIONS [1]'s direct approach to the solution of the optimal control problem.

Lemma 5.1. (i) The cost function is of the form

$$J(v,x_0) = \pi(v,v) - 2\lambda(v) + \text{terms independent of } v, \tag{5.1}$$

where

$$\pi(v,\bar{v}) = (M(T)v, LM(T)\bar{v}) + \int_0^T [(M(t)v, Q(t)M(t)\bar{v}) + (v(t), N(t)\bar{v}(t))_U]dt \tag{5.2}$$

$$\begin{cases} -\lambda(v) = (\ell + Lm(T), M(T)v) + \displaystyle\int_0^T (Q(t)m(t) + q(t), M(t)v)dt \\ \qquad + \displaystyle\int_0^T (n(t), v(t))_U dt \end{cases} \tag{5.3}$$

and

$$m(t) = \Lambda(t,0)x_0 + \int_0^t \Lambda(t,r)f(r)dr \tag{5.4}$$

$$M(t)v = \int_0^t \Lambda(t,r)G(r)v(r)dr, \tag{5.5}$$

where $(v,\bar{v}) \mapsto \pi(v,\bar{v})$ is a real continuous bilinear form defined on $L^2(0,T;U)$ with the following properties

a) $\qquad \pi(v,\bar{v}) = \pi(\bar{v},v), \ \forall \ v,\bar{v}$ in $L^2(0,T;U)$,

b) $\exists\ c > 0$ such that $\pi(v,v) \geq c|v|_U^2$, $\forall\ v$ in $L^2(0,T;U)$

and $v \mapsto \lambda(v)$ is a real continuous linear form defined on $L^2(0,T;U)$.

Proof. Directly from definitions, Proposition 2.5 and Lemma 2.2. □

Now that the problem has been reduced to equation (5.1), it is readily seen that minimizing $J(v,x_0)$ is equivalent to minimizing the form $\pi(v,v) - 2\lambda(v)$. This problem has been considered by J.L. LIONS [1, Thms 1.1 and 1.2] and his results are quoted in the following theorem.

Theorem 5.2. (J.L. LIONS). Let $\mathcal{U}_{ad}$ be a closed convex subset of a real Hilbert space $\mathcal{U}$, let π be a real continuous bilinear form defined on $\mathcal{U}$ such that

(i) $$\pi(u,v) = \pi(v,u),\quad \forall\ u,v \text{ in } \mathcal{U}, \text{ (symmetry)},$$

(ii) $$\exists\ c > 0 \text{ such that } \pi(v,v) \geq c|v|_{\mathcal{U}}^2,\quad \forall\ v \text{ in } \mathcal{U}, \text{ (coerciveness)}.$$

Let λ be a real continuous linear form defined on $\mathcal{U}$, and let

(5.6) $$C(v) = \pi(v,v) - 2\lambda(v).$$

Then there exists a unique u in $\mathcal{U}_{ad}$ such that

(5.7) $$C(u) = \text{Inf}\{C(v) \mid v \in \mathcal{U}_{ad}\}$$

and u is uniquely characterized by the inequality

(5.8) $$\pi(u,v-u) \geq \lambda(v-u),\quad \forall\ v \text{ in } \mathcal{U}_{ad}.$$

<u>Proof</u>. (i) (uniqueness). Let u_1 and u_2 be two points of $\mathcal{U}_{ad}$ for which the minimum is achieved. We can write

$$C\left(\frac{u_1+u_2}{2}\right) + C\left(\frac{u_1-u_2}{2}\right) = \tfrac{1}{2}C(u_1) + \tfrac{1}{2}C(u_2) + \tfrac{1}{2}\lambda(u_2-u_1)$$

and necessarily

$$C\left(\frac{u_1+u_2}{2}\right) = \tfrac{1}{2}C(u_1) + \tfrac{1}{2}C(u_2) - \pi\left(\frac{u_1-u_2}{2}\right)$$

$$\leq \tfrac{1}{2}C(u_1) + \tfrac{1}{2}C(u_2) - \frac{c}{4}\,|u_1-u_2|^2.$$

By convexity $(u_1+u_2)/2$ belongs to $\mathcal{U}_{ad}$ and $|u_1-u_2| = 0$.

(ii) (existence). The functional $C(v)$ is convex and continuous and for all v in $\mathcal{U}$

$$C(v) = \pi(v,v) - 2\lambda(v) \geq c|v|^2 - 2|\lambda||v|.$$

As $|v|$ goes to infinity $C(v)$ goes to $+\infty$. So there exists $M > 0$ for which the set

$$\mathcal{U}_M = \left\{v \in \mathcal{U}_{ad} : C(v) \leq M\right\}$$

is not empty. This set is closed and convex since C is continuous and convex. It is also bounded in view of the inequality of part (i) which is true for all u_1 and u_2 in $\mathcal{U}$. If we let $u_1 = v$ and $u_2 = 0$

$$\frac{c}{4}|v|^2 \le \frac{1}{2}C(v) + \frac{1}{2}C(0) - C(\frac{v}{2}) \le \frac{1}{2}C(v) \le M$$

for all v in $\mathcal{U}_M$. In particular $\mathcal{U}_M$ is weakly compact and there exists u in $\mathcal{U}_M$ for which the minimum is achieved.

(iii) (characterization of u). By definition for all v and u in $\mathcal{U}_{ad}$

$$\begin{aligned} C(v) - C(u) &= \pi(v,v) - 2\lambda(v) - [\pi(u,u) - 2\lambda(u)] \\ &= \pi(v-u,v-u) + 2[\pi(u,v-u) - \lambda(v-u)] \\ &\ge 2[\pi(u,v-u) - \lambda(v-u)]. \end{aligned}$$

If the element u in $\mathcal{U}_{ad}$ is characterized by (5.8) the L.H.S. of the above inequality is positive for all v in $\mathcal{U}_{ad}$ and the minimum of C is achieved for the element u in $\mathcal{U}_{ad}$. Conversely assume that the minimum of the cost function C is achieved for u in $\mathcal{U}_{ad}$. Then for all v in $\mathcal{U}_{ad}$ and θ in (0,1)

$$C(u) \le C((1-\theta)u + \theta v)$$

and

$$\lim_{\theta \to 0} \frac{1}{\theta}[J(u + \theta(v-u)) - J(u)] \ge 0.$$

If the limit exists

$$J'(u)\cdot(v-u) \geq 0 \quad \forall \quad v \in \mathcal{U}_{ad}.$$

But the limit exists and

$$J'(u)\cdot(v-u) = 2[\pi(u,v-u) - \lambda(v-u)].$$

This yields inequality (5.8). □

<u>Remark</u>. When $\mathcal{U} = \mathcal{U}_{ad}$ inequality (5.8) is equivalent to <u>Euler's equation</u>

$$\pi(u,w) = \lambda(w), \quad \forall \quad w \text{ in } \mathcal{U}.$$

The above theorem asserts the existence and uniqueness of the optimal control function u which is characterized by the following inequality

$$(5.9)\quad \begin{cases} (M(T)u, LM(T)(v-u)) + \int_0^T [(M(t)u, Q(t)M(t)(v-u)) + (u(t), N(t)(v(t)-u(t)))_U]dt \\ \quad + (\ell+Lm(T), M(T)(v-u)) + \int_0^T [(Q(t)m(t)+q(t), M(t)(v-u)) + (n(t), v(t)-u(t))_U]dt \\ \quad \geq 0, \quad \forall \quad v \text{ in } U_{ad}. \end{cases}$$

Let $y(t) = M(t)(v-u)$ and $x(t) = M(t)u + m(t)$. The L.H.S. of (5.9) can be rewritten in the form

$$(5.10)\quad \begin{aligned} &(Lx(T), y(T)) + \int_0^T (Q(t)x(t) + q(t), y(t))dt \\ &+ \int_0^T (N(t)u(t) + n(t), v(t) - u(t))_U dt \geq 0, \quad \forall \quad v \text{ in } U_{ad}. \end{aligned}$$

We now introduce the adjoint system

$$(5.11)\qquad p(t) = \Lambda(T,t)^*Lx(T) + \int_t^T \Lambda(r,t)^*[Q(r)x(r) + q(r)]dr.$$

The term

$$\int_0^T (Q(t)x(t) + q(t),\ y(t))dt$$

in (5.10) can be expanded as follows

$$\int_0^T (Q(t)x(t) + q(t),\ \int_0^t \Lambda(t,r)G(r)[v(r)-n(r)]dr)dt$$

$$= \int_0^T \left(\int_r^T \Lambda(t,r)^*[Q(t)x(t) + q(t)]dt,\ G(r)[v(r) - u(r)]\right)dr$$

$$= \int_0^T (p(r) - \Lambda(T,r)^*Lx(T),\ G(r)[v(r) - u(r)])dr$$

$$= \int_0^T (G(r)^*p(r),\ v(r)-u(r))_U dr - (Lx(T),\ \int_0^T \Lambda(T,r)G(r)[v(r)-u(r)]dr)$$

$$= \int_0^T (G(r)^*p(r),\ v(r)-u(r))_U dr - (Lx(T),y(T)).$$

We can now substitute this last result in (5.10) to obtain

$$(5.12)\qquad \int_0^T (N(t)u(t) + n(t) + G(t)^*p(t),\ v(t) - u(t))_U dt \geq 0,\ \forall\ \ v \text{ in } U_{ad}.$$

We have proved the following result.

<u>Theorem 5.3</u>. Let U_{ad} be a closed convex subset of $L^2(0,T;U)$.

(i) For each x_0 there exists a unique control function u in U_{ad} which minimizes $J(v,x_0)$ in U_{ad}. Moreover it is completely characterized by (5.12), where p is as defined in (5.11).

(ii) If $U_{ad} = L^2(0,T;U)$, then

$$u(t) = -N(t)^{-1}[G(t)^*p(t) + n(t)], \text{ a.e. in } [0,T], \tag{5.13}$$

where (x,p) is the unique solution in $C(0,T;U) \times C(0,T;X_W)$ of the coupled system

$$\begin{aligned} x(t) &= \Lambda(t,0)x_0 + \int_0^t \Lambda(t,r)[f(r) - G(r)N(r)^{-1}n(r)]dr \\ &\quad - \int_0^t \Lambda(t,r)G(r)N(r)^{-1}G(r)^*p(r)dr \end{aligned} \tag{5.14}$$

$$p(t) = \Lambda(T,t)^*[Lx(T) + \ell] + \int_t^T \Lambda(r,t)^*[Q(r)x(r) + q(r)]dr. \tag{5.15}$$

<u>Proof</u>. (i) is clear. (ii) Here (5.12) becomes an equality which is true for all v in $L^2(0,T;U)$. As a result

$$u(t) = -N(t)^{-1}[G(t)^*p(t) + n(t)], \text{ a.e. in } [0,T]. \tag{5.16}$$

For all v in $L^2(0,T;U)$ the system

$$\begin{aligned} \phi(t) &= \Lambda(t,0)x_0 + \int_0^t \Lambda(t,r)[B(r)v(r) + f(r)]dr \\ \psi(t) &= \Lambda(T,t)^*L\phi(T) + \int_t^T \Lambda(r,t)^*[Q(r)\phi(r) + q(r)]dr \end{aligned} \tag{5.17}$$

has a unique solution (ϕ,ψ) in $C(0,T;X) \times C(0,T;X_w)$. In particular when v is u, the optimal control function, system (5.17) still has a unique solution (x,p) in $C(0,T;X) \times C(0,T;X_w)$. But the optimal control function u is uniquely characterized by (5.16). The elimination of u in the first equation (5.17) yields (5.14). □

Remark. System (5.14)-(5.15) will be referred to as the optimality system in [0,T].

6. "Decoupling" of the optimality system.

In this section we assume that $U_{ad} = L^2(0,T;U)$ and proceed via J.L. LIONS' direct method. We first consider the problem of section 5 in the time interval $[s,T]$ for some s, $0 \leq s < T$, and study the dependence of the optimality system in $[s,T]$ with respect to the initial datum h at time s. This dependence will be characterized by an operator $P(s)$ in $\mathcal{L}(X)$ and an element $r(s)$ of X which will be shown to characterize the feedback law. To simplify our equation we introduce the following notation:

$$f'(t) = f(t) - G(t)N(t)^{-1}n(t) \text{ and } R(t) = G(t)N(t)^{-1}G(t)^*. \tag{6.1}$$

Consider the system (S) in $[s,T]$

$$\xi(t) = \Lambda(t,s)h + \int_s^t \Lambda(t,r)[G(r)v(r) + f(r)]dr \tag{6.2}$$

with the cost function

$$\begin{aligned} J_s(v,x_0) &= (\xi(T),L\xi(T)) + \int_s^T [(\xi(t),Q(t)\xi(t)) + (v(t),N(t)v(t))_U]dt \\ &\quad + 2(\xi(T),\ell) + 2\int_s^T [(\xi(t),q(t)) + (v(t),n(t))_U]dt. \end{aligned} \tag{6.3}$$

As was previously done, we introduce the adjoint system in $[s,T]$

$$\pi(t) = \Lambda(T,t)^*L\xi(T) + \int_t^T \Lambda(r,t)^*[Q(r)\xi(r) + q(r)]dr \tag{6.4}$$

and essentially obtain Theorem 5.2 (ii) in the time interval $[s,T]$:

Lemma 6.1. For each x_0 there exists a unique control function u in $L^2(s,T;U)$ which minimizes $J_s(v,x_0)$ over all v in $L^2(s,T;U)$. Moreover it is completely characterized by

$$u(t) = -N(t)^{-1}[G(t)^*\pi(t) + n(t)], \text{ a.e. in } [s,T], \tag{6.5}$$

where (ξ,π) is the unique solution in $C(s,T;X) \times C(s,T;X_w)$ of the optimality system

$$\xi(t) = \Lambda(t,s)h + \int_s^t \Lambda(t,r)\, f'(r)dr - \int_s^t \Lambda(t,r)R(r)\pi(r)dr \tag{6.6}$$

$$\pi(t) = \Lambda(T,t)^*F\xi(T) + \int_t^T \Lambda(r,t)^*[Q(r)\xi(r) + q(r)]dr. \tag{6.7}$$

Corollary 6.2. The map

$$h \mapsto \pi(s) : X \to X \tag{6.8}$$

is affine. Hence there exists a linear map $P(s) : X \to X$ and $r(s)$ in X such that

$$\pi(s) = P(s)h + r(s).\square \tag{6.9}$$

Theorem 6.3. Let (x,p) be the solution of the coupled systems (5.14)-(5.15). Then

$$p(t) = P(t)x(t) + r(t),\ 0 \le t \le T, \tag{6.10}$$

where P(t) and r(t) are defined by the following rules:

(i) we solve the system

$$(6.11)\qquad \begin{cases} \beta(t) = \Lambda(t,s)h - \int_s^t \Lambda(t,r)R(r)\gamma(r)dr \\ \gamma(t) = \Lambda(T,t)^*L\beta(T) + \int_t^T \Lambda(r,t)^*Q(r)\beta(r)dr, \end{cases}$$

then

$$(6.12)\qquad P(s)h = \gamma(s);$$

(ii) we solve the system

$$(6.13)\qquad \begin{cases} \mu(t) = \int_s^t \Lambda(t,r)f'(r)dr - \int_s^t \Lambda(t,r)R(r)\nu(r)dr \\ \nu(t) = \Lambda(T,t)^*[L\mu(T) + \ell] + \int_t^T \Lambda(r,t)^*[Q(r)\mu(r) + q(r)]dr, \end{cases}$$

then

$$(6.14)\qquad r(s) = \nu(s).$$

Proof. $P(s)$ and $r(s)$ are clearly obtained from the rules (i) and (ii) of the theorem: it suffices to decompose the map $h \mapsto \pi(s)$ of the Corollary 6.2 into its linear part and its constant part. We only need to establish identity (6.10). Consider system (6.6)-(6.7) with initial datum $x(s)$ at time s, where x is the solution of system (5.14)-(5.15) with initial datum x_0 at time 0. We denote its solution by (ξ,π).

We also define

$$\bar{\xi} \text{ (resp. } \bar{\pi}) = \text{restriction of } x \text{ (resp. } p) \text{ to } [s,T].$$

Clearly

$$\bar{\xi}(t) = \Lambda(t,s)\ \Lambda(s,0)x_0 + \int_s^t \Lambda(t,r)[f'(r) - R(r)\bar{\pi}(r)]dr$$

$$(6.15) \qquad + \int_0^s \Lambda(t,s)\ \Lambda(s,r)[f'(r) - R(r)\bar{\pi}(r)]dr$$

$$= \Lambda(t,s)\ x(s) + \int_s^t \Lambda(t,r)[f'(r) - R(r)\bar{\pi}(r)]dr$$

$$(6.16) \qquad \bar{\pi}(t) = \Lambda(T,t)^*L\bar{\xi}(T) + \int_t^T \Lambda(r,t)^*[Q(r)\bar{\xi}(r) + q(r)]dr.$$

But $(\bar{\xi},\bar{\pi})$ are solutions of system (6.6)-(6.7) in the interval $[s,T]$. By uniqueness $(\xi,\pi) = (\bar{\xi},\bar{\pi})$ and

$$p(s) = \bar{\xi}(s) = \xi(s) = P(s)x(s) + r(s).$$

This proves the theorem.□

<u>Remark</u>. The operator $P(s)$ will be referred to as the <u>feedback operator</u> and the element $r(s)$ as the <u>reference function</u>.

7. Study of the properties of the feedback operator P(s).

The feedback operator P(s) is characterized by the optimality system (6.11) of Theorem 6.3 which corresponds to the following optimal control problem in [s,T]: the controlled system is

(7.1) $$\xi(t) = \Lambda(t,s)h + \int_s^t \Lambda(t,r)G(r)v(r)dr$$

and the cost function is

(7.2) $$J_s(v,h) = (L\xi(T),\xi(T)) + \int_s^T [(Q(t)\xi(t),\xi(t)) + (N(t)v(t),v(t))_U]dt.$$

It is readily seen that the above problem corresponds to the case where f, ℓ, q and n are all equal to zero where the feedback law will be linear.

Theorem 7.1. (i) Let (β,γ) (resp. $\bar{\beta},\bar{\gamma}$) be the solution of (6.11) with initial datum h (resp. $\bar{h}$) at time s, $0 \le s < T$. Then

(7.3) $$(P(s)h,\bar{h}) = (L\beta(T),\bar{\beta}(T)) + \int_s^T [(Q(t)\beta(t),\bar{\beta}(t)) + (\gamma(t),R(t)\bar{\gamma}(t))]dt.$$

(ii) Let u be the optimal control function in [s,T] for the problem (7.1)-(7.2). Then

(7.4) $$J_s(u,h) = (P(s)h,h).$$

(iii) For all s in [0,T)

(7.5) $$P(s)^* = P(s) \ge 0$$

and there exists c > 0 (independent of s and h) such that

$$|P(s)h| \le c|h|. \tag{7.6}$$

<u>Proof</u>. (i) We already know (cf. Theorem 6.3) that $P(s)h = \pi(s)$. We now compute $(\pi(s),\bar{h})$. It is equal to

$$\begin{aligned}
&(\Lambda(T,s)^*L\beta(T) + \int_s^T \Lambda(t,s)^*Q(t)\beta(t)dt,\bar{h}) \\
&= (L\beta(T),\Lambda(T,s)\bar{h}) + \int_s^T (Q(t)\beta(t),\Lambda(t,s)\bar{h})dt \\
&= (L\beta(T),\bar{\beta}(T) + \int_s^T \Lambda(T,r)R(r)\bar{\gamma}(r)dr) \\
&+ \int_s^T (Q(t)\beta(t),\bar{\beta}(t) + \int_s^t \Lambda(t,r)R(r)\bar{\gamma}(r)dr)dt \\
&= (L\beta(T),\bar{\beta}(T)) + \int_s^T (Q(t)\beta(t),\bar{\beta}(t))dt \\
&+ \int_s^T (\Lambda(T,r)^*L\beta(T) + \int_r^T \Lambda(t,r)^*Q(t)\beta(t)dt,R(r)\bar{\gamma}(r))dr \\
&= (L\beta(T),\bar{\beta}(T)) + \int_s^T [(Q(t)\beta(t),\bar{\beta}(t)) + (\gamma(t),R(t)\bar{\gamma}(t))]dt.
\end{aligned}$$

This establishes (7.3).

(ii) Set $\bar{h} = h$ in (7.3) to obtain (7.4).

(iii) The symmetry and positivity of $P(s)$ follow from the symmetry and positivity of L, $Q(t)$ and $R(t)$. Moreover

$$(P(s)h,h) = J_s(u,h) \le J_s(0,h)$$

and

$$J_s(0,h) = (L\Lambda(T,s)h,\ \Lambda(T,s)h) + \int_s^T (Q(t)\Lambda(t,s)h,\ \Lambda(t,s)h)dt$$

$$\leq [|L| + \int_0^T |Q(t)|dt] \max_{P(0,T)} |\Lambda(t,s)|^2|h|^2,$$

where (cf. Proposition 2.5(i)) $|\Lambda(t,s)|$ is bounded by some constant C(T). Now (7.6) follows by symmetry and positivity of P(s). □

Corollary 7.2. P(s) is in $L(X)$ for all s in [0,T). □

We have established the existence and uniqueness of the operator P(s), studied its decoupling properties and shown that it is a positive self adjoint element of $L(X)$. We now denote by (β_s, γ_s) the solution of the optimality system (6.11) and study the properties of the map $(t,s) \to \beta_s(t)$. This will be used to establish some kind of continuity of P(s) as a function of s.

Definition 7.3. $\xi_T(T) = h,\ \gamma_T(T) = Lh,\ P(T) = L.$□

Lemma 7.4. (i) Given s in [0,T), the equation

$$(7.7) \qquad \beta_s(t) = \Lambda(t,s)h - \int_s^t \Lambda(t,r)R(r)P(r)\beta_s(r)dr,\ s \leq t \leq T,$$

has a unique solution in C(s,T;X). Moreover there exists a constant $c(P) \geq 1$ (independent of s and h) such that

$$(7.8) \qquad \|\beta_s\|_{C(s,T;X)} \leq c(P)|h|.$$

(ii) Given t in (0,T], the map $s \mapsto \beta_s(t):[0,t] \to X$ is continuous, and for all $\varepsilon > 0$ there exists $\delta(h,\varepsilon)$ (independent of t) such that

$$\forall \quad s_1, s_2 \in [0,T] \text{ and } |s_2 - s_1| < \delta(h,\varepsilon)$$

$$\Rightarrow |\beta_{s_2}(t) - \beta_{s_1}(t)| < \varepsilon.$$

(iii) The map $(t,s) \mapsto \beta_s(t) : \mathcal{P}(0,T) \to X$ is continuous and there exists $\Lambda_P(t,s)$ in $\mathcal{L}(X)$ such that

$$\beta_s(t) = \Lambda_P(t,s)h, \tag{7.9}$$

moreover $\Lambda_P(t,r) = \Lambda_P(t,s)\Lambda_P(s,r)$, $T \geq t \geq s \geq r \geq 0$, $\Lambda_P(s,s) = I$ (identity in $L(X)$), $T \geq s \geq 0$, and the map $\Lambda_P : \mathcal{P}(0,T) \to L(X)$ is strongly continuous.

Proof. (i) We know that the optimality system (6.11) has a unique solution (β_s, γ_s) in $C(s,T;X) \times C(s,T;X_W)$. But in that situation we know by Theorem 6.3 that

$$\gamma_s(t) = P(t)\beta_s(t). \tag{7.10}$$

The substitution of (7.10) in the first equation (6.11) yields (7.7) and the existence of a unique solution in $C(0,T;X)$. Identity (7.7) yields the following inequality

$$|\beta_s(t)| \leq |\Lambda(t,s)||h| + \int_s^t |\Lambda(t,r)||R(r)| \; c \; |\beta_s(r)|dr$$

$$\leq c(T)|h| + \int_s^t c(T)|R(r)| \; c \; |\beta_s(r)|dr.$$

Let $m(r) = c(T)|R(r)|c$. By definition m is in $L^1(0,T;\mathbb{R})$. We define for any

α, $0 < \alpha < 1$, the positive monotonically increasing continuous function g_α in $[0,T]$:

$$g_\alpha(t) = \exp\,[\alpha^{-1} \int_0^t m(r)dr].$$

It is easily verified that

$$\int_s^t m(r)g_\alpha(r)dr \le \alpha g_\alpha(t), \tag{7.11}$$

and that

$$|\beta_s(t)| \le c(T)|h| + \alpha g_\alpha(t) \max_{[s,t]} \frac{|\beta_s(r)|}{g_\alpha(r)} .$$

We can conclude from this last inequality that

$$\max_{[s,T]} \frac{|\beta_s(t)|}{g_\alpha(t)} \le \frac{1}{1-\alpha}\, c(T)|h| \max_{[s,T]} \frac{1}{g_\alpha(t)}$$

and

$$\|\beta_s\|_{C(s,T;X)} \le \frac{g_\alpha(T)}{1-\alpha}\, c(T)|h| = c(P)|h| \tag{7.12}$$

since $g_\alpha(t) \ge 1$, g_α is monotonically increasing and neither g nor α are functions of s or h.

(ii) Assume that $T > t \ge s_2 \ge s_1 \ge 0$. For all τ, $t \ge \tau \ge s_2 \ge s_1$, equation (7.7) yields the following inequality

$$|\beta_{s_2}(\tau) - \beta_{s_1}(\tau)| \le |\Lambda(\tau,s_2)h - \Lambda(\tau,s_1)h|$$

$$+ \left|\int_{s_2}^{\tau} \Lambda(\tau,r)R(r)P(r)\beta_{s_2}(r)dr - \int_{s_1}^{\tau} \Lambda(\tau,r)R(r)P(r)\beta_{s_1}(r)dr\right|.$$

We can estimate the last term in the R.H.S. of the above inequality:

$$\le \int_{s_2}^{\tau} |\Lambda(\tau,r)|\ |R(r)|\ |P(r)(\beta_{s_2}(r) - \beta_{s_1}(r))|dr$$

$$+ \int_{s_1}^{s_2} |\Lambda(\tau,r)|\ |R(r)|\ |P(r)\beta_{s_1}(r)|dr$$

$$\le \int_{s_2}^{\tau} c(T)|R(r)|\ c\ |\beta_{s_2}(r) - \beta_{s_1}(r)|dr + \int_{s_1}^{s_2} c(T)|R(r)|c\cdot c(P)|h|dr.$$

Now by using techniques analogous to the ones in part (i) we obtain

$$|\beta_{s_2}(t) - \beta_{s_1}(t)| \le \max_{[s_2,t]} |\beta_{s_2}(\tau) - \beta_{s_1}(\tau)|$$

$$\le \frac{g_\alpha(T)}{1-\alpha}\left[\max_{[s_2,t]} |\Lambda(\tau,s_2)h - \Lambda(\tau,s_1)h| + \int_{s_1}^{s_2} c(T)|R(r)|c\cdot c(P)\cdot|h|dr\right].$$

By uniform continuity of Λ in $\mathcal{P}(0,T)$ for all $\varepsilon > 0$ there exists $\delta_1(h,\varepsilon)$ such that $|s_2-s_1| < \delta_1(h,\varepsilon)$ implies

$$\max\left\{|\Lambda(\tau,s_2)h - \Lambda(\tau,s_1)h|\ \middle|\ \tau \in [s_2,t]\right\} < \varepsilon/2,$$

and there exists $\delta_2(h,\varepsilon) > 0$ such that

$$\int_{s_1}^{s_2} |R(r)|dr < \varepsilon/[2c(t)\cdot c\cdot c(P)\cdot|h|].$$

The case $s_1 \geq s_2$ is similar. To complete the proof we need to show that when $t = T$ the map $s \mapsto \beta_s(T) : [0,T] \to X$ is continuous at $s = T$. Since (7.8) gives a bound on $\beta_s(T)$ we know that the limit as s goes to T exists. Moreover for all s in $[0,T)$

$$|\beta_s(T) - \Lambda(T,s)h| \leq \int_s^T |\Lambda(T,r)R(r)\beta_s(r)|dr$$
$$\leq c(T)\cdot c(P)\cdot|h| \int_s^T |R(r)|dr$$

and

$$\lim_{s\to T} \beta_s(T) = \lim_{s\to T} \Lambda(T,s)h = h = \beta_T(T)$$

in agreement with Definition 7.3.

(iii) The proof is in two parts. We first consider the case where $t = s$. It was established in part (ii) that

$$|\beta_{s_1}(t_1) - \beta_s(s)| = |\beta_{s_1}(t_1) - h| = |\beta_{s_1}(t_1) - \beta_{t_1}(t_1)| < \varepsilon$$

for all (t_1,s_1) such that $|t_1-s_1| < \delta(h,\varepsilon)$. Finally for all (t_1,s_1) in $P(0,T)$

$$|t_1-s| + |s_1-s| < \delta(h,\varepsilon) \Rightarrow |t_1-s_1| < \delta(h,\varepsilon) \Rightarrow |\beta_{s_1}(t_1) - \beta_s(s)| < \varepsilon.$$

This shows the continuity at (s,s). When $t > s$, we notice that the set

$$N(t,s) = \{(t',s') \in P(0,T) \mid t' \geq s\} \tag{7.13}$$

is a neighborhood of (t,s). So pick (t_1,s_1) in $N(t,s)$ and consider the following inequality

$$|\beta_{s_1}(t_1) - \beta_s(t)| \le |\beta_{s_1}(t_1) - \beta_s(t_1)| + |\beta_s(t_1) - \beta_s(t)|.$$

By part (ii) there exists $\delta(h,\varepsilon) > 0$ such that

$$|s_1-s| < \delta(h,\varepsilon) \Rightarrow |\beta_{s_1}(t_1) - \beta_s(t_1)| < \varepsilon$$

and by part (i) there exists $\delta(h,\varepsilon,t,s) > 0$ such that

$$|t_1-t| < \delta(h,\varepsilon,t,s) \Rightarrow |\beta_s(t_1) - \beta_s(t)| < \varepsilon.$$

This shows the continuity at (t,s) in $P(0,T)$, $t > s$. The map $h \mapsto \beta_s(t)$ is clearly linear. It is continuous in view of inequality (7.8) and this defines an element $\Lambda_P(t,s)$ in $L(X)$. Its properties follow directly from the definition.□

We now turn to the second equation (6.11)

$$\gamma_s(t) = \Lambda(T,t)^*L\beta_s(T) + \int_t^T \Lambda(r,t)^*Q(r)\beta_s(r)dr. \tag{7.14}$$

<u>Lemma 7.5</u>. (i) Given s in [0,T), equation (7.14) has a unique solution in $C(s,T;X_w)$. Moreover there exists a constant $c_\gamma > 0$ (independent of s, t and h) such that

$$|\gamma_s(t)| \le c_\gamma \; |h|. \tag{7.15}$$

(ii) Given t in $(0,T]$, the map $s \mapsto \gamma_s(t) : [0,t] \to X$ is continuous, and for all $\varepsilon > 0$ there exists $\delta_\gamma(h,\varepsilon)$ (independent of t) such that

$$0 \le s_1,\ s_2 \le t \text{ and } |s_2 - s_1| < \delta_\gamma(h,\varepsilon) \Rightarrow |\gamma_{s_2}(t) - \gamma_{s_1}(t)| < \varepsilon.$$

(iii) The map $(t,s) \mapsto \gamma_s(t) : P(0,T) \to X$ is weakly continuous.

Proof. (i) For all k in X the function

$$t \mapsto (\Lambda(T,t)k,\ L\beta_s(T)) + \int_t^T (\Lambda(r,t)k,\ Q(r)\beta_s(r))dr$$

defined in $[s,T]$ is continuous. Moreover

$$|\gamma_s(t)| \le c(T)|L|c(P)|h| + c(T)\cdot c(P)|h| \int_0^T |Q(r)|dr = c_\gamma |h|.$$

(ii) Pick $0 \le s_1 \le s_2 \le t \le T$. Then by Lemma 7.4(ii)

$$\begin{aligned} |\gamma_{s_2}(t) - \gamma_{s_1}(t)| &\le c(T)|L|\ |\beta_{s_2}(t) - \beta_{s_1}(t)| \\ &\quad + c(T)\cdot \int_0^T |Q(r)|dr \cdot \max_{[t,T]} |\beta_{s_2}(r) - \beta_{s_1}(r)| \\ &\le c(T)\cdot[|L| + \int_0^T |Q(r)|dr]\cdot\varepsilon \end{aligned}$$

for all s_2 and s_1 such that $|s_2 - s_1| < \delta(h,\varepsilon)$. When $t = T$ we need to check the continuity of $\gamma_s(T)$ as s goes to T. It is clear that the limit exists since the map $s \mapsto \gamma_s(T) : [0,T) \to X$ is bounded by the results of part (i) and continuous by the results of part (ii). When $t = T$ equation (7.14) reduces to

$$\gamma_s(T) = L\beta_s(T)$$

and

$$\lim_{s\to T} \gamma_s(T) = L.\lim_{s\to T} \beta_s(T) = Lh = \gamma_T(T).$$

The last equality is in agreement with Definition 7.3.

(iii) The proof is similar to the proof of Lemma 7.4(iii). □

Theorem 7.6. (i) The map $P : [0,T] \to \mathcal{L}_w(X)$ is continuous and the map $t \mapsto |P(t)| : [0,T] \to \mathbb{R}$ is bounded and measurable.

(ii) P is the unique solution in $C(0,T;\mathcal{L}_w(X))$ of the following system

(7.16)
$$(P(s)h,\bar{h}) = (L\Lambda_P(T,s)h,\ \Lambda_P(T,s)\bar{h})$$
$$+ \int_s^T [(Q(t)\Lambda_P(t,s)h,\Lambda_P(t,s)\bar{h}) + (R(t)P(t)\Lambda_P(t,s)h,P(t)\Lambda_P(t,s)\bar{h})]dt$$

$\forall$ s in $[0,T]$, h and $\bar{h}$ in X,

$$(7.17)\quad \Lambda_P(t,s)k = \Lambda(t,s)k - \int_s^t \Lambda(t,r)R(r)P(r)\Lambda_P(r,s)k\,dr,$$

$\forall$ t in $[s,T]$, s in $[0,T]$, k in X.

Proof. (i) In Lemma 7.4(iii) we have established that

$$\beta_s(t) = \Lambda_P(t,s)h;$$

by decoupling of the optimality system (6.11)

$$\gamma_s(t) = P(t)\beta_s(t) = P(t)\Lambda_P(t,s)h.$$

The substitution of the above two equalities in identity (7.3) yields identity (7.16). By inspection the map $s \mapsto (P(s)h,\bar{h})$ is continuous and this establishes the weak continuity of the map P. The map $t \mapsto |P(t)| : [0,T] \to \mathbb{R}$ is bounded by Theorem 7.1(iii) and measurable by Lemma 2.2.

(ii) It suffices to show that the solution P of system (7.16)-(7.17) is unique. Let $\tilde{P}$ be another solution of system (7.16)-(7.17) in $C(0,T;L_w(X))$. Denote by $\tilde{\beta}_s$ the solution in C(s,T:X) of the following equation:

$$(7.18) \qquad \tilde{\beta}_s(t) = \Lambda(t,s)h - \int_s^t \Lambda(t,r)R(r)\tilde{P}(r)\tilde{\beta}_s(r)dr, \quad s \le t \le T.$$

There exists $\tilde{\Lambda}(t,s)$ in $L(X)$ such that

$$(7.19) \qquad \tilde{\beta}_s(t) = \tilde{\Lambda}(t,s)h,$$

where

$$(7.20) \qquad \tilde{\Lambda}(t,s)h = \Lambda(t,s)h - \int_s^t \Lambda(t,r)R(r)\tilde{P}(r)\tilde{\Lambda}(r,s)h\, dr.$$

We define

$$(7.21) \qquad z(t) = \tilde{P}(t)\tilde{\beta}_s(t) - \Lambda(T,t)^*L\tilde{\beta}_s(T) - \int_t^T \Lambda(r,t)^*Q(r)\tilde{\beta}_s(r)dr.$$

We want to show that $z(t) = 0$ in $[s,T]$. For arbitrary k in X we consider the expression $(z(t),k)$ it is equal to

$$(\widetilde{P}(t)\widetilde{\beta}_s(t),k) - (\Lambda(T,t)^*L\widetilde{\beta}_s(T) + \int_t^T \Lambda(r,t)^*Q(r)\widetilde{\beta}_s(r)dr,k).$$

By definition of a solution, equation (7.16) is verified with $\widetilde{P}$ and $\widetilde{\Lambda}$ in place of P and Λ_P. By using (7.16) to expand the term in $\widetilde{P}$ the above expression becomes

$$(L\widetilde{\Lambda}(T,t)\widetilde{\beta}_s(t), \widetilde{\Lambda}(T,t)k) + \int_s^T (Q(r)\widetilde{\Lambda}(r,t)\widetilde{\beta}_s(t), \widetilde{\Lambda}(r,t)k)dr$$

$$+ \int_s^T (\widetilde{P}(r)\widetilde{\Lambda}(r,t)\widetilde{\beta}_s(t), R(r)\widetilde{P}(r)\widetilde{\Lambda}(r,t)k)dr$$

$$- (L\widetilde{\Lambda}(T,t)\widetilde{\beta}_s(t), \Lambda(T,t)k) - \int_t^T (Q(r)\widetilde{\Lambda}(r,t)\widetilde{\beta}_s(t), \Lambda(r,t)k)dr.$$

By using identity (7.20) the above expression reduces to

$$\int_t^T (\widetilde{P}(\rho)\widetilde{\Lambda}(\rho,t)\widetilde{\beta}_s(t), R(\rho)\widetilde{P}(\rho)\widetilde{\Lambda}(\rho,t)k)d\rho$$

$$- (L\widetilde{\beta}_s(T), \int_t^T \Lambda(T,\rho)R(\rho)\widetilde{P}(\rho)\widetilde{\Lambda}(\rho,t)k\ d\rho)$$

$$- \int_t^T (Q(r)\widetilde{\Lambda}(r,t)\widetilde{\beta}_s(t), \int_t^r \Lambda(r,\rho)R(\rho)\widetilde{P}(\rho)\widetilde{\Lambda}(\rho,t)k)d\rho\ dr$$

$$= \int_t^T (R(\rho)\widetilde{P}(\rho)\widetilde{\Lambda}(\rho,t)k, \widetilde{P}(\rho)\widetilde{\Lambda}(\rho,t)\widetilde{\beta}_s(t) - \Lambda(T,\rho)^*L\widetilde{\beta}_s(T)$$

$$- \int_\rho^T \Lambda(r,\rho)^*Q(r)\widetilde{\Lambda}(r,t)\widetilde{\beta}_s(t)dr)d\rho.$$

In view of (7.21) and the last expression, it is easy to see that we finally obtain

$$(7.22)\qquad (z(t),k) = \int_t^T (R(\rho)\tilde{P}(\rho)\tilde{\Lambda}(\rho,t)k,\ z(\rho))d\rho,\quad s \le t \le T.$$

This last equation has a unique solution in $C(s,t;X_W)$ which is identically zero in $[s,T]$. Thus we have established that

$$(7.23)\qquad \tilde{\gamma}_s(t) = \tilde{P}(t)\tilde{\beta}_s(t) = \Lambda(T,t)^*L\tilde{\beta}_s(T) + \int_t^T \Lambda(r,t)^*Q(r)\tilde{\beta}_s(r)dr.$$

The elimination of the term $\tilde{P}(r)\tilde{\beta}_s(r)$ in (61) yields

$$(7.24)\qquad \tilde{\beta}_s(t) = \Lambda(t,s)h - \int_s^t \Lambda(t,r)R(r)\tilde{\gamma}_s(r)\ dr.$$

But system (7.23)-(7.24) is the optimality system (6.11). By rule (i) of Theorem 6.3

$$\tilde{P}(s)h = \tilde{\gamma}_s(s) = P(s)h \Rightarrow \tilde{P}(s) = P(s).\square$$

8. Study of the properties of the reference function r(s).

In order to study r(s) we again use the results of Theorem 6.3, Lemma 7.4 and the characterization of P(s) given in Theorem 7.6.

Lemma 8.1. Let (μ_s, γ_s) be the solution in $C(s,T;X) \times C(s,T;X_W)$ of the optimality system (6.13) of Theorem 6.3. (i) The map r belongs to $C(0,T;X_W)$. (ii) For each s in [0,T), μ_s is the unique solution in C(0,T;X) of the equation

$$(8.1) \quad \mu_s(t) = \int_s^t \Lambda_P(t,\rho)\{f'(\rho) - R(\rho)[P(\rho)\mu_s(\rho) + r(\rho)]\}d\rho, \; s \le t \le T.$$

Proof. (i) Let (μ_0, ν_0) be the solution of the optimality system (6.13) with s = 0. We know by Theorem 6.3 that

$$r(t) = \nu_0(t) - P(t)\mu_0(t), \; 0 \le t \le T.$$

In view of the properties of ν_0, μ_0 and P, r is weakly continuous (hence strongly measurable and bounded) and the right hand side of (8.1) makes sense.

(ii) We know by Theorem 6.3 that the following system has a unique solution (μ_s, ν_s) in $C(s,T;X) \times C(s,T;X_W)$:

$$(8.2) \quad \begin{cases} \mu_s(t) = \int_s^t \Lambda(t,r)[f'(r) - R(r)\nu_s(r)]dr \\ \nu_s(t) = \Lambda(T,t)^*[L\mu_s(T)+\ell] + \int_t^T \Lambda(r,t)^*[Q(r)\mu_s(r) + q(r)]dr \\ \nu_s(t) = P(t)\mu_s(t) + r(t). \end{cases}$$

If we substitute the third equation (8.2) into the first equation (8.2) we obtain equation (8.1). This proves the existence and uniqueness of the solution.□

<u>Theorem 8.2</u>. The reference function r is the unique solution in $C(0,T;X_W)$ of the equation

$$(8.3)\quad r(s) = \Lambda(T,s)*\ell + \int_s^T \Lambda(\rho,s)*\{P(\rho)[f'(\rho) - R(\rho)r(\rho)] + q(\rho)\}d\rho.$$

<u>Proof</u>. By Theorem 6.3

$$\nu_s(t) = \Lambda(T,t)*[L\mu_s(T)+\ell] + \int_t^T \Lambda(r,t)*[Q(r)\mu_s(r) + q(r)]dr.$$

Again by Theorem 6.3 and Lemma 8.1

$$\begin{aligned} r(s) = \nu_s(s) &= \Lambda(T,s)*\ell + \int_s^T \Lambda(r,s)*q(r)dr \\ &+ \Lambda(T,s)*L\int_s^T \Lambda_p(T,\rho)[f'(\rho) - R(\rho)r(\rho)]d\rho \\ &+ \int_s^T \Lambda(\zeta,s)*\{Q(\zeta)\int_s^\zeta \Lambda_p(\zeta,\rho)[f'(\rho) - R(\rho)r(\rho)]d\rho\}d\zeta \end{aligned}$$

and r(s) is equal to

$$\begin{aligned} (8.4)\qquad & \Lambda(T,s)*\ell + \int_s^T \Lambda(r,s)*q(r)dr \\ &+ \int_s^T \Lambda(\rho,s)*\{\Lambda(T,\rho)*L\Lambda_p(T,\rho)[f'(\rho) - R(\rho)r(\rho)] \\ &+ \int_\rho^T \Lambda(\zeta,\rho)*Q(\zeta)\Lambda_p(\zeta,\rho)[f'(\rho) - R(\rho)r(\rho)]d\zeta\}d\rho. \end{aligned}$$

But for all k in X we know that (cf. Theorem 6.3 and Lemma 7.4)

$$(8.5)\qquad \begin{cases} \beta_\rho(t) = \Lambda_P(t,\rho)k \\ \gamma_\rho(t) = \Lambda(T,t)^*L\beta_\rho(T) + \displaystyle\int_t^T \Lambda(\zeta,\rho)^*Q(\zeta)\beta_\rho(\zeta)d\zeta \\ \Rightarrow \gamma_\rho(\rho) = P(\rho)k. \end{cases}$$

It is easy to see that the expression between curly brackets in (8.4) is precisely $\gamma_\rho(\rho)$ of (8.5) with

$$k = f'(\rho) - R(\rho)r(\rho)$$

and as a result equation (8.4) reduces to equation (8.3). In Lemma 8.1 we have established that r is weakly continuous. The uniqueness of the solution can easily be verified.□

9. Differential equations for P and r.

In sections 7 and 8 we have derived integral equations for P and r. In this section we show that when Λ is generated by some linear differential equation, differential equations can be obtained for P and r in a rather natural and direct way.

Let V anc X be two Hilbert spaces on $\mathbb{R}$. Let V' and X' denote the topological duals of V and X, respectively. We identify the elements of X with the elements of its dual X'. We denote by $(\cdot,\cdot)$ the inner product in X and by $\langle \cdot,\cdot \rangle$ the natural pairing between V and V'. We assume that V is a dense linear subspace of H and that the canonical injection $i : V \to H$ is continuous. We denote by i^* the adjoint map of i and obtain the continuous dense injections

$$V \overset{i}{\to} X = X' \overset{i^*}{\to} V'. \tag{9.1}$$

Let Λ_P be the evolution operator introduced in Lemma 7.4 of section 7. We introduce for some k in X and some f in $L^2(0,T;X)$ the map $z : [0,T] \to X$ defined as follows

$$z(t) = \Lambda_P(T,t)^*k + \int_t^T \Lambda_P(r,t)^*f(r)dr. \tag{9.2}$$

Lemma 9.1. The map z is the unique weakly continuous solution of the integral equation

$$z(t) = \Lambda(T,t)^*k + \int_t^T \Lambda(s,t)^*[f(s)-P(s)R(s)z(s)]ds \tag{9.3}$$

and for all r in (0,T] and all h in X

$$\Lambda_P(r,t)h = \Lambda(r,t)h - \int_t^r \Lambda_P(r,s)R(s)P(s)\Lambda(s,t)hds. \tag{9.4}$$

Proof. (i) By definition of z and the properties of Λ_P (cf. Lemma 7.4) the map z is clearly weakly continuous.

(ii) (Existence). Let

$$\Delta(t) = -z(t) + \Lambda(T,t)^*k + \int_t^T \Lambda(r,t)^*f(r)dr - \int_t^T \Lambda(s,t)^*P(s)R(s)z(s)ds.$$

By definition of z

$$(9.5)\quad \Delta(t) = \Lambda(T,t)^*k - \Lambda_p(T,t)^*k + \int_t^T [\Lambda(r,t)^*-\Lambda_p(r,t)^*]f(r)dr$$

$$- \int_t^T \Lambda(s,t)^*P(s)R(s)z(s)ds.$$

But again by definition of Λ_p (cf. equations (4.9)-(5.1))

$$(9.6)\qquad \Lambda_p(r,t)h = \Lambda(r,t)h - \int_t^r \Lambda(r,s)R(s)P(s)\Lambda_p(s,t)ds.$$

When we substitute the transposed of equation (9.6) into equation (9.5), we obtain

$$(9.7)\qquad \Delta(t) = \int_t^T \Lambda_p(s,t)^*P(s)R(s)\Lambda(T,s)^*kds$$

$$+ \int_t^T dr \int_t^r ds\Lambda_p(s,t)^*P(s)R(s)\Lambda(r,s)^*f(r)$$

$$- \int_t^T \Lambda(s,t)^*P(s)R(s)z(s)ds.$$

We now change the order of integration in the second term in the R.H.S. of (9.7) and group the terms

$$(9.8)\quad \Delta(t) = \int_t^T \Lambda_p(s,t)^*P(s)R(s)[\Lambda(T,s)^*k + \int_s^T \Lambda(r,s)^*f(r)dr-z(s)]ds$$

$$+ \int_t^T [\Lambda_p(s,t)^*-\Lambda(s,t)^*]P(s)R(s)z(s)ds$$

$$= \int_t^T \Lambda_p(s,t)^*P(s)R(s)\Delta(s)ds$$

$$(9.8)\qquad + \int_t^T \Lambda_p(s,t)^*P(s)R(s)\int_s^T \Lambda(r,s)^*P(r)R(r)z(r)\,dr\,ds$$

$$+ \int_t^T [\Lambda_p(s,t)^* - \Lambda(s,t)^*]P(s)r(s)z(s)\,ds$$

$$= \int_t^T \Lambda_p(s,t)^*P(s)R(s)\Delta(s)\,ds$$

$$+ \int_t^T [\Lambda_p(s,t)^* - \Lambda(s,t)^* + \int_t^s \Lambda_p(r,t)^*P(r)R(r)\Lambda(s,r)^*dr]P(s)R(s)z(s)\,ds.$$

But in view of equation (9.6) the second term of the last equation is zero and

$$(9.9)\qquad \Delta(t) = \int_t^T \Lambda_p(s,t)^*P(s)R(s)\Delta(s)\,ds,\quad 0 \le t \le T.$$

By definition Δ is weakly continuous and the term $\Lambda_p(s,t)^*P(s)R(s)$ is uniformly bounded by some constant $c > 0$. As a result

$$(9.10)\qquad |\Delta(t)| \le \int_t^T c|\Delta(s)|\,ds,\quad 0 \le t \le T,$$

and it is now straightforward to show that $\Delta(t) = 0$ everywhere in $[0,T]$.

(iii) (Uniqueness). Pick two weakly continuous solutions z_1 and z_2. Let $\Delta(t) = z_2(t) - z_1(t)$. From equation (9.3)

$$(9.11)\qquad \Delta(t) = -\int_t^T \Lambda(r,t)^*P(r)R(r)\Delta(r)\,dr$$

and since $\Lambda(r,t)^*P(r)R(r)$ is uniformly bounded by some constant $c > 0$, equation (9.11) yields $\Delta(t) = 0$ everywhere in $[0,T]$. This prove uniqueness. □

(iv) In order to obtain (9.4) we specialize (9.2)-(9.3) to the case where f is 0. □

Remark. If we compare equation (9.4) and the transposed of equation (9.6), we notice that the role of Λ and Λ_p is interchanged in the integral term. That is

$$(9.12)\qquad \int_t^r \Lambda_p(r,s)R(s)P(s)\Lambda(s,t)h\,ds = \int_t^r \Lambda(r,s)R(s)P(s)\Lambda_p(s,t)h\,ds.$$

It is readily seen that equation (9.4) will allow us to conclude that the properties of the evolution operator Λ_p will be the same as the properties of the evolution operator Λ. We introduce the spaces

$$(9.13)\qquad \begin{cases} W(0,T;X,V') = \{y \in L^\infty(0,T;X) \mid Dy \in L^2(0,T;V')\} \\ W(0,T;V,V') = \{y \in L^2(0,T;V) \mid Dy \in L^2(0,T;V')\}, \end{cases}$$

where Dy denotes the distributional derivative of y. We shall consider the case where the map $t \mapsto \Lambda(T,t)^*k$ belongs to $W(0,T;X,V')$ and the case where that map belongs to $W(0,T;V,V')$.

9.1. Case $W(0,T;X,V')$.

We first prove a lemma which describes the properties of the elements of $W(0,T;X,V')$.

Lemma 9.2. Corresponding to each element y of $W(0,T;X,V')$, there exists a unique weakly continuous map $\bar{y} : [0,T] \to X$ such that

$$(9.14)\qquad \begin{cases} i^*\bar{y}(t) - i^*\bar{y}(s) = \int_s^t Dy(r)dr, \quad 0 \le s \le t, \\ y(s) = \bar{y}(s), \text{ a.e. in } [0,t], \text{ and } Dy = D\bar{y}. \end{cases}$$

Proof. Pick y in $W(0,T;X,V')$, the map $s \mapsto i^*y(s)$ belongs to $L^\infty(0,T;V')$ and has a distributional derivative Dy in $L^2(0,T;V')$. As a result there exists a unique y^* in $C(0,T;V')$ such that

$$i^*y(s) = y^*(s), \text{ a.e. in } [0,T].$$

Can y be redefined in a unique way as a map $\bar{y} : [0,T] \to X$ for which

$$i^*\bar{y}(s) = y^*(s), \text{ everywhere in } [0,T].$$

We first prove that for all s the map $h \mapsto \langle h, y^*(s)\rangle$ defines a continuous linear functional on X. The proof is by contradiction. Assume that there exists s in [0,T] such that

$$\sup\{\langle h,y^*(s)\rangle : h \in V, \quad |ih|_X = 1\} = \infty.$$

This means that for each M > 0

$$\exists \ h_M \in V \text{ such that } |ih_M|_X = 1 \text{ and } |\langle h_M,y^*(s)\rangle| \geq M.$$

But for all $0 < \varepsilon < M/2$, there exists $\delta > 0$ such that

$$|s'-s| < \delta \Rightarrow |\langle h_M,y^*(s')\rangle - \langle h_M,y^*(s)\rangle| \leq \varepsilon.$$

As a result for all M > 0, there exist h_M and $\delta > 0$ such that

$$|s'-s| < \delta \Rightarrow |\langle h_M,y^*(s')\rangle| \geq M/2.$$

Let S be the subset of $[s-\delta,s+\delta]$ for which $i^*y(s') = y^*(s')$. The measure of S is equal to 2δ and for all s' in S

$$|y(s')|_X \geq |(ih_M,y(s'))| = |\langle h_M,y^*(s')\rangle| \geq M/2.$$

As a result y is unbounded on a set of non-zero measure, that is, y does not belong to $L^\infty(0,T;X)$. This is in contradiction with our original hypothesis. As a result y* defines in a unique way a map $\bar{y} : [0,T] \to X$ such that $i^*\bar{y}(s) = y^*(s)$ everywhere in [0,T]. But

$$i^*\bar{y}(s) \doteq y^*(s) = i^*z(s), \text{ a.e. in } [0,T],$$

and since i* is injective

$$\bar{y}(s) = y(s), \text{ a.e. in } [0,T].$$

By hypothesis $\bar{y}$ also belongs to $L^\infty(0,T;X)$ and there exists c > 0 such that

$$|\bar{y}(s)|_X \leq c, \text{ a.e. in } [0,T].$$

We now prove that $\bar{y}$ is uniformly bounded by the constant c. Again we use a contradiction argument. Let Σ be the subset of $[0,T]$ for which $\bar{y}(s)$ is not bounded by c. Pick s in Σ

$$|\bar{y}(s)|_X = \alpha > c.$$

Given $0 < \varepsilon < \alpha - c$, there exists h in V such that

$$|ih|_X = 1 \text{ and } |(ih,z(s))| \geq |z(s)|_X - \varepsilon > c.$$

By continuity of y^*, there exists an open neighborhood $N(s)$ of s with non-zero measure such that

$$|\langle h,y^*(s')\rangle| = |(ih,z(s'))| > c.$$

Finally for all s' in $N'(s) = \{s' \in N(s) : s' \notin \Sigma\}$

$$|y(s')|_X \geq |ih|_X \; |y(s')|_X \geq |(ih,y(s'))| > c$$

and the norm of $y(s')$ in X is strictly greater than c on a set of non-zero measure ($N'(s)$ has the same measure as $N(s)$). This is in contradiction with our initial hypothesis. Thus $\bar{y}$ is uniformly bounded.

We are now ready to establish the weak continuity of $\bar{y}$. Given h in X, there exists a sequence $\{h_n\}$ in V which converges to h in X and

$$|(h,\bar{y}(s')) - (h,\bar{y}(s))| \leq |(ih_n,\bar{y}(s')) - (ih_n,\bar{y}(s))|$$
$$+ |(h-ih_n,\bar{y}(s'))| + |(h-ih_n,\bar{y}(s))|.$$

Pick n such that

$$|h-ih_n|_X \leq \varepsilon/4c$$

and $\delta > 0$ such that

$$|s'-s| \leq \delta \Rightarrow |\langle h_n,i^*\bar{y}(s')\rangle - \langle h_n,i^*\bar{y}(s)\rangle| \leq \varepsilon/2.$$

This is sufficient to establish the weak continuity of $\bar{y}$. Finally

$$i^*\bar{y}(s) = y^*(s) = i^*k - \int_s^t Dy(r)\,dr, \text{ everywhere in } [0,T],$$

and $Dy = D\bar{y}$.

<u>Lemma 9.3</u>. Let f belong to $L^2(0,T;X)$ and let $F : [0,T] \to \mathcal{L}(V,X)$ be strongly measurable and bounded. Assume that for all k in X and all t in (0,T] the map $s \mapsto \Lambda(t,s)^*k$ is the unique solution in

$$W(0,t;X,V') = \{y \in L^\infty(0,t;X) \mid Dy \in L^2(0,t;V')\} \tag{9.15}$$

of the operational differential equation

$$\begin{cases} \frac{dy}{ds}(s) + F(s)^*y(s) = 0 \text{ in } (0,t) \\ y(t) = k. \end{cases} \tag{9.16}$$

(i) For all h in V and t in (0,T]

$$\Lambda(t,s)ih = ih + \int_s^t \Lambda(t,r)F(r)hdr \tag{9.17}$$

and the map $s \mapsto \Lambda(t,s)ih$ is continuous with a distributional derivative in $L^\infty(0,t;X)$.

(ii) For all k in X and t in (0,T] the map z

$$z(s) = \Lambda_P(t,s)^*k + \int_s^t \Lambda_P(r,s)^*f(r)dr \tag{9.18}$$

is the unique solution in $W(0,t;X,V')$ of the operational differential equation

$$\begin{cases} \frac{dz}{ds}(s) + [F(s)^*-i^*P(s)R(s)]z(s) + i^*f(s) = 0 \text{ in } (0,t) \\ z(t) = k. \end{cases} \tag{9.19}$$

Moreover for all h in V and t in (0,T]

$$\Lambda_P(t,s)ih = ih + \int_s^t \Lambda_P(t,r)[F(r)-R(r)P(r)i]h. \tag{9.20}$$

<u>Proof</u>. (i) By definition the map $s \mapsto \Lambda(t,s)^*k$ is weakly continuous and necessarily coincides with the weakly continuous map of Lemma 9.2. As a result

$$i^*\Lambda(t,s)^*k = i^*k + \int_s^t F(r)^*\Lambda(t,r)^*k\,dr$$

and for all h in V we obtain (9.17) after transposition.

(ii) Let g belong to $L^2(0,t;X)$. By hypothesis on Λ, it is clear that the map

$$p(s) = \Lambda(t,s)^*k + \int_s^t \Lambda(r,s)^*g(r)\,dr \tag{9.21}$$

is the unique solution in $W(0,t;X,V')$ of the equation

$$\begin{cases} \frac{dp}{ds}(s) + F(s)^*p(s) + i^*g(s) = 0 \quad \text{in} \quad (0,t) \\ p(t) = k. \end{cases} \tag{9.22}$$

By Lemma 9.1 the map z of equation (9.18) is given by

$$z(s) = \Lambda(t,s)^*k + \int_s^t \Lambda(r,s)^*[f(r)-P(r)R(r)z(r)]\,dr \tag{9.23}$$

and by hypothesis on Λ, z belongs to $W(0,t;X,V')$. Let

$$g(r) = f(r) - P(r)R(r)z(r). \tag{9.24}$$

From the above considerations z is the unique solution in $W(0,t;X,V')$ of the equation

$$\frac{dz}{ds}(s) + F(s)^*z(s) + i^*g(s) = 0. \tag{9.25}$$

By definition the map $s \mapsto \Lambda_P(t,s)^*k$ is weakly continuous and necessarily coincides with the weakly continuous map of Lemma 9.2. As a result

$$i^*\Lambda_P(t,s)^*k = i^*k + \int_s^t [F(r)^*-i^*P(r)R(r)]\Lambda(t,r)^*k\,dr$$

and for all h in V we obtain (9.20) after transposition. □

<u>Proposition 9.4</u>. Under the hypothesis of Lemma 9.3, (i) the operator P of Theorem 7.6 is the solution in

(9.26) $\mathcal{W}(0,T;X,V') = \{K : [0,T] \to \mathcal{L}(X) \mid \forall\, h \in X,\ t \mapsto K(t)h$ belongs to $L^{\infty}(0,T;X)$; and $\exists\, DK : [0,T] \to \mathcal{L}(V,V')$, $\forall\, h \in V$, $t \mapsto DK(t)h$ belongs to $L^{\infty}(0,T;V')$ and is the distributional derivative of the map $t \mapsto K(t)ih\}$

of the operator Riccati differential equation

$$\begin{cases} \frac{d}{dt} P(t) + F(t)^*P(t)i + i^*P(t)F(t) \\ + i^*[Q(t)-P(t)R(t)P(t)]i = 0 \quad \text{in} \quad (0,T) \\ P(T) = L. \end{cases} \tag{9.27}$$

If in addition for all s in $[0,T)$ and h in V, the map $t \mapsto \Lambda(t,s)ih$ is the unique solution in

$$\{x \in L^2(s,T;V) \mid Dx \in L^2(s,T;H)\} \tag{9.28}$$

of the operational differential equation

$$\begin{cases} \frac{dx}{dt}(t) = F(t)x(t) \quad \text{in} \quad (s,T) \\ x(s) = ih, \end{cases} \tag{9.29}$$

then P is the unique solution in $\mathcal{W}(0,T;X,V')$ of equation (9.27) and (ii) the reference function r of Theorem 8.2 is the unique solution in $W(0,T;X,V')$ of the operational differential equation

$$\begin{cases} \frac{dr}{dt}(t) + [F(t)-R(t)P(t)i]^*r(t) + i^*[P(t)f'(t)+q(t)] = 0 \quad \text{in} \quad (0,T) \\ r(T) = \ell. \end{cases} \tag{9.30}$$

<u>Proof.</u> (i) By Theorem 7.6, P is given by the integral equation

$$(P(s)h,\bar{h}) = (L\Lambda_p(T,s)h,\Lambda_p(T,s)\bar{h}) + \int_s^T ([Q(t)+P(t)R(t)P(t)]\Lambda_p(t,s)h,\Lambda_p(t,s)\bar{h})dt. \tag{9.31}$$

By Lemma 7.4 (iii) for all h in X and all t in (0,T] the map $s \to \Lambda_p(t,s)h$ is continuous and by Lemma 9.3 for all h in V and t in (0,T]

$$(9.32) \qquad \Lambda_p(t,s)ih = ih + \int_s^t \Lambda_p(t,r)[F(r)-R(r)P(r)i]h\,dr.$$

As a result for all h and $\bar{h}$ in X the map $s \mapsto (P(s)h,\bar{h})$ is continuous and hence for all h in X the map $s \mapsto P(s)h$ belongs to $L^\infty(0,t;X)$. For all h and $\bar{h}$ in V we can substitute for Λ_p the R.H.S. of identity (9.32)

$$\begin{aligned}(9.33) \quad (P(s)ih,i\bar{h}) = (Lih,i\bar{h}) &+ \int_s^T (L\Lambda_p(T,r)ih,\Lambda_p(T,r)F'(r)\bar{h})\,dr \\ &+ \int_s^T (L\Lambda_p(T,r)F'(r)h,\Lambda_p(T,r)\bar{h})\,dr \\ &+ \int_s^T (Q'(t)ih,i\bar{h})\,dt \\ &+ \int_s^T (Q'(t)\int_s^t \Lambda_p(t,r)ih,\Lambda_p(t,r)F'(r)\bar{h})\,dr\,dt \\ &+ \int_s^T (Q'(t)\int_s^t \Lambda_p(t,r)F'(r)h,\Lambda_p(t,r)i\bar{h})\,dr\,dt,\end{aligned}$$

where $F'(r) = F(r) - R(r)P(r)i$ and $Q'(t) = Q(t) + P(t)R(t)P(t)$. We change the order of integration in the last two terms on the R.H.S. of the last identity and regroup the terms:

$$\begin{aligned}(P(s)ih,i\bar{h}) = (Lih,i\bar{h}) &+ \int_s^T (Q'(t)ih,i\bar{h})\,dt \\ &+ \int_s^T [(L\Lambda_p(T,r)ih,\Lambda_p(T,r)F'(r)\bar{h}) \\ &\qquad + \int_r^T (Q'(t)\Lambda_p(t,r)ih,\Lambda_p(t,r)F'(r)\bar{h})\,dt]\,dr \\ &+ \int_s^T [(L\Lambda_p(T,r)F'(r)h,\Lambda_p(T,r)i\bar{h}) \\ &\qquad + \int_r^T (Q'(t)\Lambda_p(t,r)F'(r)h,\Lambda_p(t,r)i\bar{h})\,\bar{d}t]\,dr\end{aligned}$$

$$(9.34)\qquad (P(s)ih,i\bar{h}) = (Lih,i\bar{h}) + \int_s^T([Q(t)+P(t)R(t)P(t)]ih,i\bar{h})dt$$
$$+ \int_s^T\{(P(r)ih,[F(r)-R(r)P(r)i]\bar{h}) + (P(r)[F(r)-R(r)P(r)i]h,i\bar{h})\}dr.$$

The last identity shows that

$$(9.35)\quad \begin{cases} \frac{d}{ds}(P(s)ih,i\bar{h})+(P(s)ih,[F(s)-R(s)P(s)i]\bar{h}) + (P(s)[F(s)-R(s)P(s)i]h,i\bar{h}) \\ + ([Q(s) + P(s)R(s)P(s)]ih,i\bar{h}) = 0 \quad \text{in} \quad (0,T), \\ P(T) = L. \end{cases}$$

Since (9.35) is true for all h and $\bar{h}$ in V, we obtain (9.27) and we see by inspection that the map

$$(h,\bar{h}) \mapsto \frac{d}{ds}(P(s)ih,i\bar{h})$$

defines an operator $\frac{dP}{ds}(s)$ in $\mathcal{L}(V,V')$ such that for all h in V the map $s \mapsto \left[\frac{dP}{ds}(s)\right]h$ which belongs to $L^\infty(0,T;V')$ coincides with the distributional derivative of the map $s \mapsto i^*P(s)ih$.

Assume that there exist two positive self adjoint solutions P_1 and P_2 of equation (9.27). Let $\Delta(t) = P_2(t) - P_1(t)$. Then $\Delta(T) = 0$ and

$$(\Delta(s)ih,i\bar{h}) = \int_s^T([P_1(t)R(t)P_1(t) - P_2(t)R(t)P_2(t)]ih,i\bar{h})dt$$
$$+ \int_s^T[(\Delta(t)ih,F(t)\bar{h}) + (F(t)h,\Delta(t)i\bar{h})]dt$$
$$= \int_s^T[(\Delta(t)ih,[F(t)-R(t)P_1(t)i]\bar{h}) + ([F(t)-R(t)P_1(t)i]h,\Delta(t)i\bar{h})]dt$$
$$- \int_s^T(\Delta(t)ih,R(t)\Delta(t)i\bar{h})dt$$
$$= \int_s^T[(\Delta(t)ih,[F(t)-R(t)P_2(t)i]\bar{h}) + ([F(t)-R(t)P_2(t)i]h,\Delta(t)i\bar{h})]dt$$
$$+ \int_s^T(\Delta(t)ih,R(t)\Delta(t)i\bar{h})dt.$$

As a result for all s in [0,T]

$$\int_s^T 2(\Delta(t)ih,[F(t)-R(t)P_2(t)i]h)dt \geq (\Delta(s)ih,ih)$$

$$\geq \int_s^T 2(\Delta(t)ih,[F(t)-R(t)P_1(t)i]h)dt$$

and

$$0 \geq \int_s^T 2(\Delta(t)ih,R(t)\Delta(t)ih)dt$$

$$\Rightarrow \forall \ h, \ R(t)\Delta(t)ih = 0 \ \text{ in } \ [0,T].$$

Finally for all h and $\bar{h}$ in V

(9.36) $$(\Delta(s)ih,i\bar{h}) = \int_s^T [(\Delta(t)ih,F(t)\bar{h}) + (F(t)h,\Delta(t)i\bar{h})]dt$$

and

(9.37) $$\frac{d}{ds}(\Delta(s)ih,i\bar{h}) = -[(\Delta(s)ih,F(s)\bar{h}) + (F(s)h,\Delta(s)i\bar{h})].$$

Fix s in [0,T) and consider the map

(9.38) $$t \mapsto (\Delta(t)\Lambda(t,s)ih,\Lambda(t,s)i\bar{h}).$$

In view of (9.36) the norm $|\Delta(s)|$ is bounded in [0,T] and the norm of $|\Lambda(t,s)|$ is bounded in [s,T]. The additional hypothesis on Λ makes it possible to write

(9.39) $$\Lambda(t,s)ih = ih + \int_s^t F(r)[\Lambda(r,s)ih]_V \, dr,$$

where $[\Lambda(r,s)ih]_V$ is $\Lambda(r,s)ih$ viewed as an element of V. It is now easy to see from equations (9.36) and (9.39) that the map (9.38) is absolutely continuous and that its derivative with respect to t is 0. Therefore for all t in [s,T]

$$(\Delta(t)\Lambda(t,s)ih,\Lambda(t,s)i\bar{h}) = (\Delta(T)\Lambda(T,s)ih,\Lambda(T,s)ih) = 0$$

and necessarily for all h and $\bar{h}$ in V $(\Delta(s)ih,i\bar{h}) = 0$. Since this is true for all h,$\bar{h}$ and s, we conclude that Δ is identically zero and that (9.27) has a unique solution.

(ii) By Theorem 8.2 r is the unique solution in C(0,T;X) of

$$r(s) = \Lambda(T,s)^*\ell + \int_s^T \Lambda(\rho,s)^*\{P(\rho)[f'(\rho)-R(\rho)r(\rho)]+q(\rho)\}d\rho. \tag{9.40}$$

Let

$$g(\rho) = P(\rho)[f'(\rho)-R(\rho)r(\rho)] + q(\rho). \tag{9.41}$$

From part (ii) we conclude that r is the unique solution in W(0,T;X,V') of

$$\begin{cases} \frac{dr}{ds}(s) + F(s)^*r(s) + i^*g(s) = 0 \quad \text{in} \quad (0,t) \\ r(T) = \ell. \end{cases} \tag{9.42}$$

The substitution of (9.41) in (9.42) yields (9.28). □

9.2. Case W(0,T;V,V').

Again we first quote a lemma which describes the properties of the elements of W(0,T;V,V').

Lemma 9.5. Corresponding to each element y of W(0,T;V,V'), there exists a unique continuous map $\bar{y} : [0,T] \to X$ such that

$$\begin{cases} i^*\bar{y}(t) - i^*\bar{y}(s) = \int_s^t Dy(r)dr, \quad 0 \le s \le t, \\ iy(s) = \bar{y}(s), \text{ a.e. in } [0,t], \text{ and } Dy = D(i^*\bar{y}). \end{cases} \tag{9.43}$$

Proof. Cf. LIONS-MAGENES [1]. □

We now assume that a family of continuous bilinear forms

$$(h,k) \mapsto b(t;h,k) : V \times V \to \mathbb{R}$$

is given with the following properties. For fixed h and k in V the map

$$t \mapsto b(t;h,k) : [0,T] \to \mathbb{R}$$

is measurable and bounded. We can define the operators F(t) and F*(t) in $\mathcal{L}(V,V')$ as follows

$$(9.44)\qquad \begin{cases} \langle k, F(t)h \rangle = b(t;h,k) \\ \langle h, F^*(t)k \rangle = b(t;k,h). \end{cases}$$

At this point it is helpful to list hypotheses and introduce definitions which will be used throughout this section.

Hypothesis I. For all k in X and all t in (0,T] we assume that the map $s \mapsto (\Gamma^t k)(s) = \Lambda(t,s)^* k$ belongs to $L^2(0,t;V)$ and that

$$(9.45)\qquad \exists\ c > 0,\ \ \forall\ \ t \in (0,T],\ \ \forall\ \ k \in X,\ \|\Gamma^t k\|_{L^2(0,t;V)} \le c|k|.\ \square$$

Hypothesis II. For all k in X the map

$$(9.46)\qquad (t,s) \mapsto \Lambda(t,s)^* k : P(0,T) \to V$$

is measurable. □

We now fix t in (0,T]. Denote by St(0,t;X) the vector space of all step maps. We can define the linear map $\bar{\Gamma}^t : St(0,t;X) \to L^1(0,t;V)$ as follows. Given ϕ in St(0,t;X) of the form

$$(9.47)\qquad \phi(s) = \sum_{n=1}^{N} k_n \chi_{E_n}(s),$$

we define

$$(9.48)\qquad (\bar{\Gamma}^t \phi)(s) = \sum_{n=1}^{N} (\Gamma^t k_n)(s) \chi_{E_n}(s).$$

It is easy to check that (by Hypothesis I)

$$(9.49)\qquad \|\bar{\Gamma}^t \phi\|_{L^1(0,t;V)} \le c\, \|\phi\|_{L^2(0,t;X)}.$$

It follows from the last inequality that the map $\bar{\Gamma}^t$ can be lifted to a unique continuous linear map (also denoted $\bar{\Gamma}^t$) from $L^2(0,t;X)$ to $L^1(0,t;V)$. As a result we can define a map $(\bar{\Gamma}^t)^* : L^\infty(0,t;V') \to L^2(0,t;X)$

$$(9.50)\qquad \int_0^t ((\bar{\Gamma}^t)^* g)(s), f(s))\, ds = \int_0^t \langle (\bar{\Gamma}^t f)(s), g(s) \rangle\, ds.$$

When g belongs to $L^\infty(0,t;X)$, $(\bar{\Gamma}^t)^*g$ reduces to

$$((\bar{\Gamma}^t)^*g)(s) = \Lambda(t,s)g(s) \tag{9.51}$$

and for this reason the map $(\bar{\Gamma}^t)^*$ will be denoted by

$$s \mapsto \bar{\Lambda}(t,s)g(s). \tag{9.52}$$

Moreover for all f in $L^2(0,t;X)$ and all g in $L^\infty(0,t;V')$

$$\begin{aligned} |\langle \bar{\Gamma}^t f, g \rangle| &\le \| \bar{\Gamma}^t f \|_{L^1(0,t;V)} \| g \|_{L^\infty(0,t;V')} \\ &\le c \| f \|_{L^2(0,t;X)} \| g \|_{L^\infty(0,t;V')} \end{aligned}$$

by Hypothesis I and

$$\| (\bar{\Gamma}^t)^* g \|_{L^2(0,t;X)} \le c \| g \|_{L^\infty(0,t;V')}. \tag{9.53}$$

For each ϕ in $St(P(0,T);X)$ we can construct the map $\Gamma\phi : P(0,T) \to V$

$$(\Gamma\phi)(t,s) = \sum_{i=1}^{N} \Lambda(t,s)^* k_i \chi_{E_i}(t,s), \quad \phi = \sum_{i=1}^{N} k_i \chi_{E_i}. \tag{9.54}$$

By Hypothesis II, $\Gamma\phi$ is measurable and

$$\begin{aligned} \| \Gamma\phi \|_{L^1(P(0,T);V)} &\le \sum_{i=1}^{N} \int_P |\Lambda(t,s)^* k_i|_V \chi_{E_i}(t,s)\, dP \\ &\le \sum_{i=1}^{N} \int_0^T dt \int_0^t ds\ |\Lambda(t,s)^* k_i|_V \chi_{E_i}(t,s) \\ &\le \sum_{i=1}^{N} \int_0^T dt\ c|k_i| \left[\int_0^t \chi_{E_i}(t,s)\, ds\right]^{\frac{1}{2}} \\ &\le c T^{\frac{1}{2}} \sum_{i=1}^{N} |k_i| m(E_i)^{\frac{1}{2}} \\ &\le c T^{\frac{1}{2}} \| \phi \|_{L^2(0,T;X)}. \end{aligned}$$

(by Hypothesis I)

As a result Γ can be lifted to a unique continuous linear map

$$(9.55) \qquad \Gamma : L^2(P(0,T);X) \to L^1(P(0,T);V).$$

We can now define a map $\Gamma^* : L^\infty(P(0,T);V') \to L^2(P(0,T);X)$ by

$$(\Gamma^* g,f) = \int_0^T dt \int_0^t ds \,\langle (\Gamma^t f(t,\cdot))(s), g(t,s) \rangle$$

and

$$(9.56) \qquad \| \Gamma^* g \|_2 \le cT^{\frac{1}{2}} \| g \|_\infty.$$

Again the above map will be denoted

$$(t,s) \mapsto \bar{\Lambda}(t,s)g(t,s)$$

since it is a lifting of the map $(t,s) \mapsto \Lambda(t,s)g(t,s)$ form g in $L^\infty(P(0,T);X)$ to $L^\infty(P(0,T);V')$.

Consider equation (9.4) of Lemma 9.1. For all h in X

$$(9.57) \qquad \Lambda_p(t,s)h = \Lambda(t,s)h - \int_s^t \Lambda_p(t,r)R(r)P(r)\Lambda(r,s)h\,dr.$$

In view of the previous considerations we can define for each t in (0,T] a map $\Pi^t : L^\infty(0,t;V') \to L^2(0,t;X)$ as follows

$$(\Pi^t g)(s) = \bar{\Lambda}(t,s)g(s) - \int_s^t \Lambda_p(t,r)R(r)P(r)\bar{\Lambda}(r,s)g(s)\,dr.$$

Necessarily

$$\| \Pi^t g \|_2 \le c \| g \|_\infty + \left[\int_0^t \left| \int_s^t \Lambda_p(t,r)R(r)P(r)\bar{\Lambda}(r,s)g(s)\,dr \right|^2 ds \right]^{\frac{1}{2}}.$$

The term $\Lambda_p(t,r)R(r)P(r)$ is uniformly bounded by some constant $c' > 0$ and the above inequality reduces to

$$\| \Pi^t g \|_2 \le c \| g \|_\infty + c' T^{\frac{1}{2}} \left[\int_0^t ds \int_s^t dr \; |\bar{\Lambda}(r,s)g(s)|^2 \right]^{\frac{1}{2}}$$

$$\| \Pi^t g \|_2 \leq [c + cc'T] \| g \|_\infty \tag{9.58}$$

by inequality (9.56). Moreover in view of the definition of Γ for all g in $L^2(P(0,T);X)$

$$(\Pi g)(t,s) = (\Gamma g)(t,s) - \int_s^t \Lambda_P(t,r)R(r)P(r)(\Gamma g)(r,s)dr \tag{9.59}$$

and the map $\Pi : L^\infty(P(0,T);V') \to L^2(P(0,T;X))$ is linear and continuous.

<u>Lemma 9.6.</u> Let F* and F be as defined in identity (9.44).

(i) Assume that Hypotheses I and II are verified and that for all k in X and t in (0,T] the map $s \mapsto \Lambda(t,s)^*k$ is the unique solution in $W(0,t;V,V')$ of the operational differential equation

$$\begin{cases} \dfrac{dy}{ds}(s) + (F^*y)(s) = 0 \quad \text{in} \quad (0,t) \\ y(t) = k. \end{cases} \tag{9.60}$$

For all h in V, t in (0,T] and s in [0,t]

$$\Lambda(t,s)ih = ih + \int_s^t \bar{\Lambda}(t,r)(Fh)dr, \tag{9.61}$$

where the map Fh in $L^\infty(0,t;V')$ is defined as

$$(Fh)(s) = F(s)h. \tag{9.62}$$

As a result the map $s \mapsto \Lambda(t,s)ih$ has a distributional derivative in $L^2(0,t;X)$ which coincides with the map

$$s \mapsto \bar{\Lambda}(t,s)(Fh). \tag{9.63}$$

For all k in X and g in $L^2(0,t;X)$ the map

$$p(s) = \Lambda(t,s)^*k + \int_s^t \Lambda(r,s)^*g(r)dr \tag{9.64}$$

is the unique solution in $W(0,t;V,V')$ of the operational differential equation

$$(9.65)\qquad \begin{cases} \dfrac{dp}{ds}(s) + (F^*p)(s) + i^*g(s) = 0 \text{ in } (0,t) \\ p(t) = k. \end{cases}$$

(ii) Under the hypotheses of part (i), for all t in (0,T], all k in X, and g in $L^2(0,t;X)$ the map

$$(9.66)\qquad p(s) = \Lambda_p(t,s)^*k + \int_s^t \Lambda_p(r,s)^*g(r)dr$$

is the unique solution in $W(0,t;V,V')$ of the operational differential equation

$$(9.67)\qquad \begin{cases} \dfrac{dp}{ds}(s) + [(F^*-i^*PRi)p](s) + i^*g(s) = 0 \text{ in } (0,t) \\ p(t) = k, \end{cases}$$

where

$$(9.68)\qquad [(F^*-i^*PRi)p](s) = F^*(s)p(s) - i^*P(s)R(s)p(s).$$

For all h in V, t in (0,T] and s in [0,t]

$$(9.69)\qquad \Lambda_p(t,s)ih = ih + \int_s^t \bar{\Lambda}_p(t,r)[(F-i^*RPi)h]dr.$$

As a result the map $s \mapsto \Lambda_p(t,s)ih$ has a distributional derivative in $L^2(0,t;X)$ which coincides with the map

$$(9.70)\qquad s \mapsto \bar{\Lambda}_p(t,s)[(F-i^*RPi)h].$$

(iii) For all k in X and t in (0,T] the map

$$(9.71)\qquad s \mapsto (\Gamma_p^t k)(s) = \Lambda_p(t,s)^*k$$

belongs to $L^2(0,t;V)$ and

$$(9.72)\qquad \exists\, c > 0,\ \forall\, t \in (0,T],\ \forall\, k \in X,\ \|\Gamma_p^t k\|_{L^2(0,t;V)} \le c|k|.$$

(iv) For all k in V the map

$$(9.73) \qquad (t,s) \mapsto \Lambda_p(t,s)^*k : \mathcal{P}(0,T) \to V$$

is measurable.

(v) For all t in (0,T] the map $\Pi^t : L^\infty(0,t;V') \to L^2(0,t;X)$ defined by

$$(9.74) \qquad (\Pi^t g)(s) = \bar{\Lambda}_p(t,s)g$$

is linear and continuous and

$$(9.75) \qquad \exists\, c > 0,\ \forall\, t \in (0,T],\ \| \Pi^t g \|_2 \le c \| g \|_\infty .$$

(vi) The linear map $\Pi : L^\infty(\mathcal{P}(0,T);V') \to L^2(\mathcal{P}(0,T);X)$ defined by

$$(9.76) \qquad (\Pi g)(t,s) = \bar{\Lambda}_p(t,s)g$$

is continuous.

Proof. (i) By Lemma 9.5

$$i^*\Lambda(t,s')^*k - i^*\Lambda(t,s)^*k = - \int_s^{s'} (F^*y)(r)\,dr.$$

For all h in V and s in [0,t]

$$(\Lambda(t,t)ih,k) - (\Lambda(t,s)ih,k) = - \int_s^t \langle h,(F^*y)(r) \rangle\, dr$$

or

$$(\Lambda(t,s)ih-ih,k) = \int_s^t \langle y(r),(Fh)(r) \rangle\, dr$$

$$= \int_s^t \langle [\Lambda(t,r)^*k]_V,(Fh)(r) \rangle\, dr$$

$$= \int_s^t (k,\bar{\Lambda}(t,r)(Fh))\,dr.$$

This proves identity (9.61). Pick g in $L^2(0,t;X)$ and consider the map (9.64).

By Hypothesis II, $p : [0,t] \to V$ is measurable and

$$\| p \|_{L^2(0,t;V)} \le \| \Gamma^t k \|_{L^2(0,t;V)} + \left[\int_0^t \left| \int_s^t \Lambda(r,s)^* g(r) dr \right|^2 ds \right]^{\frac{1}{2}}.$$

But

$$\int_0^t \left| \int_s^t \Lambda(r,s)^* g(r) dr \right|^2 ds \le \int_0^t ds \left[\int_s^t dr \; |\Lambda(r,s)^* g(r)| \right]^2$$

$$\le t \int_0^t ds \int_s^t dr \; |\Lambda(r,s)^* g(r)|^2$$

$$\le t \int_0^t dr \int_0^r ds \; |\Lambda(r,s)^* g(r)|^2$$

$$\le t \int_0^t dr \; \| \bar{\Gamma}^r g(r) \|^2_{L^2(0,t;V)}.$$

By Hypothesis I the above expression can be majored by

$$t \int_0^t c^2 \; |g(r)|^2 dr \le tc^2 \| g \|^2_{L^2(0,t;X)}.$$

We can now conclude that p belongs to $L^2(0,t;V)$. In view of equation (9.61) for all k in X

$$i^*\Lambda(t,s)k = i^*k + \int_s^t F^*(r)[\Lambda(t,r)^*k]_V dr. \tag{9.77}$$

We can now substitute (9.71) into (9.64) to obtain

$$i^*p(s) = i^*k + \int_s^t F^*(r)[\Lambda(t,r)^*k]_V dr$$

$$+ \int_s^t \{i^*g(r) + \int_s^r F^*(\rho)[\Lambda(r,\rho)^*g(r)]_V d\rho\} dr$$

$$= i^*k + \int_s^t i^*g(r) dr$$

$$+ \int_s^t F^*(r)[\Lambda(t,r)^*k + \int_r^t \Lambda(\rho,r)^*g(\rho)d\rho]_V dr$$

$$= i^*k + \int_s^t [i^*g(r) + F^*(r)p(r)]dr.$$

This shows that p has a distributional derivative equal to

$$-[i^*g + F^*p] \tag{9.78}$$

in $L^2(0,t;V')$. The uniqueness follows from the uniqueness of solution to (9.60) in $W(0,t;V,V')$.

(ii) By Lemma 9.1 the map p defined by identity (9.66) is the unique weakly continuous solution of the integral equation

$$p(s) = \Lambda(t,s)^*k + \int_s^t \Lambda(r,s)^*[g(r)-P(r)R(r)p(r)]dr. \tag{9.79}$$

By Hypotheses I and II it is readily seen that p belongs to $L^2(0,t;V)$ and has a distributional derivative in $L^2(0,t;V')$ equal to

$$-[F^*p - i^*g'],$$

where

$$g'(r) = g(r) - P(r)R(r)p(r).$$

The solution p to equation (9.67) is unique in $W(0,t;V,V')$ since the solution p to

$$\begin{cases} \dfrac{dp}{ds} + F^*p + i^*g' = 0 \\ p(t) = k \end{cases} \tag{9.80}$$

is unique in $W(0,t;V,V')$. Now we can repeat the argument of part (i) to obtain identity (9.69).

(iii) to (iv) From the considerations preceding Lemma 9.6. □

Proposition 9.7. Let the hypotheses of Lemma 9.6 be verified.

(i) The operator P of Lemma 7.6 is the solution in

(9.81) $\mathcal{W}(0,T;V,V') = \{K : [0,T] \to \mathcal{L}(X)$ weakly continuous. For all h in V the map $t \mapsto i^*P(t)ih$ has a distributional derivative in $L^1(0,T;V')$. The map $t \mapsto K(t)h$ can be lifted to a unique continuous linear map $\bar{K}^* : L^2(0,T;X) \to L^1(0,T;V)$. The map $\bar{K}^*$ can also be lifted to a unique continuous linear map $\bar{K} : L^\infty(0,T;V') \to L^2(0,T;X)\}$.

of the operator Riccati differential equation

$$
(9.82) \quad \begin{cases} \dfrac{d}{dt}[i^*P(t)ih] + i^*(\bar{P}(Fh))(t) + F^*(t)(\bar{P}^*(ih))(t) \\ \quad + i^*[Q(t)-P(t)R(t)P(t)]ih = 0, \text{ in } (0,T), \ \forall\, h \in V, \\ P(t) = L. \end{cases}
$$

(ii) The reference function r of Theorem 8.2 is the unique solution in $W(0,T;V,V')$ of the operational differential equation

$$
(9.83) \quad \begin{cases} \dfrac{dr}{dt}(t) + [(F^*-i^*PRi)r](t) + i^*[P(t)f'(t)+q(t)] = 0 \text{ in } (0,T) \\ r(T) = \ell. \end{cases}
$$

Proof. (i) By Theorem 7.6, P is given by the integral equation

$$
(9.84) \quad (P(s)h,\bar{h}) = (L\Lambda_p(T,s)h,\Lambda_p(T,s)\bar{h}) + \int_s^T ([Q(t)+P(t)R(t)P(t)]\Lambda_p(t,s)h,\Lambda_p(t,s)\bar{h})dt.
$$

By Lemma 7.4 (iii) for all h in X and all t in (0,T] the map $s \mapsto \Lambda_p(t,s)h$ is continuous and by Lemma 9.6 for all h in V and t in (0,T]

$$(9.85)\qquad \Lambda_p(t,s)ih = ih + \int_s^t \bar{\Lambda}_p(t,r)[F-i^*RP]h\,dr.$$

As a result for all h and $\bar{h}$ in X, the map $s \mapsto (P(s)h,\bar{h})$ is continuous and hence for all h in X the map $s \mapsto P(s)h$ is weakly continuous. We now rewrite equation (9.84) in the form

$$(9.86)\qquad P(s)h = \Lambda_p(T,s)^*L\Lambda_p(T,s)h + \int_s^T \Lambda_p(t,s)^*[Q(t)+P(t)R(t)P(t)]\Lambda_p(t,s)h\,dt.$$

In view of Lemma 9.6 (v) for all g in $L^\infty(0,T;V')$ the map $\bar{P} : L^\infty(0,T;V') \to L^2(0,T;X)$ defined by

$$(9.87)\qquad (\bar{P}g)(s) = \Lambda_p(T,s)^*L\bar{\Lambda}_p(T,s)g + \int_s^T \Lambda_p(t,s)^*[Q(t)+P(t)R(t)P(t)]\bar{\Lambda}_p(t,s)g\,dt$$

is linear and continuous. It coincides with the map $s \to P(s)g(s)$ when g belongs to $L^\infty(0,T;X)$. Similarly using the results preceeding Lemma 9.6 for all f in $L^2(0,T;X)$ the map $\bar{P}^* : L^2(0,T;X) \to L^1(0,T;V)$ defined by

$$(9.88)\qquad (\bar{P}^*f)(s) = [\Lambda_p(T,s)^*L\Lambda_p(T,s)f(s)]_V + \int_s^T [\Lambda_p(t,s)^*(Q(t)+P(t)R(t)P(t))\Lambda_p(t,s)f(s)]_V\,dt$$

is also linear and continuous. Moreover for all f in $L^2(0,T;X)$ and g in $L^\infty(0,T;V')$

$$(9.89)\qquad \langle \bar{P}^*f,g \rangle_{L^1(0,T;V)\times L^\infty(0,T;V')} = (f,\bar{P}g)_{L^2(0,T;X)}.$$

For all h and $\bar{h}$ in V we can substitute for Λ_p the R.H.S. of identity (9.85) in equation (9.84)

$$P(s)ih,i\bar{h}) = (Lih,i\bar{h}) + \int_s^T (L\Lambda_p(T,r)ih,\bar{\Lambda}_p(T,r)(F'\bar{h})dr$$

$$+ \int_s^T (L\bar{\Lambda}_p(T,r)(F'h),\Lambda_p(T,r)i\bar{h})dr$$

$$+ \int_s^T (Q'(t)ih,i\bar{h})dt$$

$$+ \int_s^T (Q'(t) \int_s^t \Lambda_p(t,r)ih,\bar{\Lambda}_p(t,r)(F'\bar{h}))drdt$$

$$+ \int_s^T (Q'(t) \int_s^t \bar{\Lambda}_p(t,r)(F'h),\Lambda_p(t,r)i\bar{h})drdt,$$

where $F' = F-i^*RPi$ and $Q'(t) = Q(t) + P(t)R(t)P(t)$. By Lemma 9.6 (ii) we can change the order of integration in the last two terms on the R.H.S. of the last identity and regroup the terms:

$$(P(s)ih,i\bar{h}) = (Lih,i\bar{h}) + \int_s^T (Q'(t)ih,i\bar{h})dt$$

$$+ \int_s^T [(L\Lambda_p(T,r)ih,\bar{\Lambda}_p(T,r)(F'\bar{h}))$$

$$+ \int_r^T (Q'(t)\Lambda_p(t,r)ih,\bar{\Lambda}_p(t,r)(F'\bar{h})dt]dr$$

$$+ \int_s^T [(L\bar{\Lambda}_p(T,r)(F'h),\Lambda_p(T,r)i\bar{h})$$

$$+ \int_r^T (Q'(t)\bar{\Lambda}_p(t,r)(F'h),\Lambda_p(t,r)i\bar{h})dt]dr.$$

We obtain

$$(P(s)ih,i\bar{h}) = (Lih,i\bar{h}) + \int_s^T ([Q(t)+P(t)R(t)P(t)]ih,i\bar{h})dt$$

$$+ \int_s^T [(ih,(\bar{P}(F'\bar{h}))(r)) + \langle (\bar{P}^*(i\bar{h}))(r),F'(r)h \rangle]dr.$$

The terms on the last line are equal to

$$(ih,(\bar{P}(F\bar{h}-i^*RPi\bar{h}))(r)) + \langle (\bar{P}^*(i\bar{h}))(r),F(r)h-i^*R(r)P(r)ih \rangle$$

$$= (ih,(\bar{P}(F\bar{h}))(r)-P(r)R(r)P(r)i\bar{h}) + \langle h,F(r)^*(\bar{P}^*(i\bar{h}))(r) \rangle$$

$$- (P(r)i\bar{h},R(r)P(r)ih).$$

Finally

$$(P(s)ih,i\bar{h}) = (Lih,i\bar{h}) + \int_s^T ([Q(t)-P(t)R(t)P(t)]ih,i\bar{h})dt$$

$$+ \int_s^T \langle h,i^*(\bar{P}(F\bar{h}))(r) + F^*(r)(\bar{P}^*(i\bar{h}))(r) \rangle dr$$

and for all h in V the map $t \mapsto i^*P(t)ih$ has a distributional derivative

(9.89) $$\frac{d}{dt} i^*P(t)ih + i^*[Q(t)-P(t)R(t)P(t)]ih + i^*(\bar{P}(Fh))(t) + F^*(t)(\bar{P}^*(ih))(t) = 0$$

which belongs to $L^1(0,T;V')$.

(ii) By Theorem 8.2 r is the unique solution in $C(0,T;X)$ of

(9.90) $$r(s) = \Lambda(T,s)^*\ell + \int_s^T \Lambda(\rho,s)^*\{P(\rho)[f'(\rho)-R(\rho)r(\rho)]+q(\rho)\}d\rho.$$

Let

(9.91) $$g(\rho) = P(\rho)[f'(\rho)-R(\rho)r(\rho)] + q(\rho).$$

From Lemma 9.6 (i) we conclude that r is the unique solution in $W(0,T;V,V')$ of

(9.92) $$\begin{cases} \frac{dr}{ds}(s) + (F^*r)(s) + i^*g(s) = 0 \quad \text{in} \quad (0,t) \\ r(T) = \ell. \end{cases}$$

The substitution of (9.91) in (9.92) yields (9.83). □

Remark. The derivation of the Riccati differential equation for the case W(0,T;V,V') is not as "clean" as for the case W(0,T;X,V'). Proposition 9.7 is not complete; the question of uniqueness has not been dealt with. This section was meant to cover the parabolic case, but a certain amount of work is still required to lighten the presentation of the results. However we shall see in Chapter 7 that the derivation of the Riccati differential equation is not necessary to construct numerical approximation to the operator P(s). This will be done by the method of J.C. NÉDELEC which does not require a direct approximation of the Riccati differential equation.

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