Automatica, Vol. 3, pp. 135-149. Pergamon Press, 1966. Printed in Great Britain.

PAPER II

SUCCESSIVE APPROXIMATION METHODS FOR THE SOLUTION OF OPTIMAL CONTROL PROBLEMS

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1. INTRODUCTION

IN THIS paper we present some successive approximation methods for the solution of a general class of optimal control problems. The class of problems considered is known as the Bolza Problem in the Calculus of Variations [1]. The algorithms considered are extensions of the gradient methods due to KELLEY [2] and BRYSON [3] and similar to the methods proposed by MERRIAM [4, 5]. MERRIAM approaches the problem from the Hamilton-Jacobi viewpoint and restricts himself to the simplified Bolza problem. The algorithm presented is formally equivalent to Newton's Method in Function Space [6, 7] and indeed in some problems it would be better to use Newton's Method.

The development in this paper is formal and indicates how we solve these problems on a digital computer. However, under the assumptions we have made a rigorous treatment of these successive approximation methods can be given. We shall do this elsewhere.

The paper may be divided into 8 sections. In Section 3 we formulate the problem and state the assumptions we have made. In Section 4 we state the first-order necessary conditions of optimality. These are the Euler-Lagrange equations and the transversality condition.

Section 5 is devoted to Second Variation Successive Approximation Methods and certain modifications to it.

In Section 6 we show how the second variation method is formally equivalent to Newton's Method and also indicate how the linear two point boundary value problem arising in Newton's Method can be solved in essentially the same way as in the Second Variation Method.

In Section 7 we point out certain advantages and disadvantages of the Second Variation Method.

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[3] A. E. BRYSON and W. F. DENHAM: A steepest ascent method for solving optimum programming problems. *J. Appl. Mech.* 247–257 (1962).
[4] C. W. MERRIAM, III: *Optimization Theory and the Design of Feedback Control Systems*. McGraw Hill, New York (1964).
[5] C. W. MERRIAM, III: An algorithm for the iterative solution of a class of two point boundary value problems. *S.I.A.M. J. Contr.* A2. 1–10 (1964).
[6] R. H. MOORE: Newton's Method and Variations, in: *Nonlinear Integral Equations*. ed. by P. M. ANSELONE. University of Wisconsin Press (1964).
[7] M. L. STEIN: On methods for obtaining solutions of fixed end point problems in the calculus of variations. *J. Res. Nat. Bur. Stand.* 50, May (1953).

In Section 8 we indicate that algorithms and problems presented in this paper may be considered to be special cases of a more general class of problem.

Some numerical work using these methods has been done. Detailed results will be presented elsewhere.

2. NOTATION

Throughout we shall use vector matrix notation. All vectors are column vectors. Components of a vector will be denoted by subscripts. Superscript T denotes transposed matrix. The symbol $< .,. >_{E^n}$ denotes inner-product in Euclidean n-space. Usually we shall only write $< .,. >$. For a scalar-valued function $F(x_1, x_2, \dots . x_n)$.

$$F_x(\bar{x}) = \begin{Bmatrix} \frac{\partial F}{\partial x_1} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial F}{\partial x_n} \end{Bmatrix}^T, \qquad \text{where the partial derivatives}$$

are evaluated at $x = \bar{x}$.

For a vector valued function $f(x_1 \dots . x_n)$, where f is an m-vector.

$$f_x(\bar{x}) = \begin{Bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots\cdots & \frac{\partial f_1}{\partial x_n} \\ & & \\ \frac{\partial f_m}{\partial x_1} & & \frac{\partial f_m}{\partial x_n} \end{Bmatrix} \qquad \text{an } m \times n \text{ matrix}$$

and the partial derivatives are again evaluated at $x = \bar{x}$.
Similarly for the scalar-valued function F

$$F_{xx}(\bar{x}) = \begin{Bmatrix} \frac{\partial^2 F}{\partial x_1^2} & \frac{\partial^2 F}{\partial x_1 \partial x_2} & \frac{\partial^2 F}{\partial x_1 \partial x_n} \\ & & \\ \frac{\partial^2 F}{\partial x_n \partial x_1} & & \frac{\partial^2 F}{\partial x_n^2} \end{Bmatrix}$$

Dot indicates differentiation.

3. FORMULATION OF THE PROBLEM

We consider the following Bolza problem. Find the optimal control function u and the corresponding optimal trajectory x so that the performance functional

$$P[x(t_0), u] = F[x(t_f), t_f] + \int_{t_0}^{t_f} L(x(t), u(t), t)\mathrm{d}t \tag{1}$$

is minimised, subject to the constraints

$$\frac{dx}{dt}=f[x(t), u(t), t]; \qquad x(t_0)=c \tag{2}$$

$$G[x(t_f), t_f]=0 \tag{3}$$

Here $x(t)\varepsilon E^n$, $u(t)\varepsilon E^m$, f is a function mapping E^{n+m+1} to E^n and G is a function mapping E^{n+1} to E^p, $p \leq n$. The time t_f may be explicitly or implicitly specified.

Assumptions. i. All functions are assumed to have continuous second derivatives.

ii. The p-terminal constraints are assumed to be independent.

iii. The system is assumed to be locally completely controllable uniformly in $(t_0, t_f]$ along any trajectory $\bar{x}$ corresponding to an admissible* control $\bar{u}$ i.e. for the linearized system

$$\delta\dot{x}=\bar{f}_x(t)\delta x+\bar{f}_u(t)\delta u; \qquad \delta x(t_0)=0$$

we have

$$\int_{t_0}^{t}\Phi(t, \tau)\bar{f}_u(\tau)f_u^T(\tau)\Phi^T(t, \tau)d\tau>0 \tag{4}$$

for all $t\varepsilon(t_0, t_f]$ and where $\Phi(t, \tau)$ is the solution of

$$\frac{d\Phi}{dt}(t, t_0)=\bar{f}_x(t)\Phi(t, t_0); \qquad \Phi(t_0, t_0)=I \tag{5}$$

The local controllability assumption ensures that the solution of the accessory minimization problem is normal (in the sense of Classical Calculus of Variations).

4. FIRST ORDER NECESSARY CONDITIONS

For the problem formulated in Section 3 the Euler-Lagrange equations and Transversality conditions may be derived in the usual way.

Let u^0 and x^0 be the optimal control and optimal trajectory and let $\lambda^0(t)$ be an n-vector of Lagrange multiplier functions and μ^0 be a p-vector of constants which are the multipliers corresponding to the terminal constraints.

Define

$$H^0=H[x^0(t), u^0(t), \lambda^0(t), t]=L[x^0(t), u^0(t), t]+<\lambda^0(t), f[x^0(t), u^0(l), t]> \tag{6}$$

$$\psi^0=\psi[x^0(t_f), t_f)=F[x^0(t_f), t_f]+<\mu^0, G[x^0(t_f), t_f]> \tag{7}$$

Then u^0 and x^0 satisfy

Euler-Lagrange Equations

$$\dot{x}(t)=f[x(t), u(t), t]=H_\lambda(t); \qquad x(t_0)=c \tag{8}$$

$$\dot{\lambda}(t)=-H_x(t); \qquad \lambda(t_f)=\psi_x(t_f) \tag{9}$$

$$H_u(t)=0 \tag{10}$$

$$G[x(t_f), t_f]=0 \tag{11}$$

* It is assumed that u belongs to a bounded open set $\hat{\Omega}\subseteq$ E.

and *Transversality Condition*

$$\Omega(t_f)=H(t_f)+\psi_t(t_f)=0 \tag{12}$$

5. SECOND VARIATION SUCCESSIVE APPROXIMATION METHOD

We shall consider three different cases of the problem formulated in Section 3.

Case (i). We assume that terminal constraints are absent and the terminal time t_f is fixed. This is the simplified Bolza problem of the Calculus of Variations.

Let us assume that we have chosen a nominal control function u and obtained the corresponding nominal trajectory x by integrating the system dynamic equations in the forward direction. We can now integrate the Euler-Lagrange equation $\dot{\lambda}=-\bar{H}_x$; with the boundary condition $\lambda(t_f)=\bar{F}_x(t_f)$ where the bar indicates that H and F are evaluated at the nominal control and nominal trajectory. The performance functional may now be re-written as

$$P[x(t_0), u]=\bar{F}(t_f)+\int_{t_0}^{t_f}[\bar{H}(t)-<\bar{\lambda}(t), \bar{x}(t)>]\mathrm{d}t \tag{13}$$

Expanding the performance functional P in a Taylor's Series and retaining terms up to the second order we obtain the following expressions for the first and second variations of P

$$\delta P=\int_{t_0}^{t_f}[<\bar{H}_u(t), \delta u(t)>]\mathrm{d}t \tag{14}$$

$$\tfrac{1}{2}\delta^2 P=\tfrac{1}{2}\int_{t_0}^{t_f}[<\bar{H}_{uu}(t)\delta u(t), \delta u(t)>+<\bar{H}_{xx}(t)\delta x(t), \delta x(t)> +2<\bar{H}_{ux}(t)\delta x(t), \delta u(t)>]\mathrm{d}t+\tfrac{1}{2}<\bar{F}_{xx}(t_f)\delta x(t_f), \delta x(t_f)> \tag{15}$$

In obtaining the above expressions we have performed the usual integration by parts.

At this point we have to introduce the following *assumption:* The matrix of partial derivatives H_{uu} is positive definite; F_{xx} and $H_{xx}-H_{xu}H_{uu}^{-1}H_{ux}$ are positive semi-definite. This implies that there are no points conjugate to $t=t_f$ in the interval $[t_0, t_f)$.

The improvement in control δu is obtained by minimising

$$\delta P+\tfrac{1}{2}\delta^2 P=\tfrac{1}{2}<\bar{F}_{xx}(t_f)\delta x(t_f), \delta x(t_f)>+\int_{t_0}^{t_f}[<\bar{H}_u(t), \delta u(t)>]\mathrm{d}t +\tfrac{1}{2}\int_{t_0}^{t_f}(<\bar{H}_{uu}(t)\delta u(t), \delta u(t)>+<\bar{H}_{xx}(t)\delta x(t), \delta x(t)>+<\bar{H}_{ux}(t)\delta x(t), \delta u(t)>)\mathrm{d}t \tag{16}$$

where δu and δx are related by the linearized system,

$$\delta\dot{x}(t)=\bar{f}_x(t)\delta x(t)+\bar{f}_u(t)\delta u(t); \qquad \delta x(t_0)=0 \tag{17}$$

This is a new variational problem. In view of the assumption we have just made this problem has a weak relative minimum. The Euler-Lagrange Equations of this auxiliary minimization problem are

$$\delta\dot{x}(t)=\bar{f}_x(t)\delta x(t)+\bar{f}_u(t)\delta u(t); \qquad \delta x(t_0)=0 \tag{18}$$

$$\Delta\dot{\lambda}(t) = -\bar{H}_{xx}(t)\delta x(t) - \bar{H}_{xu}(t)\delta u(t) - \bar{f}_x^T(t)\Delta\lambda(t); \qquad \Delta\lambda(t_f) = \bar{F}_{xx}(t_f) \tag{19}$$

$$\delta u(t) = -(\bar{H}_{uu}(t))^{-1}[\bar{H}_u(t) + \bar{H}_{ux}(t)\delta x(t) + \bar{f}_u^T(t)\Delta\lambda(t)] \tag{20}$$

$\Delta\lambda$ is the multiplier for the auxiliary minimization problem.
Substituting (20) into (19) and (18), we obtain

$$\delta\dot{x}(t) = A(t)\delta x(t) + B(t)\Delta\lambda(t) + v(t); \qquad \delta x(t_u) = 0 \tag{21}$$

$$\Delta\dot{\lambda}(t) = -C(t)\delta x(t) - A^T(t)\Delta\lambda(t) - w(t); \qquad \Delta\lambda(t_f) = \bar{F}_{xx} \tag{22}$$

where

$$\left.\begin{aligned} A &= \bar{f}_x - \bar{f}_u\bar{H}_{uu}^{-1}\bar{H}_{ux} \\ B &= -\bar{f}_u\bar{H}_{uu}^{-1}\bar{f}_u^T \\ C &= \bar{H}_{xx} - \bar{H}_{xu}\bar{H}_{uu}^{-1}\bar{H}_{ux} \\ v &= -\bar{f}_u\bar{H}_{uu}^{-1}\bar{H}_u \\ w &= -\bar{H}_{xu}\bar{H}_{uu}^{-1}\bar{H}_u \end{aligned}\right\} \tag{23}$$

Before proceeding further it is necessary to show that this choice of δu reduces the value of the performance functional (assuming that the linearization of the system dynamics and the Taylor's series expansion are valid) i.e. we have to show $\delta P + \frac{1}{2}\delta^2 P$ is negative. Using (18), (19) and (20) and substituting in (16), it may be shown that the value of $\delta P + \frac{1}{2}\delta^2 P$ corresponding to the choice of δu is given by

$$\delta P + \tfrac{1}{2}\delta^2 P = -\tfrac{1}{2}\int_{t_0}^{t_f} \{<\bar{H}_u(t) + \bar{f}_u^T(t)\Delta\lambda(t),\ \bar{H}_{uu}^{-1}(t)[\bar{H}_u(t) + \bar{f}_u^T(t)\Delta\lambda(t)]>\}\mathrm{d}t$$
$$-\int_{t_0}^{t_f} <C(t)\delta x(t),\ \delta x(t)> dt \tag{24}$$

which is negative.

In general, the linearization and second order expansion of the performance functional will not be valid and it is necessary to introduce a parameter ε, $0 < \varepsilon \leqslant 1$ in the following way to reduce the step size.

$$\delta u = -\varepsilon(\bar{H}_{uu}(t))^{-1}(\bar{H}_u(t) + \bar{f}_u^T(t)\Delta\lambda(t)) - \bar{H}_{uu}^{-1}\bar{H}_{ux}(t)\delta x(t) \tag{25}$$

With this choice of δu,

$$\delta P + \tfrac{1}{2}\delta^2 P = -\tfrac{1}{2}\varepsilon^2\int_{t_0}^{t_f} [<\bar{H}_u(t) + \bar{f}_u^T(t)\Delta\lambda(t),\ \bar{H}_{uu}^{-1}(t)(\bar{H}_u(t) + \bar{f}_u^T(t)\Delta\lambda(t))>]\mathrm{d}t$$
$$-\int_{t_0}^{t_f} <C(t)\delta x(t),\ \delta x(t)> \mathrm{d}t \tag{26}$$

which is negative for $0 < \varepsilon \leqslant 1$.

The linear two-point boundary value problem (21)—(22) may be solved in various ways. The most advantageous way appears to be to introduce the linear transformation

$$\Delta\lambda(t) = \mathrm{l}(t) + K(t)\delta x(t) \tag{27}$$

where $l(t)$ is an n-vector and $K(t)$ is an $n\times n$ symmetric matrix. Differentiating (27) and equating with the right hand side of (22) we get

$$\dot{l}(t)+[K(t)B(t)+A^T(t)]l(t)+K(t)v(t)+w(t)$$
$$+[\dot{K}(t)+K(t)B(t)K(t)+K(t)A(t)+A^T(t)K(t)+C(t)]\delta x(t)=0 \qquad (28)$$

Since (28) is true for arbitrary δx we have

$$\dot{l}(t)+[K(t)B(t)+A^T(t)]l(t)+K(t)v(t)+w(t)=0; \qquad l(t_f)=0 \qquad (29)$$

$$\dot{K}(t)+K(t)B(t)K(t)+K(t)A(t)+A^T(t)K(t)+C(t)=0; \qquad K(t_f)=\bar{F}_{xx} \qquad (30)$$

Equation (30) is a matrix Riccati Equation and its properties have been extensively studied in the literature. In particular the equation is stable when integrated in the backwards direction. The solution of equation (30) is defined everywhere in $[t_0, t_f]$ in view of our conjugate point assumption [8].

The computing algorithm for solving the Lagrange problem may now be summarised as follows:

i. Guess the control function u and integrate the system equation $\dot{x}(t)=f[x(t), u(t), t]$ forwards with $x(t_0)=c$.
Store u and the corresponding trajectory x.

ii. Integrate the Euler-Lagrange equation $\dot{\lambda}=-H_x$ backwards with $\lambda(t_f)=F_x$. Calculate H_u, H_{uu}, H_{ux} and H_{ux} and also evaluate $(H_{uu})^{-1}$ along the trajectory. Simultaneously integrate the differential equations for $l(t)$ and $K(t)$ with the proper boundary conditions. Store $r(t)=H_{uu}^{-1}(t)[H_u(t)+f_u^T(t)l(t)]$ and the feedback gain matrix

$$M(t)=H_{uu}^{-1}(t)\lfloor H_{ux}(t)+f_u^T(t)K(t)\rfloor$$

where $0<\varepsilon\leqslant 1$.

iii. Repeat (i) using

$$u_{new}(t)=u_{old}(t)-\varepsilon r(t)-M(t)[x_{new}(t)-x_{old}(t)]$$

iv. Repeat (ii).

v. Stop computation when $\|H_u\|\leqslant\varepsilon$, where ε is a suitably chosen small number and $\| \;\|$ is a suitable norm.

Note: Some adjustment scheme for ε must be included in the computer program.

For many problems the assumption on H_{uu} and the non-existence of conjugate points may not be satisfied. In such cases a better estimate of the control function may be obtained by using a *gradient method.* Alternatively the following *successive approximation scheme* may be used till the assumption on H_{uu} is satisfied. For this development it is assumed

F_{xx} is positive semi-definite

L_{uu} is positive definite

$L_{xx}-L_{ux}^TL_{uu}^{-1}L_{ux}$ is positive semi-definite

We again choose a nominal control function u and integrate the system equations to obtain the corresponding nominal trajectory x. The performance functional is now expanded in Taylor's series and terms up to the second order are retained.

[8] S. K. MITTER: On second order necessary conditions and sufficient conditions for a class of optimal control problems. To be published.

We obtain

$$\delta P = <F_x(t_f), \delta x(t_f)> + \int_{t_0}^{t_f} (<L_u(t), \delta u(t)> + <L_x(t), \delta x(t)>)dt$$

$$\tfrac{1}{2}\delta^2 P = \tfrac{1}{2}<F_{xx}(t_f)\delta x(t_f), \delta x(t_f)> + \tfrac{1}{2}\int_{t_0}^{t_f} (<L_{uu}(t)\delta u(t), \delta u(t)> + <L_{xx}(t)\delta x(t), \delta x(t)> + 2<L_{ux}(t)\delta x(t), \delta u(t)>)dt$$

The control improvement δu is obtained by minimising $\delta P + \tfrac{1}{2}\delta^2 P$ subject to the constraint

$$\delta \dot{x} = f_x(t)\delta x(t) + f_u(t)\delta u(t); \qquad \delta x(t_0) = 0$$

In view of our assumptions on F and L the strengthened Legendre condition and the conjugate point condition are automatically satisfied for this auxiliary minimization problem.

The Euler-Langrange equations of this problem are

$$\delta \dot{x}(t) = f_x(t)\delta x(t) + f_u(t)\delta u(t); \qquad \delta x(t_0) = 0 \tag{31}$$

$$\Delta \dot{\lambda}(t) = -L_x(t) - L_{xx}(t)\delta x(t) - L_{xu}(t)\delta u(t) - f_x^T(t)\Delta\lambda(t); \Delta\lambda(t_f) = F_x(t_f) + F_{xx}(t)\delta x(t_f) \tag{32}$$

$$\delta u(t) = -L_{uu}^{-1}(t)(L_u(t) + L_{ux}(t)\delta x(t) + f_u^T(t)\Delta\lambda(t) \tag{33}$$

Substituting (33) into (31) and (32) we get

$$\delta \dot{x}(t) = A(t)\delta x(t) + B(t)\Delta\lambda(t) + v(t) \tag{34}$$

$$\Delta\dot{\lambda}(t) = -C(t)\delta x(t) - A^T(t)\Delta\lambda(t) - w(t) \tag{35}$$

where

$$A(t) = f_x(t) - f_u(t)L_{uu}^{-1}(t)L_{ux}(t)$$
$$B(t) = -f_u(t)L_{uu}^{-1}(t)f_u^T(t)$$
$$C(t) = L_{xx}(t) - L_{xu}(t)L_{uu}^{-1}(t)L_{ux}(t)$$
$$v(t) = -f_u(t)L_{uu}^{-1}(t)L_u(t)$$
$$w(t) = L_x(t) - L_{xu}(t)L_{uu}^{-1}(t)L_u(t)$$

This linear two point boundary value problem is solved in exactly the same way as for the second variation case by assuming $\Delta\lambda(t) = l(t) + K(t)\delta x(t)$. The differential equations for $l(t)$ and $K(t)$ are

$$\dot{l}(t) + [K(t)B(t) + A^T(t)]l(t) + K(t)v(t) + w(t) = 0; \qquad l(t_f) = F_x(t_f) \tag{36}$$

$$\dot{K}(t) + K(t)B(t)K(t) + K(t)A(t) + A^T(t)K(t) + C = 0; \qquad K(t_f) = F_{xx}(t_f) \tag{37}$$

Note however that the boundary condition of the l equation is different from that in the second variation case and there is no λ equation to integrate.

It is easily shown that this choice of δu gives a value of

$$\delta P+\tfrac{1}{2}\delta^2 P=-\tfrac{1}{2}\int_{t_0}^{t_f}[<\bar{L}_u(t)+\bar{f}_u(t)\Delta\lambda(t),\ \bar{L}_{uu}^{-1}(t)[\bar{L}_u(t)+\bar{f}_u(t)\Delta\lambda(t)]>$$
$$+<(\bar{L}_{xx}(t)-\bar{L}_{xu}(t)\bar{L}_{uu}^{-1}(t)\bar{L}_{ux}(t))\delta x(t),\ \delta x(t)>]\mathrm{d}t$$

which is negative in view of the assumption on L. To reduce step size the parameter ε, $0<\varepsilon\leqslant 1$ is introduced as follows

$$\delta u(t)=-\bar{L}_{uu}^{-1}(t)(\varepsilon\bar{L}_u(t)+\bar{L}_{ux}(t)\delta x(t)+\varepsilon\bar{f}_u^T(t)\Delta\lambda(t))$$

Case (ii). We assume that terminal constraints are present but the final time t_f is given explicitly.

We define the ψ function as in Section 4

$$\psi[x(t_f),\ t_f]=F(x(t_f),\ t_f)+<\nu,\ G(x(t_f),\ t_f)>$$

A nominal control $\bar{u}$ is chosen and the corresponding trajectory $\bar{x}$ is obtained by integrating the system equations forward. The multiplier ν is estimated and the Euler-Lagrange equation $\dot{\lambda}=-\bar{H}_x$ is integrated backwards with boundary condition $\lambda(t_f)=\bar{\psi}_x(t_f)$ δu is now chosen so that

$$\delta P+\tfrac{1}{2}\delta^2 P=\tfrac{1}{2}<\bar{\psi}_{xx}(t_f)\delta x(t_f),\ \delta x(t_f)>+\int_{t_0}^{t_f}<\bar{H}_u(t),\ \delta u(t)>\mathrm{d}t$$
$$+\tfrac{1}{2}\int_{t_0}^{t_f}(<\bar{H}_{uu}(t)\delta u(t),\ \delta u(t)>+<\bar{H}_{xx}(t)\delta x(t),\ \delta x(t)>+2<\bar{H}_{ux}(t)\delta x(t),\ \delta u(t)>)\mathrm{d}t \tag{38}$$

is minimized, subject to the constraints,

$$\delta\dot{x}(t)=\bar{f}_x(t)\delta x(t)+\bar{f}_u(t)\delta u(t);\qquad \delta x(t_0)=0 \tag{39}$$

$$\bar{G}(t_f)+\bar{G}_x(t_f)\delta x(t_f)=0 \tag{40}$$

The solution to this problem is similar to that of Case (i) the only difference being in the boundary condition of $\Delta\lambda$. The relevant equations are

$$\delta\dot{x}(t)=A(t)\delta x(t)+B(t)\Delta\lambda(t)+v(t);\qquad \delta x(t_0)=0 \tag{41}$$

$$\Delta\dot{\lambda}(t)=-C(t)\delta x(t)-A^T(t)\Delta\lambda(t)-w(t);\qquad \Delta\lambda(t_f)=\bar{\psi}_{xx}(t_f)\delta x(t_f)+G_x^T(t_f)\Delta\nu \tag{42}$$

where $\Delta\nu$ is a p-vector of constants, being the multiplier for the terminal constraints, while the definitions of A, B etc. are the same as in equation (23).

The linear two-point boundary value problem we have to solve is given by equations (40), (41) and (42). The solution is again analogous to that of Case (i).

Introduce the linear transformations

$$\Delta\lambda(t)=l(t)+K(t)\delta x(t)+N(t)\Delta\nu$$
$$\delta G\ \ =m(t)+N^T(t)\delta x(t)+P(t)\Delta\nu$$

where l is an n-vector, m a p-vector, K a $n\times n$ matrix, N a $n\times p$ matrix, P a $p\times p$ matrix. Here $\delta G=G_x(t_f)\delta x(t_f)$ is the amount by which the terminal conditions have been missed. If this quantity is large we may introduce a parameter ε_2 where $0<\varepsilon_2\leqslant 1$ to specify the desired change in terminal condition i.e. $\delta G=-\varepsilon_2\bar{G}$.

In exactly the same way as for Case (i) we deduce, by equating the coefficients of δx and Δv to zero the following differential equations

$$\dot{l}(t)+[K(t)B(t)+A^T(t)]l(t)+K(t)v(t)+w(t)=0; \qquad l(t_f)=0 \tag{45}$$

$$\dot{K}(t)+K(t)B(t)K(t)+K(t)A(t)+A^T(t)K(t)+C(t)=0; \qquad K(t_f)=\bar{\psi}_{xx}(t_f) \tag{46}$$

$$\dot{N}(t)+(A^T(t)+K(t)B(t)N(t)=0; \qquad N(t_f)=\bar{G}_x^T(t_f) \tag{47}$$

$$\dot{m}(t)+N^T(t)[B(t)l(t)+v(t)]=0; \qquad m(t_f)=0 \tag{48}$$

$$\dot{P}(t)+N^T(t)B(t)N(t)=0; \qquad P(t_f)=0 \tag{49}$$

To proceed with the improvement process it is necessary to determine Δv. Having integrated equations (45)–(49) backwards, we may determine Δv from

$$\Delta v=P^{-1}(t_0)[\delta G-m(t_0)-N^T(t_0)\delta x(t_0)] \tag{50}$$

The conjugate point assumption ensures that $P^{-1}(t_0)$ exists. This procedure for determining Δv is somewhat similar to that of Breakwell, Speyer and Bryson [9]. They, however, solve the linear two-point boundary value problem in a different way. It is thought that our method has advantages from the point of view of numerical stability.

Substituting (43) and (50) into the expression for δu, we obtain

$$\delta u=-\bar{H}_{uu}^{-1}(t)[\varepsilon_1\bar{H}_u(t)+\varepsilon_1 f_u^T(t)l(t)+f_u^T(t)N(t)P^{-1}(t_0)[\delta G-\varepsilon m(t_0)-N^T(t_0)\delta x(t_0)]$$
$$-\bar{H}_{uu}^{-1}(t)[\bar{H}_{ux}(t)+\varepsilon f_u^T(t)K(t)]\delta x(t) \tag{51}$$

We have introduced ε_1, ε_2, $0<\varepsilon_1, \varepsilon_2\leqslant 1$ to reduce step size. It may be shown that by suitable choice of ε_1 and ε_2, $\delta P+\frac{1}{2}\delta^2 P$ is negative.

The computing algorithm is almost the same as for *Case* (*i*). We however need to guess the initial value of v and store $N(t)$ for all values of t.

Case (*iii*). The terminal time t_f is not given explicitly. In this case the linear two-point boundary value is more complicated.

The control function u and the terminal time t_f is guessed and the nominal trajectory x is obtained by integrating the system equations. A value of v is guessed and the Euler-Lagrange equation $\dot{\lambda}=-H_x$ is integrated backwards with the boundary condition, $\lambda(t_f)=\bar{\psi}_x(t_f)$. The improvement δu is calculated by minimising $\delta P+\frac{1}{2}\delta^2 P$ subject to certain constraints.

Let us first calculate $\delta P+\frac{1}{2}\delta^2 P$. For the end-point we now have to consider dependent and independent variations, namely

$$\Delta x(t_f)=\delta x(t_f)+\dot{x}(t_f)\mathrm{d}t_f.$$

In this case,

$$\delta P=(\bar{H}(t_f)+\bar{\psi}_t(t_f))\mathrm{d}t_f+\int_{t_0}^{t_f}<\bar{H}_u(t),\ \delta u(t)>\mathrm{d}t \tag{52}$$

[9] J. V. Breakwell, J. L. Speyer and A. E. Bryson: Optimization and control of nonlinear systems using the second variation. *S.I.A.M. J. Contr.* **A 1**, (1963).

$$\tfrac{1}{2}\delta^2 P=\tfrac{1}{2}(<\bar{\psi}_{xx}(t_f)\Delta x(t_f),\ \Delta x(t_f)>+<\bar{\psi}_x(t_f),\ \Delta^2 x(t_f)>+\bar{\psi}_{tt}(t_f)\mathrm{d}t_f^2$$
$$+\bar{\psi}_t\mathrm{d}^2 t_f+2<\bar{\psi}_{xt},\ \Delta x(t_f)\mathrm{d}t_f>)+<\bar{H}_u(t_f),\ \delta u(t_f)\mathrm{d}t_f>$$
$$+\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\bar{H}(t)-<\bar{\lambda}(t),\ \bar{\dot{x}}(t)>]_{t=t_f}dt_f^2$$
$$+\tfrac{1}{2}[\bar{H}(t_f)-<\bar{\lambda}(t_f),\ \bar{\dot{x}}(t_f)>]\mathrm{d}^2 t_f$$
$$+\tfrac{1}{2}\int_{t_0}^{t_f}[<\bar{H}_{uu}(t)\delta u(t),\ \delta u(t)>+<\bar{H}_{xx}(t)\delta x(t),\ \delta x(t)>$$
$$+2<\bar{H}_{ux}(t)\delta x(t),\ \delta u(t)>]\mathrm{d}t \qquad (53)$$

where

$$\Delta^2 x(t_f)=\dot{x}(t_f)\mathrm{d}^2 t_f+\ddot{x}(t_f)\mathrm{d}t_f^2+2\delta\dot{x}(t_f)\mathrm{d}t_f \qquad (54)$$

The auxiliary minimization problem to be solved is Minimise $\delta P+\tfrac{1}{2}\delta^2 P$, subject to the constraints

$$\delta\dot{x}(t)=\bar{f}_x(t)\delta x(t)+\bar{f}_u(t)\delta u(t);\qquad \delta x(t_0)=0 \qquad (55)$$

$$\bar{G}(t_f)+\bar{G}_x(t_f)\delta x(t_f)+\bar{G}_x(t_f)\dot{x}(t_f)\mathrm{d}t_f+\bar{G}_t(t_f)\mathrm{d}t_f=0 \qquad (56)$$

If we use $\lambda(t_f)=\bar{\psi}_x(t_f)$ and neglect terms in $\mathrm{d}^2 t_f$ and solve this variational problem, we obtain the following Euler-Lagrange Equations

$$\delta\dot{x}(t)=A(t)\delta x(t)+B(t)\Delta\lambda(t)+v(t);\qquad \delta x(t_0)=0 \qquad (57)$$

$$\Delta\dot{\lambda}(t)=-C(t)\delta x(t)-A^T(t)\Delta\lambda(t)-w(t) \qquad (58)$$

$$\Delta\lambda(t_f)=\bar{\psi}_{xx}(t_f)\delta x(t_f)+\bar{G}_x^T(t_f)\Delta v+(\bar{\psi}_{xx}(t_f)\bar{\dot{x}}(t_f)+\bar{\psi}_{xt}(t_f)+\bar{f}_x^T(t_f)\bar{\psi}_x(t_f))\mathrm{d}t_f \qquad (59)$$

$$\delta G=\bar{G}_x(t_f)\delta x(t_f)+(\bar{G}_x(t_f)\dot{x}(t_f)+\bar{G}_t(t_f))\mathrm{d}t_f \qquad (60)$$

$$\delta\hat{\Omega}=<\bar{\psi}_{xx}(t_f)\bar{\dot{x}}(t_f)+\bar{\psi}_{xt}(t_f)+\bar{f}_x^T(t_f)\bar{\psi}_x(t_f),\ \delta x(t_f)>$$
$$+<\bar{G}_x(t_f)\bar{\dot{x}}(t_f)+\bar{G}_t(t_f),\ \Delta v>+s(t_f)\mathrm{d}t_f \qquad (61)$$

where

$$s(t_f)=[<\bar{\psi}_{xx}\dot{x},\ \dot{x}>+<\psi_x,\ f_x\dot{x}+f_u\dot{u}+f_t>+<L_x,\ \dot{x}>-<L_u,\ \dot{u}>$$
$$+L_t+\psi_{tt}+2<\psi_{2t},\ \dot{x}>]_{t=t_f}$$

The linear two-point boundary value problem 57–60 is solved in exactly the same way as previously by introducing

$$\Delta\lambda(t)=l(t)+K(t)\delta x(t)+N(t)\Delta v+p(t)\mathrm{d}t_f$$
$$\delta G=m(t)+N^T(t)\delta x(t)+P(t)\Delta v+q(t)\mathrm{d}t_f$$
$$\delta\hat{\Omega}=n(t)+<p(t),\ \delta x(t)>+<q(t),\ \Delta v>+s(t)\mathrm{d}t_f$$

where

$$\delta G=-\varepsilon_2\bar{G}(t_f)\qquad 0<\varepsilon_2\leqslant 1$$
$$\delta\hat{\Omega}=-\varepsilon_3(\bar{H}(t_f)+\bar{\psi}_t(t_f))\qquad 0<\varepsilon_3\leqslant 1$$

The differential equations for l, K, N, m and P are the same as that given by equations (45)–(49).

The equation for p, q, n, and s are obtained in the same way as in the previous two cases. They are:

$$\dot{p}(t)+[A^T(t)+K(t)B(t)]p(t)=0 \tag{62}$$

$$\dot{q}(t)+N^T(t)B(t)p(t)=0 \tag{63}$$

$$\dot{n}(t)+<p(t), B(t)l(t)+v(t)> =0 \tag{64}$$

$$\dot{s}(t)+<p(t), B(t)p(t)> =0 \tag{65}$$

Δv and dt_f are determined by integrating equations (45)–(49) and equations (62)–(68) backwards and solving

$$\delta G=m(t_0)+N^T(t_0)\delta x(t_0)+P(t_0)\Delta v+q(t_0)dt_f \tag{69}$$

$$\delta\Omega=n(t_0)+<p(t_0), \delta x(t_0)> + <q(t_0), \Delta v> +s(t_0)dt_f \tag{70}$$

6. RELATIONSHIPS WITH NEWTON'S METHOD

For simplicity we consider the case when there are no terminal constraints present. The method and conclusions are valid for the general Bolza problem. Solving the variational problem by Newton's Method means solving the Euler-Lagrange Equations (8), (9) and (10). The method consists in guessing a nominal control function, a nominal trajectory and a nominal multiplier function and then linearizing equations (8), (9) and (10) round the guessed functions. A linear two-point boundary value problem is then solved which yields corrections to the guessed functions. The linear two-point boundary value problem to be solved is

$$\dot{\bar{x}}+\delta\dot{x}=\bar{f}(t)+\bar{f}_x(t)\delta x(t)+\bar{f}_u(t)\delta u(t); \qquad \delta x(t_0)=0$$

$$\dot{\bar{\lambda}}+\delta\dot{\lambda}=-\bar{H}_x(t)-\bar{H}_{xx}(t)\delta x(t)-\bar{H}_{xu}(t)\delta u(t)-\bar{H}_{x\lambda}(t)\delta\lambda(t); \qquad \delta\lambda(t_f)=0$$

$$\bar{H}_u(t)+\bar{H}_{uu}(t)\delta u(t)+\bar{H}_{ux}(t)\delta x(t)+\bar{H}_{u\lambda}(t)\delta\lambda(t)=0$$

But for the fact that the system equations and the Euler-Lagrange equations are not satisfied by the initially guessed functions, these equations are precisely the same as equations (18), (19) and (20). Thus the methods we have used in solving equations (18), (19) and (20) may be used in solving the linear two-point boundary value problem in Newton's Method. As we have indicated previously from the viewpoint of numerical stability it is advantageous to solve the two-point boundary value problem in the way we have indicated. In problems where there is a constraint of the form $x(t_f)=a$ it may be better to use Newton's Method since we can guess the nominal trajectory to satisfy the boundary condition.

7. A DISCUSSION OF VARIOUS METHODS OF SOLVING OPTIMAL CONTROL PROBLEMS

A number of methods have been proposed for the solution of two-point boundary value problems arising in optimal control problems. These may be subdivided into three main classes:

i. Boundary Condition Iteration Method
ii. Control function Iteration Method
iii. Newton type Iteration Methods

The choice of the method to be adopted depends on the problem and on the nature of the application. Each problem will have a certain structure and exhibit certain stability properties, although in a non-linear problem it might be very difficult to isolate either. Further the nature of the control application may impose various constraints. For example, if on-line control is envisaged, rapidity of convergence may over-ride other factors. For some problems it may be necessary to obtain extremely accurate trajectories, while in others convergence of the performance functional to within a pre-assigned tolerance may be sufficient. In spite of this, certain advantages and disadvantages of each of these methods may be pointed out and certain recommendations made.

i. *Boundary condition iteration*

In this method, typically the control function u is eliminated from the first two Euler-Lagrange equations by solving $H_u=0$ and the resulting first two Euler-Lagrange equations are solved by iteration on one of the unknown boundary values say, $\lambda(t_0)$. A suitable scalar terminal error function $V\{x[t_f, \lambda(t_0)], \lambda[t_f, \lambda(t_0)]\}$ is then constructed. The boundary value $\lambda(t_0)$ is then adjusted till the error function goes to zero. The adjustment requires the computation of the gradient of V. Systematic methods for doing this are available [10]. These methods have certain computer programming advantages. Computer logic is simple and fast storage requirements are small. In problems where the method is successful accurate trajectories are obtained. The main disadvantage is the inherent instability of one of the Euler-Lagrange equations. To determine whether the method is applicable a preliminary analysis of the problem may possibly be carried out in the following way: let the unforced system equation be linearized round the given initial condition. An eigen-value analysis of the linearized system matrix could now be made. If the matrix turns out to be essentially self-adjoint boundary iteration methods are quite suitable. If not and if t_f-t_0 is substantially greater than the dominant system time-constant, severe instabilities may be encountered.

ii. *Control function iteration*

Control function iteration methods using both gradient techniques and steepest descent technique have been proposed in the literature. In these methods the control function is successively improved till $\|H_u\|\to 0$, where $\|\quad\|$ is some suitable norm of the H_u function. The primary advantage of this method is that computations are always performed in the stable direction. However convergence tends to be intolerably slow in a certain neighbourhood of the optimum. To improve convergence the size-step cannot be increased since this leads to instability. The iteration methods we have presented in this paper may be considered to be direct extensions of gradient or steepest descent techniques. We have stated previously that the second variation method is formally equivalent to Newton's method in function space. In a suitable neighbourhood of the optimum convergence is therefore quadratic. Computations here are also always performed in the stable direction. In fact in a suitable neighbourhood of the optimum, the inherent stability properties of linear feedback control systems inhibits the propagation of numerical errors. As a by-product we obtain linear time-varying feedback gains for neighbouring optimum feedback control.

[10] M. LEVINE: A steepest descent technique for synthesizing optimal control programmes. Paper 4, *Conf. Advances In Automatic Control.* Nottingham, April (1965).

On the other hand the conditions that H_{uu} be positive definite and that there be no conjugate point for the trajectories occurring in successive auxiliary minimization problems may be too strong. In such cases it may be necessary to get better estimates of the control function by using gradient methods or use the alternative successive approximation method we have indicated in conjunction with the second variation method. Numerical difficulties may also be encountered in integrating the matrix Ricatti equations, specially if the dynamic system is unstable. It is also to be noted that the matrix H_{uu} is to be inverted. Computer storage requirements are also greater since the feedback gain matrices have to be stored.

Some computational effort may be saved. For example, it is not necessary to compute H_{uu}^{-1} at every iteration. In fact in practice this may be held constant after two or three iterations. Convergence will necessarily be slower.

For ordinary minimization problem some very efficient computational algorithms have recently been proposed [11]. These algorithms may be considered to lie somewhere between gradient and Newton's method. A distinctive feature of these methods is that use is made of information generated in previous iterations. Generalisations of these methods to function spaces should be possible.

In this paper we have not considered inequality constraints. The assumption was made that these could be approximated by means of penalty functions. Extensions of the techniques presented here to problems with inequality constraints on control and state variables appear to be possible. The auxiliary minimization problem then has additional linear inequality constraints. In this case the corresponding dual maximization problem could be solved to obtain the improvement in control function.

iii. *Newton's method*

Newton's method was first proposed by HESTENES [12] to solve fixed end point problems of the Calculus of Variations. A complete analysis of the method for this class of problems was given by STEIN [13]. In the context of function space, the method dates back to KANTOROVICH [14]. KALABA [15] has also used this method for a special class of problems and called it "quasi-linearisation". Recently the method has been applied to some optimal control problems by KOPP and MCGILL [18]. They eliminated the control function u from the first two Euler-Lagrange equations by using the equation $H_u = 0$. The linearised Euler-Lagrange equations are then integrated for n-linearly independent boundary conditions. The unknown boundary value $\delta\lambda(t_0)$ is found by using linear interpolation and a matrix inversion. Improvements $\delta x(t)$ and $\delta\lambda(t)$ are then obtained by one more integration.

If the linear two-point boundary value problem is solved in this way, the method suffers from the instability disadvantages of boundary integration methods.

In our view, the methods advocated in this paper could be used to solve the linear two-point boundary value problem arising within Newton's Method.

[11] R. FLETCHER and M. J. D. POWELL: A rapidly convergent descent method for minimization. *Computer J.* 6 (1963).
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[13] M. L. STEIN: ref. cit.
[14] L. V. KANTOROVICH: On Newton's method. *Trudy Mat. Inst. Steklov* 28, 104–144 (1949).
[15] R. KALABA: On nonlinear differential equations, the maximum operation and monotone convergence *J. Math. Mech.* 8, 519–574 (1959).
[16] S. K. MITTER: Programming in function space. To be published.
[17] L. COLLATZ: *Funktionalanalysis and Numerische Mathematok*, Springer, Berlin (1964).
[18] R. E. KOPP and R. MCGILL: Several trajectory optimization techniques. in: *Computing Methods in Optimization Problems*. ed. by A. V. BALAKRINSHNAN and L. W. NEUSTADT. Academic, New York (1964).

8. SOME GENERALISATIONS

In the introduction we have stated that the derivations in this paper are formal. In this section we briefly indicate how a mathematically rigorous treatment of the successive approximation methods can be given. To this end, it is convenient to consider the following general problem:

We are given a dynamical system whose behaviour is governed by the following operator equation

$$g(u, x)=0_x \ldots (71), \text{ where } g: \mathscr{X}\times U\to\mathscr{X}$$

is a non-linear mapping from the Hilbert Space $\mathscr{X}\times U$ to the Hilbert Space $\mathscr{X}$, and 0_x is the null element of $\mathscr{X}$. U is to be thought as the control space and $\mathscr{X}$ the state space.

Let Ω be an open subset of $\mathscr{X}\times U$. The problem of optimal control is to find a point $z^0=(u^0, x^0)\varepsilon\Omega$ which satisfies equation (71), such that the functional

$$f(u, x), f\colon \mathscr{X}\times U\to R$$

is a minimum.

We follow the notation and terminology of DIENDONNÉ [19].

Assumptions. i.

$$f\varepsilon C^2(\Omega; R); \qquad g\varepsilon C^2(\Omega; \mathscr{X})$$

i.e. the mappings f and g possess continuous second Frechet Derivatives.

ii. The mapping $Dg(u^0, x^0)\varepsilon\mathscr{L}(\mathscr{X}\times U; \mathscr{X})$ is onto.

iii. $D_xg(u^0, x^0)$ is a linear homeomorphism of $\mathscr{X}$ onto $\mathscr{X}$.

We are then able to prove [16].

Theorem 1. (Necessary condition). Under the above assumptions, necessary conditions for f to have a minimum at (u^0, x^0) subject to $g(u, x)=0_x$ are

$$<D_uf(u^0, x^0), (t_1, 0)>_u+<\lambda^0, D_ug(u^0, x^0)\cdot(t_1, 0)>_u=0 \qquad (72)$$

$$<D_xf(u^0, x^0), (0, t_2)>_x+<\lambda^0, D_xg(u^0, x^0)\cdot(0, t_2)>_x=0 \qquad (73)$$

and $g(u^0, x^0)=0_x$ (74), where $<>_u$ and $<>_x$ denote inner products in Hilbert Spaces U and $\mathscr{X}$ and λ^0 is a unique element of $\mathscr{X}$.

Theorem 2. (Necessary condition). If the hypotheses of Theorem 1 hold at the point (u^0, x^0), then a necessary condition for $f(x)$ to have a minimum at (u^0, x^0) subject to $g(u, x)=0_x$ is

$$\inf_{\substack{\|t\|=1 \\ Dg(u^0, x^0)\cdot(t_1, t_2)=0_x}} D^2h(u^0, x^0)\cdot(t_1, t_2)\geq 0$$

where $h=f+<\lambda, g>$, $t=(t_1, t_2)$ and $\|t\|=\max(\|t_1\|, \|t_2\|)$.

[19] J. DIENDONNÉ: *Foundations of Modern Analysis*. Academic, New York (1960).

Theorem 3. (Sufficient Condition). Let all the hypotheses of Theorem 1 hold and let the necessary conditions of Theorem 1 be satisfied. Further let

$$\inf_{\substack{\|t\|=1 \\ Dg(u^0,\, x^0)\cdot(t_1,\, t_2)=0_x}} D^2h(u^0, x^0)\cdot(t_1, t_2)>0, \text{ where } h=f+\langle\lambda, g\rangle.$$

Then there is an open connected neighbourhood N of (u^0, x^0), such that $f(u, x)>f(u^0, x^0)$ for every $(u^0, x^0)\varepsilon N$.

Theorems 1–3 are generalisations of familiar theorems in the minimisation of a function of n variables subject to p constraints. In order to solve the minimisation problem, we therefore have to solve the set of equations

$$\left.\begin{aligned} D_xh(u, x, \lambda)&=0 \\ D_\lambda h(u, x, \lambda)&=0 \\ D_uh(u, x, \lambda)&=0 \end{aligned}\right\} \tag{74}$$

One way of solving this set of equations is to use Newton's Method in Function Space. At each iteration step we have to solve the set of linear equations given by

$$\begin{aligned} &D_xh(\bar{u}, \bar{x}, \bar{\lambda})+D^2_xh(\bar{u}, \bar{x}, \bar{\lambda})\cdot(0, \delta x, 0)+D^2_{xu}h(\bar{u}, \bar{x}, \bar{\lambda})\cdot(\delta u, 0, 0) \\ &\qquad\qquad +D^2_{x\lambda}h(\bar{u}, \bar{x}, \bar{\lambda}).(0, 0, \delta\lambda)=0 \\ &D_\lambda h(\bar{u}, \bar{x}, \bar{\lambda})+D^2_{\lambda x}h(\bar{u}, \bar{x}, \bar{\lambda})\cdot(0, \delta x, 0)+D^2_{\lambda u}h(\bar{u}, \bar{x}, \bar{\lambda})\cdot(\delta u, 0, 0)=0 \\ &D_uh(\bar{u}, \bar{x}, \bar{\lambda})+D^2_{ux}h(\bar{u}, \bar{x}, \bar{\lambda})\cdot(0, \delta x, 0)+D^2_uh(\bar{u}, \bar{x}, \bar{\lambda})\cdot(\delta u, 0, 0) \\ &\qquad\qquad +D^2_{u\lambda}h(\bar{u}, \bar{x}, \bar{\lambda})\cdot(0, 0, \delta\lambda)=0 \end{aligned}$$

Based on the work of STEIN and COLLATZ [17] sufficient conditions for the Newton Process to converge can be given.

The variational problem we have considered in this paper may be recast into this form by writing the Euler-Lagrange equations in integral form.

9. CONCLUSIONS

In this paper we have considered some successive approximation methods for the solution of a general class of optimal control problems. The methods we have presented are formally equivalent to Newton's Method in function space. The main advantage of the methods are rapidity of convergence and stable computation. However in many problems, it may be necessary to resort to Gradient or other methods to obtain a sufficiently good estimate of the nominal control function. The method directly provides neighbouring optimal feedback gains.

For the variational problems treated here, it has been assumed that inequality constraints on control and state variables are either absent or adequately approximated by means of penalty functions. The results presented here extend, in part to cases where bounds on the control variable and state variable are present. We shall cover this in a subsequent paper.

Acknowledgement—This research was supported by the Central Electricity Research Laboratories, Leatherhead, England.