GENERAL POLE ASSIGNMENT BY OUTPUT FEEDBACK AND SOLUTION OF LINEAR MATRIX EQUATIONS FROM AN ALGEBRAIC VIEWPOINT

by

Theodore Euclid Djaferis

B.S. University of Massachusetts

May 1974

M.S. Massachusetts Institute of Technology

February 1977

E.E. Massachusetts Institute of Technology

February 1978

SUBMITTED IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

AT THE

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Signature of Author ....Theodore Djaferis..............

Department of Electrical Engineering ~~and Computer~~ Science June 22, 1979.

Certified by ....

Thesis Supervisor.

Accepted by ....

Chairman, Departmental Committee on Graduate Students.

GENERAL POLE ASSIGNMENT BY OUTPUT FEEDBACK
AND SOLUTION OF LINEAR MATRIX EQUATIONS
FROM AN ALGEBRAIC VIEWPOINT

by

Theodore Euclid Djaferis

Submitted to the Department of Electrical Engineering and Computer Science on June 22 1979 in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Abstract

For some linear strictly proper system given by its transfer function, two dynamic output feedback problems can be posed. The first one is that of using dynamic output feedback to assign the closed loop characteristic polynomial and the second that of assigning the closed loop invariant factors. In the first part of this thesis we are concerned with these problems and their inter-relationships. The formulation is done in the frequency domain and the investigation carried out from an algebraic point of view, in terms of linear equations over rings of polynomials. Several results are expressed by exploiting the notion of Genericity.

In the second part of the thesis we undertake the study of a family of linear matrix equations and give necessary and sufficient conditions for the existence of a unique solution, while showing a connection with the Hilbert-Nullstellensatz. The basic idea is that the set of matrices, with elements in some field can be thought of as a module over some polynomial quotient ring. Emphasis is also given in suggesting algorithms for constructing the solution, which make use of finite algebraic procedures which are easily implemented on a digital computer.

Thesis Supervisor: Sanjoy Mitter.

Title: Professor of Electrical Engineering.

Acknowledgment

It is my pleasure to thank Professor Mitter, my thesis supervisor, for his continued guidance and encouragement. His supervision was vital to the successful completion of the thesis research.

I am grateful to Professors Chris Byrnes, Nils Sandell and Dr. Bernard Levy, my thesis readers, for their stimulating comments.

This research was conducted at the M.I.T. Laboratory for Information and Descision Systems with partial support provided by the Energy Research and Development Administration under grant E.R.D.A.-E(49-]8)-2087 and by the National Aeronautics and Space Administration under grant N.A.S.A.-NGL22-009-124.

Table of Contents

Chapter 1

Introduction

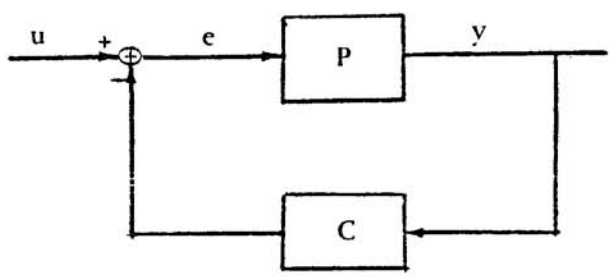
Several problems in Control Theory can be formulated in terms of linear equations and in many others it is essential that a linear equation be solved in order to obtain their solution. We mention a few which are closely related to what we will be concerned with. In the Wiener-Hopf design of Optimal Controllers [34], one encounters the equation $A(s)X(s)+B(s)Y(s)=I$. In the Synthesis of Linear Multivariable Regulators [3,8] equations $A(s)X(s)+B(s)Y(s) = I$ and $A(s)X(s)+Y(s)B(s) = F(s)$ appear. In the Model Following problem [22,27] one can formulate the problem using the linear equation $H(s)X(s) = M(s)$. The Lyapunov Equation $PA+A'P = Q$ appears in Stability Theory, Optimal Control, Stochastic Control and in the solution of the Algebraic Riccati equation when Newton's method is used [18].

This thesis is divided in two major parts. In the first one we are interested in formulating the problem of General Pole Assignment by dynamic output feedback, in terms of transfer matrices and linear equations. This is done by exploiting the notion of matrix fraction representation. The similar problem utilizing state feedback has been investigated [24]. The present approach is appealing due to the fact that the state does not enter into the analysis. This means that in computations no state estimate need be calculated. Instead only the output is used for feedback. We also will be concerned with an equation of the form $X(s)A(s)+Y(s)B(s) = \Phi(s)$, and will be interested in solutions which have the property that $X^{-1}(s)Y(s)$ exists and is proper.

In the second part we investigate a method for solving equations which belong to a certain class of linear matrix equations which includes the Lyapunov equations $PA+A'P = Q$, $P-A'PA = Q$ and the Sylvester equation $PA+BP = Q$. The approach uses module theoretic ideas and readily lends itself to computations. This research is a continuation of work done in my Masters Thesis [12].

General Pole Assignment.

Placing the poles of a closed loop system at desired locations using feedback is a problem that has been seriously studied and finds application in many areas of Control Theory. For several years most of the work done delt with state feedback. Recently researchers have turned to the question of dynamic output feedback. In Chapter 2 we work with the following feedback system:



The $m \times \ell$ $(m \geq \ell)$ matrix P is the input-output transfer function of a strictly proper plant and C $(\ell \times m)$ that of some dynamic compensator. Both P and C have elements in $R(s)$, the field of rational functions in the indeterminate s over the reals R.

The closed loop transfer function is

$$G = P(I+CP)^{-1}.$$

The condition $m \geqslant \ell$ is not restrictive because the situation $m \lneq \ell$ can be treated in a similar manner, and dual results obtained.

The transfer function P is assumed to be given. We are interested in the following two problems and their inter-relationships.

(The Characteristic Polynomial Problem).Let ϕ be some polynomial in R[s]. What are necessary and sufficient conditions for the existence of a proper compensator C, so that if χ is the characteristic polynomial of the closed loop system, then χ is a factor of ϕ. A variant of this problem is the investigation of the situation in which χ is equal to ϕ.

(The Invariant Factor Problem).Let Φ be an $\ell x \ell$ diagonal matrix with elements in R[s]. What are necessary and sufficient conditions for the existence of a proper compensator C, so that if Ψ is the closed loop invariant factor matrix, Ψ is equivalent to Φ. A variant of this problem is to let $\Phi=(\phi_i)$ be in Smith form and require that $(\Psi=(\psi_i))$ ψ_i divides ϕ_i, $1 \leqslant i \leqslant \ell$,or more specefically that $\psi_i = \phi_i$.

Similar problems have been treated by Rosenbrock [24], in the state feedback situation where necessary and sufficient conditions were given involving the degrees of the diagonal polynomials of Φ, for the existence of a constant compensator C. In later work [26] an attept was made to generalize this result to the output feedback case with a proper dynamic C.

In our approach we will use some of the results given in [26]. We also give a clear proof of Rosenbrock's earlier result.

The problem of assigning the closed loop poles of a system by feedback is of fundamental importance in System Theory. It has received, and rightly so, a great deal of attention from both the frequency and state space point of view. Therefore justification about carrying out research on this topic is self evident.

The situation is different with the problem of assigning the closed loop invariant factors. It is a problem that becomes much more pronounced in the theory of multivariable systems, a subject which is less understood and still in the developmental stage.

It is clear that from a mathematical standpoint, that the invariant factors of a transfer function, determine the deeper structure of a system. If $P = C(sI-A)^{-1}B$ with (A,B,C) minimal, then A can be written in Companion form as:

$$\bar{A} = \begin{bmatrix} C_1 & & & 0 \\ & C_2 & & \\ & & \ddots & \\ 0 & & & C_\ell \end{bmatrix}, \quad \bar{A} = T^{-1}AT,$$

where $\psi_i = \det(sI-C_i)$ are the invariant factors of P.

Beyond this, the system theoretic significance of the invariant factors is much less understood at the present time.

There does exist a relationship between the degrees of the invariant factors and the controllability or observability indecies for a certain class of systems. For example let P be an $\ell x \ell$ upper-triangular transfer function with the following properties:

$$P = \begin{bmatrix} \frac{a_{11}}{\phi_1} & \frac{a_{12}}{\phi_2} & \cdots & \frac{a_{1\ell}}{\phi_\ell} \\ & \frac{a_{22}}{\phi_2} & \cdots & \frac{a_{2\ell}}{\phi_\ell} \\ & 0 & & \\ & & & \frac{a_{\ell\ell}}{\phi_\ell} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1\ell} \\ & a_{22} & \cdots & a_{2\ell} \\ & 0 & \ddots & \\ & & & a_{\ell\ell} \end{bmatrix}}_{N} \underbrace{\begin{bmatrix} \phi_1 & & & \\ & \phi_2 & & 0 \\ & 0 & \ddots & \\ & & & \phi_\ell \end{bmatrix}}_{D}^{-1}$$

where:

1) P is stricly proper, a_{ij}, ϕ_j coprime.

2) $\phi_\ell | \phi_{\ell-1}, \phi_{\ell-1} | \phi_{\ell-2}, \ldots, \phi_2 | \phi_1, \phi_i$ monic.

3) $a_{11} | a_{22}, \ldots, a_{\ell-1\ell-1} | a_{\ell\ell}, a_{ii}$ monic.

4) $a_{jj} | a_{ij} \quad 1 \leqslant j \leqslant \ell, 1 \leqslant i \leqslant \ell$.

It can be easily shown that there exists some unimodular matrix E such that $EP = \mathrm{diag}(\frac{a_{ii}}{\phi_i})$ which is the Smith–McMillan form of P. Therefore the ϕ_i are the invariant factors of P. On the other hand the matrices N,D are right coprime and $\begin{bmatrix} D \\ N \end{bmatrix}$ is column proper which means that the controllability indecies of P are the degrees of the invariant factors. This is quite clear in the single input single output situation.

From another point of view, the invariant factors are closely related with the definition of transmission zeros of a plant [11]. Let P be an $m \times \ell$ $(m \geq \ell)$ plant, with Smith-McMillan form given by M_p.

$$M_p = \begin{bmatrix} \frac{\varepsilon_1}{\phi_1} & & & \\ & \frac{\varepsilon_2}{\phi_2} & 0 & \\ 0 & & \ddots & \\ & & & \frac{\varepsilon_\ell}{\phi_\ell} \\ & & 0 & \end{bmatrix}$$

The ϕ_i are the invariant factors of P and the zeros of P are associated with the zeros of the polynomials ε_i. Suppose that $\varepsilon_i \neq 0$. Then [11] z in $\mathbb{C}$ is a zero of P of order m iff $\varepsilon_\ell(\cdot)$ has a zero of order m at z. The significance of this order, roughly speaking, is that the system completely blocks the transmission of some input of the form

$\sum_{k=0}^{\sigma} g_k t^k \exp(zt)$ for $\sigma = 0,1,2,\cdots m-1$. For $\sigma = m$, there is an input of this form for which the output is non-zero and proportional to exp(zt). Therefore if two systems P and $\overline{P}$ have the same characteristic polynomial (ie $\chi=\phi_1\phi_2\cdots\phi_\ell, \overline{\chi}=\overline{\phi}_1\overline{\phi}_2\cdots\overline{\phi}_\ell$, $\chi = \overline{\chi}$) but different invariant factors and zeros, the transmission blocking properties of the two systems would be different.

In Chapter 2 several versions of the Characteristic Polynomial and Invariant Factor Problems will be formulated, as well as the Denominator Matrix Problem. It will be seen that degree constraints on ϕ and the elements of Φ are generally not enough to ensure the desired characteristics. We are therefore urged to introduce the very natural idea of Genericity.

I believe that the approach taken here clarifies many earlier results, including those of Rosenbrock and Hayton [26], is general enough to encompass several known results and introduces new ideas which are used constructively such as Genericity, Generalized Resultants, Construction of acceptable solutions to the equation $X D_{RP} + Y N_{RP} = \Phi$. All of the procedures used are constructive and can be programmed on a digital computer. It would be quite interesting to implement these algorithms using MACSYMA as is done in [12], for some of the equations in Chapter 3.

One of the ideas that will be extensively used is that of matrix fraction representation. If P is a matrix over R(s) then it can be expressed as:

$$P = N_1 D_1^{-1} = D_2^{-2} N_2$$

where N_1, D_1, D_2, N_2, are all matrices over R[s]. In the single-input, single-output case we always did think of the rational function $P = \frac{n}{d} = nd^{-1}$ as a ratio of two polynomials and associated with this is the construction of the ring (actually a field) of fractions of R[s], a form of commutative localization. We will see that because of certain properties of the ring of square matrices this idea can be generalized in the matrix case and is a form of non-commutative localization [28]. To my knowledge this is the first time that this relationship is clearly and explicitly demonstrated.

Linear Matrix Equations

In the second part of this research we present an algebraic method for obtaining the exact solution P, to equations of the type

$$\sum_{i=0}^{s} \sum_{j=0}^{t} g_{ij} B^i P A^j = Q \qquad (1.1)$$

Matrices B(mxm), A(nxn) and A(mxn) are given and have entries in some field F, the g_{ij}'s are elements of F. The method is founded on the observation that the set MN of mxn matrices over F, form an $F[x,y]/\Psi$ - module where Ψ is some ideal in F[x,y]. The Hilbert-Nullstelensatz [35] is then employed to show that existence of a unique solution to (1.1)is equivalent to the condition that a certain element in $F[x,y]/\Psi$ has an inverse. The method uses polynomial arithmetic and is very well suited for computer implementation and has the added advantage that it allows for parametric studies. Said in a different way, this means that the theory and the method do not involve the concept of an eigenvalue. This is of great importance in the investigation of systems over rings.

The work in this chapter is a continuation of our research done for my Masters Thesis. Algorithms for the Lyapunov equation in particular PA + A'P =Q have already been constructed and programmed on MACSYMA [12].

Throughout this thesis we take an algebraic point of view, and we demonstrate once again its importance in the treatment of control theory problems. It is therefore one more indication of the power of the algebro-geometric approach which has been used in recent years.

Chapter 2

General Pole Assignment by Output Feedback

2.1 Formulation

In recent years there has been a resurgence of interest in frequency domain techniques for the analysis of linear systems. One idea that has been repeatedly used is the Matrix Fraction Representation of Transfer Functions. For single-input single-output systems the transfer function is a rational function, which is nothing else but an element of the ring (field) of quotients of the integral domain R[s]. This idea can be extended in the case of commutative rings and modules [2] and is a form of Localization. The idea can be generalized even further for non-commutative rings which satisfy certain conditions [28]. What we will now show is that matrices with elements in R[s] do satisfy these conditions and that Matrix Fraction Representation is a form of non-commutative Localization. Our principal reference is [28].

Let $\underline{A}$ be a ring with identity and $\underline{T}$ a multiplicatively closed subset of $\underline{A}$ (ie if s,t in $\underline{T}$ then $st \in \underline{T}$ and $1 \in \underline{T}$). Let $\underline{A}[\underline{T}^{-1}]$ denote the right ring of fractions of $\underline{A}$ with respect to $\underline{T}$. From [28] we have that $\underline{A}[\underline{T}^{-1}]$ exists iff

S_1) if s in $\underline{T}$ and $a \in \underline{A}$ there exist $t \in \underline{T}$, $b \in \underline{A}$ such that $sb = at$.

S_2) if $sa = 0$ with $s \in \underline{T}$ then $at = 0$ for some t in $\underline{T}$.

both hold.

When $\underline{A}[\underline{T}^{-1}]$ exists it has the form

$$\underline{A}[\underline{T}^{-1}] = \underline{A} \times \underline{T} / \sim$$

where $\sim$ is the equivalence relation defined as:

$(a,s) \sim (b,t)$ if there exist $c,d \in \underline{A}$ such that $ac = bd$ and $sc = td \in \underline{T}$.

Addition in $\underline{A}[\underline{T}^{-1}]$ is given by:

$$(a,s) + (b,t) = (ac+bd,u) \qquad u = sc = td \text{ in } \underline{T}.$$

Multiplication:

$$(a,s)(b,t) = (ac,tu) \qquad sc = bu \text{ and } u \text{ in } \underline{T}.$$

In our case let $\underline{A} = F^{nxn}[s]$ F a field and $\underline{T}$ be

$$\underline{T} = \{G \in \underline{A} | \quad \det G \neq 0.$$

$\underline{T}$ is multiplicatively closed, and one can show that conditions S_1, S_2 are satisfied (see Appendix A). Therefore $\underline{A}[\underline{T}^{-1}]$ exists.

Proposition 2.1 Let $(F(s))^{nxn}$ be the ring of nxn rational matrices. Then $(F(s))^{nxn}$ is isomorphic to $\underline{A}[\underline{T}^{-1}]$.

Now as in the commutative case we also see that for each right $\underline{A}$-module $\underline{M}$, we can define its right module of fractions with respect to $\underline{T}$ a multiplicatively closed set of $\underline{A}$, denoted by $\underline{M}[\underline{T}^{-1}]$, having a right $\underline{A}[\underline{T}^{-1}]$-module structure. It can be shown that [28]

$$\underline{M}[\underline{T}^{-1}] \simeq \underline{M} \times \underline{T} / \sim$$

where $(x,s) \sim (y,t)$ if there exist $c,d \in \underline{A}$ such that $xc = yd$ and $sc = td \in \underline{T}$.

The operations of addition in $\underline{M}[\underline{T}^{-1}]$ and right multiplication by elements of $\underline{A}[\underline{T}^{-1}]$ are given by:

$$(x,s)+(y,t) = (xc+yd,u) \qquad u = sc = td \in \underline{T}$$

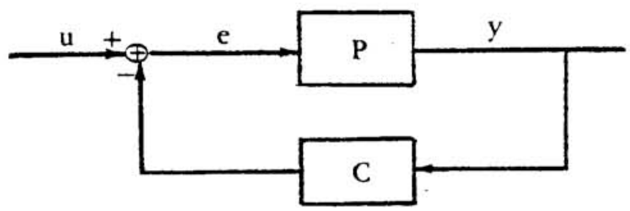
$$(x,s)\cdot(b,t) = (xc, tu) \qquad (b,t) \in \underline{A}[\underline{T}^{-1}],\ sc = bu,\ u \in \underline{T}.$$

Proposition 2.2 The right $(F(s))^{n\times n}$- modules $(F(s))^{m\times n}$ and $F^{m\times m}[s]\ [\underline{T}^{-1}]$ are isomorphic.

The proofs of the above two Propositions can be found in Appendix A.

The concept of Matrix Fraction Representations will play a very important role in our investigation of pole assignment by output feedback.

Assume that we have the following feedback system:



where P is an $m\times\ell$ $(m\geq\ell)$ strictly proper input-output transfer function and C some $\ell\times m$ proper dynamic conpensator. Both P and C have elements in R(s) the field of rational functions in s over the reals R. The closed loop input-output transfer function G is

$$e = u - C\cdot y$$

$$y = P\cdot e$$

$$\Rightarrow \quad e = u - CPe \;, \quad (I + CP)e = u$$

$$\Rightarrow \quad y = P(I + CP)^{-1}u$$

$$\Rightarrow \quad G = P(I + CP)^{-1}$$

where we assume that $(I + CP)^{-1}$ exists.

Since P is a rational matrix(ie a matrix with elements which are rational functions) we have that P [10] can be factored as follows:

$$P = B\,A^{-1} = D^{-1}N$$

where B,A,D,N are polynomial matrices.

Notation:

$P = B_{RP} A_{RP}^{-1}$ some right fraction representation of P.

$= A_{LP}^{-1} B_{LP}$ some left fraction representation of P.

$= N_{RP} D_{RP}^{-1}$ some right coprime fraction representation of P.

$= D_{LP}^{-1} N_{LP}$ some left coprime fraction representation of P.

The closed loop transfer function G can then be expressed in the following ways:

Form 1

$$G = P\,(I + CP)^{-1}$$
$$= B_{RP} A_{RP}^{-1} (I + A_{LC}^{-1} B_{LC} B_{RP} A_{RP}^{-1})^{-1}$$
$$= B_{RP} A_{RP}^{-1} (A_{LC}^{-1}(A_{LC}A_{RP} + B_{LC} B_{RP}^{-1}) A_{RP}^{-1})^{-1}$$
$$= B_{RP} (A_{LC} A_{RP} + B_{LC} B_{RP})^{-1} A_{LC}$$

Since the representations for P and C are not coprime, unnecessary dynamics are present (ie hidden modes are introduced). We have that if χ the characteristic polynomial [10] of G, then χ is a factor of $\det(A_{LC} A_{RP} + B_{LC} B_{RP})$. If K is a greatest common left divisor of A_{LC}, B_{LC} and L a greatest common right divisor of B_{RP}, A_{RP} then

$$\chi = \alpha \frac{\det(A_{LC} A_{RP} + B_{LC} B_{RP})}{\det K \cdot \det L} \qquad \alpha \text{ , constant.}$$

Form 2

$$G = N_{RP} (A_{LC} D_{RP} + B_{LC} N_{RP})^{-1} D_{LC}$$

As in form 1 if χ is the characteristic polynomial of the feedback system and K a greatest common left divisor of

A_{LC}, B_{LC} then

$$\chi = \frac{\det(A_{LC}D_{RP} + B_{LC}N_{RP})}{\det K} \quad ,\alpha \text{ constant.}$$

Form 3

$$G = N_{RP}(D_{LC} + N_{LC}N_{RP})^{-1}D_{LC} = N_{RP}\Omega^{-1}D_{LC}$$

From [7,10] we have that Ω has the special property that $\det \Omega = 0$ is the characteristic equation of the feedback system (ie the roots of $\det \Omega(s)$ are the internal poles of the system).

Form 4 (least order)

$$G = \tilde{N}_{RP}\tilde{\Omega}^{-1}\tilde{D}_{LC}$$

where $\tilde{N}_{RP}$, $\tilde{\Omega}$ are right coprime.

$\tilde{D}_{LC}$, $\tilde{\Omega}$ are left coprime.

From [14, 24, 25] we have that the system matrix

$$\begin{bmatrix} I & 0 & 0 \\ 0 & \tilde{\Omega} & \tilde{D}_{LC} \\ 0 & -\tilde{N}_{RP} & 0 \end{bmatrix}$$

is of least order and therefore that $\tilde{\Omega}$ is equivalent to the invariant factor matrix of the closed loop system G. By invariant factor matrix of G we mean the following: Let M_G be the Smith-McMillan form of G

$$M_G = \begin{bmatrix} \frac{\varepsilon_1}{\psi_1} & & 0 & \\ & \ddots & & \\ & 0 & & \frac{\varepsilon_\ell}{\psi_\ell} \\ & & 0_{m-\ell \quad \ell} & \end{bmatrix} \qquad \psi_i \text{ monic}, \; \psi_i \mid \psi_{i-1} \quad 1 \leqslant i \leqslant \ell.$$

Then $\begin{bmatrix} \psi_1 & & & \\ & \psi_2 & & 0 \\ & & \ddots & \\ & 0 & & \psi_\ell \end{bmatrix}$ is the <u>invariant factor matrix</u> of G and ψ_i are the invariant factors.

<u>Remark</u>. We want to justify the statement that $\tilde{\Omega}$ is equivalent to Ψ . We use $\theta(\cdot)$ to denote the degree of a polynomial.
Let G be a strictly proper $m\times\ell$ matrix $(m\geq\ell)$ and $M_G=\text{diag}(\frac{\varepsilon_i}{\psi_i})$ its Smith-McMillan form.

$$M_G = E\ G\ H \qquad\qquad E,H \text{ unimodular}$$

$$\Rightarrow \qquad G = E^{-1}M_G H^{-1}$$

$$= E^{-1}\underbrace{\begin{bmatrix} \varepsilon_1 & & & \\ & \varepsilon_2 & & 0 \\ & 0 & \ddots & \\ & & & \varepsilon_\ell \\ & & 0 & \end{bmatrix}}_{A(m\times\ell)} \cdot \underbrace{\begin{bmatrix} \psi_1 & & \\ & \ddots & 0 \\ 0 & & \\ & & \psi_\ell \end{bmatrix}^{-1}}_{\Psi(\ell\times\ell)} \underbrace{H^{-1}}_{B(\ell\times\ell)} = A\Psi^{-1}B$$

We call Ψ the invariant factor matrix of P.
Let $G = A_1\Psi_1^{-1}B_1$ with $A_1(m\times\ell)$, $\Psi_1(\ell\times\ell)$, $B_1(\ell\times\ell)$ be some other least order representation of G.
We then have that Ψ_1 and Ψ are equivalent. (In the case when $\theta(\det\Psi) = n\geq m$, $n\geq\ell$ we can use Theorem 3.1 p.106 in [24]).
Let Q,Q_1 be the system matrices (Rosenbrock sense)

$$Q = \begin{matrix} \\ r\left\{ \right. \\ \\ \end{matrix}\begin{bmatrix} \overbrace{I_q \quad\quad 0}^{r} & 0 \\ 0 \quad\quad \Psi & B \\ 0 \quad\ -A & 0 \end{bmatrix} \qquad\qquad Q_1 = \begin{matrix} \\ r\left\{ \right. \\ \\ \end{matrix}\begin{bmatrix} \overbrace{I_q \quad\quad 0}^{r} & 0 \\ 0 \quad\quad \Psi_1 & B_1 \\ 0 \quad\ -A_1 & 0 \end{bmatrix}$$

where $r \geq n$. From the work of Fuhrman we have that Q and Q_1 are strictly system equivalent (Fuhrman sense). But from [25] Theorem 6 we have that Q, Q_1 are strictly system equivalent (Rosenbrock sense), which means that

$$\begin{bmatrix} I_q & 0 \\ 0 & \Psi \end{bmatrix}, \quad \begin{bmatrix} I_q & 0 \\ 0 & \Psi_1 \end{bmatrix} \quad \text{are equivalent.}$$

Therefore Ψ and Ψ_1 are equivalent.

We are now in a position to formulate the following problems.

Problem 1

Let P be a strictly proper plant and ϕ a monic polynomial. What are necessary and sufficient conditions for the existence of a proper compensator C, so that if χ is the characteristic polynomial of the closed system then χ is a factor of ϕ.

Problem 2

Let P be a strictly proper plant and ϕ a monic polynomial. What are necessary and sufficient conditions for the existence of a proper compensator C, so that if χ is the characteristic polynomial of the closed loop system then $\chi = \phi$.

Both of these problems will be referred to as subcases of the Characteristic Polynomial Problem.

Problem 3

Let P be an $m \times \ell$ strictly proper plant and Φ an $\ell \times \ell$ diagonal matrix in Smith form. What are necessary and sufficient conditions for the existence of a proper compensator C so that the closed loop invariant factor matrix $\Psi = (\psi_i)$ has the property that ψ_i divides ϕ_i for $1 \leq i \leq \ell$.

Problem 4

Let P be an m$\times\ell$ strictly proper plant and Φ an $\ell\times\ell$ diagonal matrix in Smith form. What are necessary and sufficient conditions for the existence of a proper compensator C so that the closed loop invariant factor matrix Ψ has the property that $\Psi = -\Phi$.

This problem is clearly a special case of Problem 3. Here we require that the invariant factors be placed exactly, at given locations. Rosenbrock in his book [24] delt with this problem when static, state feedback is used. Problems 3 and 4 will be referred to as subcases of the Invariant Factor Problem.

Problem 5

Let $P = N_{RP}D_{RP}^{-1}$ be an m$\times\ell$ strictly proper plant and Φ an $\ell\times\ell$ matrix. What are necessary and sufficient conditions for the existence of a polynomial solution X, Y of $XD_{RP} + YN_{RP} = \Phi$ for which $X^{-1}Y$ exists and is proper with N_{RP}, Φ right coprime and X,Φ left coprime.

This last problem will be referred to as the Denominator Matrix Problem. Clearly if such an acceptable solution does exist then $C = X^{-1}Y$ is a proper dynamic compensator and $N_{RP}\ \Phi^{-1}X$ a least order representation of G, which means that if Ψ is the invariant factor matrix of the closed loop system then Ψ and Φ are equivalent.

Remark In a recent paper [26], Rosenbrock and Hayton are concerned with the following problem.

Let P be an m$\times\ell$ strictly proper plant and Φ an $\ell\times\ell$ diagonal matrix in Smith form. What are sufficient conditions for the existence of a proper compensator $C = A_{LC}^{-1}B_{LC}$ so that for

$$Q = \begin{bmatrix} I & 0 & 0 \\ 0 & A_{LC}D_{RP}+B_{LC}N_{RP} & A_{LC} \\ 0 & -N_{RP} & 0 \end{bmatrix}$$

a matrix (not necessarily of least order) which gives rise to the closed loop transfer function G we have that $A_{LC}D_{RP}+B_{LC}N_{RP}$ and Φ are equivalent.

Now this does not guarantee that we have placed the invariant factors of the closed loop system since $A_{LC}D_{RP}+B_{LC}N_{RP}$ and N_{RP} may have a non unimodular right divisor and A_{LC} and $A_{LC}D_{RP} + B_{LC}N_{RP}$ a non-unimodular left divisor. That this is actually the case can be seen from the following example.

In the paper they say that if $\theta(\phi) = 2\theta(d_p) -1$ where $P = n_p d_p^{-1}$ is a strictly proper plant (siso) then a proper compensator exists such that det $T = \alpha\phi$ where α is a constant and $\begin{bmatrix} T & U \\ -V & 0 \end{bmatrix}$ a matrix giving rise to G.

Let $P = \dfrac{1}{(s-1)^2}$, $\phi = s^3-2s^2$.

Then using their procedure for constructing the compensator we find that $a_c = s$ $b_c = -s \Rightarrow C = -1$

And also

$$\begin{bmatrix} I_2 & 0 & 0 \\ 0 & s^3-2s^2 & s \\ 0 & -1 & 0 \end{bmatrix}$$ is a system matrix giving rise to

$$G = 1\,(s^3 - 2s^2)^{-1}s.$$

Clearly det $T = s^3 - 2s^2 = \phi$

But we can see that this ϕ is not the characteristic polynomial (invariant polynomial) χ of the closed loop system since

$$\chi = \frac{s^3 - 2s^2}{s}.$$

We see that degree conditions on ϕ are not by themselves sufficient to ensure that the closed loop invariant polynomial be equal to ϕ. We see that we first have to solve the equation

$$xd_p + yn_p = \phi$$

for x,y such that $x^{-1}y$ is proper and then require something additional.

We have suggested several problems, classifying them as the Characteristic Polynomial, Invariant Factor and Denominator Matrix Problem. We have commented on their system theoretic significance (of major significance in Stability Theory). We will now show the significance of the equation $XD_{RP}+YN_{RP} = \Phi$ and then investigate the existence of a special kind of polynomial solutions (X,Y) to this equation.

2.2 Method of Solution

One way in which we may proceed is to utilize the form in which the closed loop transfer function G has been expressed:

$$G = N_{RP}(A_{LC}D_{RP}+ B_{LC}N_{RP})^{-1}A_{LC}$$

$$= N_{RP}\ (D_{LC}D_{RP}+ N_{LC}N_{RP})^{-1}D_{LC}$$

$$= \tilde{N}_{RP}\tilde{\Omega}^{-1}\tilde{D}_{LC} \quad .$$

Suppose now that we are investigating the characteristic polynomial problem. What we then want to do is give some matrix Φ with $\det \Phi = \phi$ and see whether any polynomial solution (X,Y) exists to the equation

$$XD_{RP} + YN_{RP} = \Phi \qquad (2.1)$$

with the additional requirements that $\det X \neq 0$ and $X^{-1}Y$ be proper matrix. (Since N_{RP}, D_{RP} are right coprime we already know that a polynomial solution (X,Y) does exist). Now then if such an (X,Y) does exist, let K be a greatest common left divisor of X and Y. We then have that the closed loop characteristic polynomial χ is expressed by

$$\chi = \frac{\phi}{q} \quad , \quad q = \det K.$$

In a similar manner if we are looking at the invariant factor problem and Φ is some matrix in Smith form, we want to see whether a polynomial solution (X,Y) exists to equation (2.1) with $\det X \neq 0$ and $X^{-1}Y$ proper. This will ensure that if $\Psi = (\psi_i)$ is the closed loop invariant factor matrix $\psi_i | \phi_i$ $1 \leq i \leq \ell$. If in addition N_{RP} and Φ are right coprime and X and Φ left coprime then $\psi_i = \phi_i$. The association of this equation with the denominator matrix problem is evident. Our analysis evolves around the solution of this equation.

2.3 The Equation $XD_{RP} + YN_{RP} = \Phi$

We need to answer the following question: For which Φ does there exist a polynomial solution (X,Y) to $XD_{RP} + YN_{RP} = \Phi$ with the additional properties that $\det X \neq 0$ and $X^{-1}Y$ is proper. Such a solution will be called <u>acceptable</u>.

We have that N_{RP}, D_{RP} are $m \times \ell$ and $\ell \times \ell$ matrices respectively which are coprime. It is known [24] that a polynomial solution (X,Y) does exist for all $\ell \times \ell$ polynomial matrices Φ. The solution is not unique and as a matter of fact we have:

<u>Proposition 2.3</u> [23] Let (U,V) be a polynomial solution to $UD_{RP} + VN_{RP} = I$. Then all polynomial solutions (X,Y) of $XD_{RP} + YN_{RP} = \Phi$ can be expressed as:

$$X = \Phi U - NN_{LP}$$
$$X = \Phi V + ND_{LP} \qquad (2.2)$$

where N is a polynomial matrix.

<u>proof</u>: The equation (considered over the field of rational functions) is linear and the general solution is given by:

$$X = X_1 + X_0$$
$$Y = Y_1 + Y_0$$

where (X_1, Y_1) is some particular solution of (2.1) and (X_0, Y_0) any solution of the homogeneous equation

$$X_0 D_{RP} + Y_0 N_{RP} = 0 \qquad (2.3)$$

Now let $X_1 = \Phi U$, $Y_1 = \Phi V$. We have polynomial $X = X_1 + X_o$ $Y = Y_1 + Y_0$ iff X_0, Y_0 are polynomial solutions of (2.3) (ie not rational). We then want to know which are the polynomial solutions (2.3).

Let $KD_{RP} + LN_{RP} = 0$ be some polynomial solution of (2.3). We want to show that $K = NN_{LP}$, $L = ND_{LP}$ for some polynomial N.

Since $KD_{RP} = -LN_{RP}$ and for any rational N $NN_{LP}D_{RP} = -ND_{LP}N_{RP}$ we have

$$(K-NN_{LP})\, D_{RP} = -(L-ND_{LP})\, N_{RP}\ .$$

Let $N = LD_{LP}^{-1}$ then

$$(K-NN_{LP})\, D_{RP} = 0\ .$$

Since det $D_{RP} \neq 0$ it is not a zero divisor and so $K = NN_{LP}$. Therefore $K = NN_{LP}$, $L = ND_{LP}$. We now must show that N is actually a polynomial matrix.

Since N_{LP}, D_{LP} are left coprime [24] we have that the Smith form of $[D_{LP}, N_{LP}]$ is $[I, 0]$.

$\Rightarrow$ E,H unimodular in $E[D_{LP},N_{LP}]\, H = [I,0]$

$\Rightarrow$ $[D_{LP},N_{LP}] = E^{-1}[I,0]\, H^{-1}$

$\Rightarrow$ $[L,K] = N[D_{LP},N_{LP}]$
$= NE^{-1}[I,0]\, H^{-1}$

$\Rightarrow$ $[L,K]\, H = NE^{-1}[I,0] = [N\ 0] \begin{bmatrix} E^{-1} & 0 \\ 0 & I \end{bmatrix}$

$\Rightarrow$ $[L,K]\, H \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix} = [N,0]$

But the left hand side is a product of polynomial matrices therefore N is a polynomial matrix.□

It is easy to see that any X,Y of the form

$$X = \Phi U - NB_{LP}$$
$$Y = \Phi V + NA_{LP}$$

for some polynomial N is a solution to (2.1)

We can also go ahead and state

Proposition 2.4 Let Φ be an $\ell x \ell$ polynomial matrix with det $\Phi \neq 0$. Then there exists a polynomial solution (X,Y) to equation (2.1) for which det $X \neq 0$.

proof:

Let $P = \frac{N}{d(s)}$ where d(s) is the least common multiple of all the denominators of P. Let E,H be unimodular matrices such that ENH = S is in Smith form. Then

$$P = E^{-1}\frac{NH^{-1}}{d} = E^{-1}MH^{-1}$$

where M is the Smith-McMillan form of P [24].

$$M = \begin{bmatrix} \frac{a_1}{b_1} & & & & 0 \\ & \ddots & & & \\ & & \frac{a_q}{b_q} & & \\ & 0 & & \ddots & \\ & & & & \frac{a_\ell}{b_\ell} \\ & & 0 & & \end{bmatrix}$$

where $a_1, a_2 \cdots a_q$ are non-zero, $a_{q+1} = \cdots a_\ell = 0$, $b_{q+1} = \cdots b_q = 1$ and a_i, b_i coprime $1 \leq i \leq q$.

$$\text{Now } M = \underbrace{\begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_\ell \\ & 0 & \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_\ell \end{bmatrix}}_{B}^{-1}$$

where A and B are right coprime

This means that $P = E^{-1}A(HB)^{-1} = \bar{N}_{RP} \cdot \bar{D}_{RP}^{-1}$ is a right coprime representation of P. Now if $P = N_{RP}D_{RP}^{-1}$ there exists a unimodular matrix F such that

$$N_{RP} = \bar{N}_{RP}F$$
$$D_{RP} = \bar{D}_{RP}F$$

Since A and B are right coprime there exist polynomial U,V which are diagonal and

$$UB + VA = I$$

with det $U \neq 0$.

Let $U_1 = UH^{-1}$ $V_1 = VE$ then

$$UH^{-1}HB + VEE^{-1}B = I$$

$$\Rightarrow \quad U_1\bar{D}_{RP} + V_1\bar{N}_{RP} = I$$

Let $U_2 = F^{-1}U_1$ $V_2 = F^{-1}V_1$

$$\Rightarrow \quad U_1\bar{D}_{RP}F + V_1\bar{N}_{RP}F = F$$

$$\Rightarrow \quad F^{-1}U_1D_{RP} + F^{-1}V_1N_{RP} = I$$

$$\Rightarrow \quad U_2D_{RP} + V_2N_{RP} = I$$

Let $X = \Phi U_2$ $Y = \Phi V_2$ Then

$$XD_{RP} + YN_{RP} = \Phi \text{ with } \det X \neq 0. \square$$

The only condition we have imposed so far is for det $\Phi \neq 0$. And we have shown that we can construct a polynomial solution X,Y satisfying $XD_{RP} + YN_{RP} = \Phi$ with det $X \neq 0$. But we have not said anything about $X^{-1}Y$ being proper. This is the critical point. To accomplish this we will need to apply more conditions on Φ and investigate the matter further. In fact the next part will concentrate on this issue. To do this we will exploit the form in which the general polynomial solution X,Y has been expressed.

$$X = \Phi U - NB_{LP}$$

$$Y = \Phi V + NA_{LP}$$

In this part of the section we will develop some theory that will help us with the construction of desirable polynomial solutions to equation (2.1).

Let R(s) be the field of rational functions in s over R. It is a set of equivalence classes, denoted by [a,b]. We pick a representative from each equivalence class $(\bar{a},\bar{b})$ where $\bar{a},\bar{b}$ are coprime and $\bar{b}$ is monic. This representative is unique. We define two subsets of R(s) in the following manner

$$P_s = \{ [a,b] \mid \bar{a} \text{ polynomial}, \ \bar{b} = 1 \}$$

$$S_p = \{ [a,b] \mid \theta(\bar{a}) < \theta(\bar{b}) \} \quad \{[0,1]\}$$

The set P_s is isomorphic to R[s] and S_p is the set of strictly proper rational functions in which we have also included the zero element.

We can now define two functions

$$P_0 : R(s) \to P_s \qquad\qquad P_1 : R(s) \to S_p$$

Let $(\bar{a},\bar{b})$ the representative of [a,b] and let $\bar{a} = n\bar{b} + r$ where n,r are the quotient and remainder.

Now define

$$P_0([a,b]) = [n,1]$$

$$P_1([a,b]) = [r,\bar{b}]$$

The functions are well defined.

We can now state:

<u>Proposition 2.5</u> Every element x in R(s) can be expressed uniquely as $x = x_0 + x_1$ where x_0 in P_s and x_1 in S_p.

proof: For $x = [a,b]$ we let $x_0 = P_0([a,b])$ and $x_1 = P_1([a,b])$ (ie $[a,b] = [n,1] + [r,\bar{b}]$).

Suppose that $[a,b] = [q,1] + [\bar{g},\bar{p}]$.
We are allowed to choose $(\bar{g},\bar{p})$ to represent the equivalence class.
Then

$$[n\bar{b} + r,\bar{b}] = [q\bar{p} + \bar{g},\bar{p}]$$

$$\Rightarrow \quad (n\bar{b} + r)\,\bar{p} = +\bar{b}q\bar{p} + \bar{b}\bar{g}$$

$$\Rightarrow \quad (n\bar{p} - q\bar{p} - \bar{g})\,\bar{b} + r\bar{p} = 0$$

Since $\bar{b},r$ are coprime then $\bar{p} = -z\bar{b}$

$$np - q\bar{p} - \bar{g} = z\cdot r$$

$\Rightarrow$ $z|g$, $z|p$ therefore z is a constant.

Since $\theta(\bar{g}) < \theta(\bar{p})$ we must have $n=q$, $z=-1$, which implies that $\bar{p}=\bar{b}$ $\bar{g}=r$.

From the above we can now say that: the fraction $\frac{a}{b}\neq 0$ is proper iff $R_0([a,b])$ is constant, the fraction $\frac{a}{b}\neq 0$ is strictly proper iff $P_0([a,b]) = 0$.

The above can also be applied to matrices over R(s) in the following way:

$$P_0(M) = (P_0(m_{ij})) \qquad P_1(M) = (P_1(m_{ij})).$$

Clearly

$$M = P_0(M) + P_1(M)$$

Here are some properties of P_0, P_1 .

Property 1 Let $[a,b]$, $[c,d]$ be two rational functions. Then $P_0([a,b] + [c,d]) = P_0([a,b]) + P_0([c,d])$

Let A,B be two compatible matrices with rational entries.

Then $P_0(A+B) = P_0(A) + P_0(B)$.

<u>proof</u>: Let $a = n_1 b + r_1$ $\quad c = n_2 d + r_2$.

we know that $P_0([a,b]) = [n_1,1]$

$$P_0([c,d]) = [n_2,2]$$

but $ad = n_1 bd + r_1 d$ $\quad \Rightarrow ad + cb = (n_1+n_2)bd + r_1 d + r_2 b$

$cb = n_2 bd + r_2 b$

therefore

$$\begin{aligned} P_0([a,b]+[c,d]) &= P_0([ad+cb,bd]) \\ &= [n_1+n_2,1] \\ &= [n_1,1] + [n_2,1] \\ &= P_0([a,b]) + P_0([c,d]) \end{aligned}$$

Clearly $P_0(A+B) = P_0(A)+P_0(B)$

<u>Property 2</u> Let [a,b], [c,d] be proper rational functions.

Then $P_0([a,b][c,d]) = P_0([a,b])\cdot P_0([c,d])$.

Let A,B be two proper compatible matrices with rational entries.

Then $P_0(AB) = P_0(A)\cdot P_0(B)$.

<u>proof</u>: $ac = n_1 n_2 bd + r_1 n_2 d + n_1 r_2 b + r_1 r_2$

Since [a,b] [c,d] are proper n_1, n_2 are constants therefore $n_1 n_2$ is the quotient $r_1 n_2 d + n_1 r_2 b + r_1 r_2$ the remainder of bd|ac.

$$\begin{aligned} = \qquad P_0([ac,bd]) &= [n_1 n_2,1] \\ &= [n_1,1]\cdot[n_2,1] \\ &= P_0([a,b])\cdot P_0([c,d]) \end{aligned}$$

Let $P_0(A) = A_0$ $\quad P_1(A) = A_1$ $\quad P_0(B) = B_0$ $\quad P_1(B) = B_1$

Then $P_0(AB) = P_0((A_0+A_1)\ (B_0+B_1))$

$= P_0(A_0B_0+A_0B_1+A_1B_0+ A_1B_1)$

$= P_0(A_0B_0) + P_0(A_0B_1) + P_0(A_1B_0) + P_0(A_1B_1)$

$= A_0B_0$

$= P_0(A)\cdot P_0(B)$

<u>Property 3</u> Let A be a square matrix, which is proper. Then if $\det P_0(A)\neq 0$ we have $\det A\neq 0$.

<u>proof</u>: A is proper so that

$$A = A_0 + A_1 \text{ where } A_0 \text{ is constant.}$$

Suppose that $\det A_0\neq 0$

Then $\det A = \det A_0 + \frac{a(s)}{b(s)}$, $\frac{a(s)}{b(s)}$ strictly proper.

$\Rightarrow \det A\neq 0$.

<u>Property 4</u> Let K be a constant and A some compatible matrix. Then $P_0(KA) = KP_0(A)$

<u>proof</u>:

$$P_0(K(A_0+A_1)) = P_0(KA_0) + P_0(KA_1) = P_0(KA_0) = KA_0=KP_0(A).$$

<u>Lemma 2.6</u> Let $\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \cdots \geqslant \alpha_\ell \geqslant 0$ and $\beta_1 \geqslant \beta_2 \geqslant \cdots \beta_\ell \geqslant 0$ be two sets of non-negative integers with $c_i=\alpha_i+\beta_i$ and $c_{k+1}=c_{k+2}=\cdots c_\ell=0$. Let $c = c_1+c_2+\cdots c_k$ and ϕ be a polynomial of degree c Then there exists an $\ell x \ell$ polynomial matrix Φ such that i) $\det \Phi=\alpha\phi$ (α constant) and

ii) $P_0(\mathrm{diag}(s^{-\beta_i})\ \Phi\ \mathrm{diag}(s^{-\alpha_i})\ ) = J$ (2.4)

where J is diagonal with constant entries .

<u>proof</u>: We will give a constructive proof in which we will construct a matrix Φ which has elements on the diagonal of degree

c_i and elements in column ℓ and row i of degree less than c_i, and det $\Phi=\alpha\phi$. This will immediately satisfy (2.4)

We at first factor ϕ to irreducible terms over R[s] and then form polynomials $\phi_1,\phi_2\cdots\phi_k$, with $\phi=\phi_1\cdot\phi_2\cdot\cdots\cdot\phi_k$. Some may be equal to 1.

From the matrix Φ_1

$$\Phi_1 = \begin{bmatrix} \bar{\phi}_1 & & & & & \\ & \bar{\phi}_2 & & & & \\ & & \ddots & & 0 & \\ & & & \bar{\phi}_k & & \\ & 0 & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix}$$

where $\bar{\phi}_i = \frac{1}{\ell c(\phi_i)}\ \phi_i$ are monic.

$$\Phi_1 = \begin{bmatrix} \Phi_2 & 0 \\ 0 & I_{\ell-k} \end{bmatrix}$$

If $\theta(\phi_i)=c_i$ then we are done since $\Phi = \Phi_1$ satisfies i) and ii). If not then we must have $\theta(\phi_i)$ c_i and $\theta(\phi_j)<c_j$ for at least one pair $i\neq j$. By repeated applications of Lemma 4.1 [24] on Φ_2 we will arrive at a matrix $\Phi_3= E_1\Phi_2 H_1$ (E_1,H_1 unimodular) with diagonal elements of degree c_i and elements in column i of degree less than c_i.

$$\Phi_3 = \begin{bmatrix} e_{11} & e_{12} & e_{13}\cdots & e_{1k} \\ e_{21} & e_{22} & e_{23}\cdots & e_{2k} \\ & & \vdots & \\ e_{k1} & e_{k2}\cdots & & e_{kk} \end{bmatrix}$$

If elements in row i are of degree less than c_i we are done. If not then we start with column 1 and using elementary column operations transform it so that $\theta(e_{i1})<c_i \quad 1\leq i\leq k$.

Column 1

Let $r = \max(\theta(e_{21}),\theta(e_{31})\ldots\theta(e_{k1}))$ and suppose that $c_1>r\geq c_2$.

Let

$$
\begin{array}{lll}
e_{21}=q_{21}e_{22}+p_{21} & p_{21}=0 \quad \text{or} & \theta(p_{21})\leq r-c_2 \\
e_{31}=q_{31}e_{33}+p_{31} & p_{31}=0 \quad \text{or} & \theta(p_{31})\leq r-c_3 \\
\vdots & \vdots & \\
e_{k1}=q_{k1}e_{kk}+p_{k1} & p_{k1}=0 \quad \text{or} & \theta(p_{k1})\leq r-c_k
\end{array}
$$

Multiply column j $2\leq j\leq k$ by q_{ji} and subtrack from column 1.

The new column 1 is $(\bar{e}_{11}\bar{e}_{12}\cdots\bar{e}_{k1})$.

$$
\begin{bmatrix} \bar{e}_{11} \\ \bar{e}_{21} \\ \\ e_{k1} \end{bmatrix} = \begin{bmatrix} e_{11} \\ e_{21} \\ \\ e_{k1} \end{bmatrix} - q_{21}\cdot \begin{bmatrix} e_{12} \\ e_{22} \\ \\ e_{k2} \end{bmatrix} - q_{31}\cdot \begin{bmatrix} e_{13} \\ e_{23} \\ \\ e_{k3} \end{bmatrix} \cdots q_{k1}\cdot \begin{bmatrix} e_{1k} \\ e_{2k} \\ \\ e_{kk} \end{bmatrix}
$$

Now $\theta(\bar{e}_{11}) = c_1$

$\theta(\bar{e}_{21}) \leq r-1$

$\vdots$

$\theta(\bar{e}_{k1}) \leq r-1$

The new max, $\bar{r} = \max(\theta(\bar{e}_{21}),\cdots\theta(\bar{e}_k)) \leq r-1$

We can continue doing the above until we make $r=\max(\theta(\bar{\bar{e}}_{21})\cdots$
$\cdots\theta(\bar{\bar{e}}_{k1})) \leq c_2-1$.

We now have $c_2 > r \geqslant c_3$.

Let
$$\bar{\bar{e}}_{31} = \bar{q}_{31} e_{33} + \bar{p}_{31} \qquad \bar{p}_{31} = 0 \quad \text{or } \theta(\bar{p}_{31}) \leqslant r - c_3$$
$$\bar{\bar{e}}_{41} = \bar{q}_{41} e_{44} + \bar{p}_{41} \qquad \vdots$$
$$\bar{\bar{e}}_{k1} = \bar{q}_{k1} e_{kk} + \bar{p}_{k1} \qquad \bar{p}_{k1} = 0 \quad \text{or } \theta(\bar{p}_{k1}) \leqslant r - c_k$$

Multiply column j $3 \leqslant j \leqslant k$ by $\bar{q}_{j1}$ and subtract from column 1. The new column 1 is $(\bar{\bar{\bar{e}}}_{11}, \cdots \bar{\bar{\bar{e}}}_{k1})$.

$$\begin{bmatrix} \bar{\bar{\bar{e}}}_{11} \\ \bar{\bar{\bar{e}}}_{21} \\ \\ \bar{\bar{\bar{e}}}_{k1} \end{bmatrix} = \begin{bmatrix} \bar{\bar{e}}_{11} \\ \bar{\bar{e}}_{21} \\ \\ \bar{\bar{e}}_{k1} \end{bmatrix} - q_{31} \cdot \begin{bmatrix} e_{13} \\ e_{23} \\ \\ e_{k3} \end{bmatrix} - \cdots - q_{k1} \cdot \begin{bmatrix} e_{1k} \\ \\ \\ e_{kk} \end{bmatrix}$$

Now
$$\theta(\bar{\bar{\bar{e}}}_{11}) = c_1$$
$$\theta(\bar{\bar{\bar{e}}}_{21}) < c_2$$
$$\theta(\bar{\bar{\bar{e}}}_{31}) \leqslant r-1$$
$$\vdots$$
$$\theta(\bar{\bar{\bar{e}}}_{k1}) \leqslant r-1$$

and $\bar{r} = \max(\theta(\bar{\bar{\bar{e}}}_{31}), \cdots \theta(\bar{\bar{\bar{e}}}_{k1})) \leqslant r-1$. We can continue doing this until $r' = \max(\theta(e'_{31}), \cdots \theta(e'_{k1})) \leqslant c_3 - 1$

Continuing in this manner we can make $\theta(e_{21}) < \bar{c}_2$ $\theta(e_{31}) < c_3 \cdots$ $\cdots \theta(e_{k1}) < c_k$ and be through with column 1.

We subsequently forget column 1 and work on column 2 in the same manner using columns 3, 4, $\cdots$k. At the end of this procedure we will have formed Φ_4 such that

$$\Phi_4 = \Phi_3 H$$

with H unimodular and with elements on the diagonal of degree c_i and of diagonal elements in row i and column i of degree less than c_i.

Remark

This Lemma says the following. Let $\Phi=(\phi_i)$ be an $\ell x\ell$ diagonal matrix with $\theta(\phi_i)=d_i$. Let $c_1\geqslant c_2\geqslant\cdots c_\ell\geqslant 0$ be a set of non-negative integers with $\sum_{i=1}^{\ell} d_i = \sum_{i=1}^{\ell} c_i$.

Then unimodular matrices E and H exist such that a matrix $\overline{\Phi}=E\Phi H$ can be constructed with the property that the diagonal entries of $\overline{\Phi}$ have degrees c_i and all other elements in column i and row i have degrees less than c_i.

Lemma 2.7 [26] Let $Q=[T,U]$ be a rational matrix with T $\ell x\ell$, U ℓxm and suppose there exist non-negative integers $\alpha_i\geqslant 0$ such that

$$P_0(\mathrm{diag}(s^{-\alpha_i})[T,U]) = H = [H_2,H_1]$$

where H is constant and of rank ℓ.

i) Then $T^{-1}U$ exists and is proper iff det $H_2\neq 0$.

ii) Then $T^{-1}U$ exists and is strictly proper iff det $H_2\neq 0$ and $H_1=0$.

proof:

Suppose that det $H_2\neq 0$. We also have that $P_0(\underbrace{\mathrm{diag}(s^{-\alpha_i})T}_{M})=H_2$. Then M^{-1} exists and is proper.

We see this from the following:

$$\det M = \det H_2 + \frac{p(s)}{q(s)} = \frac{a(s)}{b(s)} \neq 0$$

where $\frac{p(s)}{q(s)}$ is strictly proper

and $\theta(a(s)) = \theta(b(s))$.

Now $$M^{-1} = \frac{\mathrm{Adj}(M)}{\det M} = \mathrm{Adj}(M) \cdot \frac{b(s)}{a(s)}$$

But M is proper and so is Adj(M). Therefore M^{-1} exists and is proper.

Now $$T^{-1}U = T^{-1}\mathrm{diag}(s^{-\alpha_i})^{-1}\ \mathrm{diag}(s^{-\alpha_i})U$$
$$= (\mathrm{diag}(s^{-\alpha_i})T)^{-1}(\mathrm{diag}(s^{-\alpha_i})U)$$
$$= M^{-1}\cdot(\mathrm{diag}(s^{-\alpha_i})\ U)$$

It is a product of proper matrices therefore $T^{-1}U$ is proper.

If in addition $H_1=0$ $P_0(T^{-1}U) = P_0(M^{-1})\cdot P_0(\mathrm{diag}(s^{-\alpha_i})U$

$\Rightarrow P_0(T^{-1}U) = 0$ $\Rightarrow$ $T^{-1}U$ is strictly proper.

On the other hand suppose that T^{-1} exists and $T^{-1}U$ is proper.

Show that $\det H_2 \neq 0$.

$$P_0(\mathrm{diag}(s^{-\alpha_i})U) = P_0(\mathrm{diag}(s^{-\alpha_i})\ TT^{-1}U)$$
$$= P_0(\mathrm{diag}(s^{-\alpha_i})T)\cdot P_0(T^{-1}U)$$

$\Rightarrow$ $H_1 = H_2\cdot F$. $F = P_0(T^{-1}U)$ F a constant.

$\Rightarrow$ $H = [H_2,H_1] = H_2[I,F]$

if $\det H_2=0$ then $H_2[I,F]$ cannot have rank ℓ. Therefore $\det H_2\neq 0$.

Now if in addition $T^{-1}U$ is strictly proper

then $P_0(T^{-1}U) = 0$

$\Rightarrow$ $P_0(\mathrm{diag}(s^{-\alpha_i})U) = H_1 = 0$.

Lemma 2.8. Let Q be an $\ell x(\ell+m)$ polynomial matrix of rank ℓ. Then there exists a unimodular matrix M such that MQ is row proper, (ie there exist $\alpha_i \geqslant 0$ such that $P_0(\mathrm{diag}(s^{-\alpha_i})MQ) = H$ with H having full rank).

This is a generalization of Theorem 2.5.7 in Wolovich [33].

proof:

Let β be the maximum degree of all $\ell x \ell$ minors of Q. Let r_i be the degrees of the rows of Q. We must have $r = \sum_{i=1}^{\ell} r_i \;\; \beta$ since if $r \;\; \beta$ we would have a matrix Q_1 such that the degree of each row being less than or equal to r_i but with $\theta(\det Q_1) = \beta$ which is impossible.

Let $Q^r = \mathrm{diag}(s^{r_i})\Gamma_r$, where Γ_r is the high order coefficient matrix of Q. If Γ_r is rank ℓ we are done because we let $\alpha_i = r_i$ and $M = I$. Suppose then that Γ_r has rank less than ℓ. This means that the ℓ rows of Γ_r are linearly dependent. This implies that there exist constants c_i such that $\sum_{i=1}^{\ell} c_i\gamma_i = 0$ (γ_i the i^{th} row of Γ_r), with not all c_i equal to zero. Now since Q is rank ℓ none of the γ_i's are all zero. This consequently means that at least two c_i are non-zero. Let $r_j = \max(r_1, r_2, \ldots, r\;)$.

$$=> \qquad \sum_{i=1}^{\ell} c_i s^{r_j - r_i} s^{r_i}\gamma_i = 0. \qquad (2.5)$$

Let

$$U_1 = \begin{bmatrix} 1 & 0 & & & & \\ 0 & 1 & & & 0 & \\ \vdots & \vdots & \ddots & & & \\ c_1 s^{r_j - r_1} & \cdots & & c_j & c_{j+1}s^{r_j - r_i} \cdots & c_\ell s^{r_j - r_\ell} \\ & 0 & & & \ddots & \vdots \\ & & & & & 1 \end{bmatrix}$$

be a unimodular matrix.

We see that by multiplying Q by U_1 on the left, has the effect of replacing row j of Q by $\sum_{i=1}^{\ell} c_i s^{r_j - r_i} d_i$ (d_i i^{th} row of Q) and because of (2.5) the degree of row j is less than r_j. Let $Q_1 = U_1 Q$ and repeat the above process until you obtain a Q' with Γ_r' being full rank. This completes the proof.

We are now in a position to state the following basic Theorem. It expresses a necessary and sufficient condition for a polynomial solution (X,Y) of (2.1) to also be an acceptable solution.

Theorem 2.9. [26] Let P be a strictly proper mx rational matrix and $P = N_{RP} D_{RP}^{-1}$ a right coprime representation which in addition has the property that $\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix}$ is column proper and the column degrees are ordered $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_\ell \geqslant 0$. Let Φ be a nonsingular $\ell x \ell$ matrix with $q = \theta(\det\Phi) - \theta(\det D_{RP}) \geqslant 0$. Let X,Y be a polynomial solution of $XD_{RP} + YN_{RP} = \Phi$.
Then $X^{-1}Y$ exists and is proper iff there exists a unimodular matrix M and indecies $d_i \geqslant 0$ satisfying $\sum_{i=1}^{\ell} d_i = q$ such that

$$P_0(\operatorname{diag}(s^{-d_i}) M [Y \ \Phi \operatorname{diag}(s^{-\alpha_i})]) \text{ is a constant.} \qquad (2.6)$$

proof:

(sufficiency) Suppose that (2.6) is satisfied. This means that both $\operatorname{diag}(s^{-d_i})MY$ and $\operatorname{diag}(s^{-d_i})M\Phi\operatorname{diag}(s^{-\alpha_i})$ are proper.

Now

$$\underbrace{\operatorname{diag}(s^{-d_i})M[X \ Y]}_{[\bar{X} \ \bar{Y}]} \underbrace{\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix} \operatorname{diag}(s^{-\alpha_i})}_{\begin{bmatrix} W \\ Z \end{bmatrix}} = \underbrace{\operatorname{diag}(s^{-d_i})M\Phi\operatorname{diag}(s^{-\alpha_i})}_{F}$$

By assumption we have that F is proper which means that $P_0(F)=F_0$ is a constant. Now $\det F = \det F_0 + \frac{a(s)}{b(s)}$, where $\frac{a(s)}{b(s)}$ is strictly proper. Since $\theta(\det\Phi) = q+\alpha$ $(\alpha=\alpha_1+\alpha_2+ \dots +\alpha_\ell)$ then $\det F = \frac{\det\Phi}{s^{q+\alpha}}$ is proper (not strictly proper), which means that $\det F_0 \neq 0$.

By assumption we also have that (Let $P_0(A)=A_0$ $P_1(A)=A_1$)

$$\begin{bmatrix} W \\ Z \end{bmatrix} = \begin{bmatrix} D_{RP} \\ R_{RP} \end{bmatrix} \operatorname{diag}(s^{-\alpha_i}) = \begin{bmatrix} W_0+W_1 \\ Z_1 \end{bmatrix} \quad \text{with } \det W_0 \neq 0.$$

This follows from the fact that $\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix}$ is column proper.

From Lemma 2.7 we have that $\det W_0 \neq 0$ and that $Z_0=0$.

By assumption we also have that $\overline{Y}_0 (=P_0(\overline{Y}))$ is a constant.

Now

$$[\overline{X},\overline{Y}] \begin{bmatrix} W \\ Z \end{bmatrix} = F$$

$$\Rightarrow \quad [\overline{X}_0+\overline{X}_1,\ Y_0+Y_1] \begin{bmatrix} W_0+W_1 \\ Z_1 \end{bmatrix} = [F_0+F_1]$$

$$\Rightarrow \quad \overline{X}_0(W_0+W_1) + \underbrace{\overline{X}_1 W_0 + X_1 W_1 + Y_0 Z_1 + Y_1 Z_1}_{\text{strictly proper matrix}} = F_0 + F_1$$

$\overline{X}_0$ is a polynomial matrix in general.

We will show that i) $\det \overline{X}_0 \neq 0$ and in fact that $\overline{X}_0$ is a constant, (ie $P_0(\operatorname{diag}(s^{-\alpha_i})X)$ is a constant).

Now $\overline{X}_0(W_0+W_1)$ is proper and

$$\overline{X}_0(W_0+W_1)=F_0+V_1, \quad V_1=P_1(\overline{X}_0(W_0+W_1)) \quad F_0=P_0(\overline{X}_0(W_0+W_1))$$

Since $\det F_0 \neq 0$, from property 3 of P_0 we have that

$\det(\overline{X}_0(W_0+W_1)) \neq 0$

$\Rightarrow \det \overline{X}_0 \neq 0$.

Now $\overline{X}_0(W_0+W_1) = F_0+V_1$

$\Rightarrow$ $\overline{X}_0(I+C) = F_0W_0^{-1}+V_1W_0^{-1}$ $\qquad C=W_1W_0^{-1}$ strictly proper.

Suppose that $\overline{X}_0$ was a polynomial matrix. Then it has some row i, which has polynomial entries. Let r_{ij} be the element of largest degree $\delta_i>0$ in row i. The ij^{th} element of $\overline{X}_0(I+C)$ then becomes

$$q=r_{ij}+r_{i1}c_{ij}+r_{i2}c_{2j}+\cdots r_{i\ell}c_{\ell j}$$

with each c_{ij} strictly proper.

$$P_0(q) = P_0(r_{ij})+P_0(r_{i1}c_{ij})+\cdots+P_0(r_{i\ell}c_{\ell j})$$
$$= [n,1]$$

where $\theta(n)=\delta_i>0$

This is a contradiction because $F_0W_0^{-1}+V_1W_0^{-1}$ is proper.

Therefore $\overline{X}_0$ is a constant.

We have that

$$P_0(\mathrm{diag}(s^{-d_i})M[X,Y]) = H = [\overline{X}_0 \;\; \overline{Y}_0]$$

where H is constant and det $\overline{X}_0\neq 0$. Therefore $X^{-1}Y$ exists and is proper.

(necessity)

Suppose now that $X^{-1}Y$ exists and is proper. Show that there exists unimodular M and indecies

d_i, $\sum_{i=1}^{\ell} d_i = q$ satisfying (2.6).

$$[X,Y]\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix} = \Phi$$

Since $\det\Phi \neq 0$ $[X,Y]$ must have rank ℓ. From Lemma 2.8 then, there exist $d_i \geqslant 0$ and M such that

$$P_0(\operatorname{dig}(s^{-d_i})M[X,Y]) = H_1 \quad \text{with } H_1 \text{ having rank } \ell.$$

Now $P_0(\operatorname{diag}(s^{-d_i})\ M[X,Y]) = P_0([\overline{X}_0+\overline{X}_1 \quad \overline{Y}_0+\overline{Y}_1])$
$= [\overline{X}_0, \overline{Y}_0] = H_1$

and since $(MX)^{-1}(MY)$ is proper from Lemma 2.7 we have that $\det \overline{X}_0 \neq 0$.

We already know that $P_0\left(\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix} \operatorname{diag}(s^{-\alpha_i})\right) = \begin{bmatrix} H_2 \\ 0 \end{bmatrix}$

where $\det H_2 \neq 0$.

Therefore

$$\operatorname{diag}(s^{-d_i})M[X,Y]\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix} \operatorname{diag}(s^{-\alpha_i}) = \underbrace{\operatorname{diag}(s^{-d_i})M\Phi\operatorname{diag}(s^{-\alpha_i})}_{Q}$$

$$\Rightarrow \quad P_0(\operatorname{diag}(s^{-d_i})M[X,Y]\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix} \operatorname{diag}(s^{-\alpha_i})) = P_0(Q)$$

$$\Rightarrow \quad \overline{X}_0\ H_2 = P_0(Q)$$
with $\det \overline{X}_0 H_2 \neq 0$

$$\Rightarrow \quad \det Q = \frac{e(s)}{f(s)} \quad \text{with } \theta(e) = \theta(f) \Rightarrow \theta(\det\Phi) = \sum_{i=1}^{\ell} d_i + \sum_{i=1}^{\ell} \alpha_i$$

Condition 2.6 is clearly satisfied.

It is very interesting to see the form Theorem 2.9 takes in the Single Input Single Output (SISO) case.

Corollary 2.10 Let P be a strictly proper rational function and $P=n_p d_p^{-1}$ a coprime representation (in fact it is a minimal coprime representation). Let be a polynomial with $q=\theta(\phi)-\theta(d_p)\geq 0$ and let x,y be a solution of $xd_p+yn_p=\phi$

Then $x^{-1}y$ exists and is proper iff

$$P_0((s^{-q})[y,\phi s^{-\alpha}])\text{ is constant.}$$

Remark

The above Theorem is quite reasonable for the following reason: We do have that x,y is a polynomial solution of $xd_p+yn_p=\phi$ with $\theta(d_p)=\alpha$ and $\theta(\phi)=\alpha+q$.

Necessity. If $x^{-1}y$ is proper then it is clear that $\theta(x)+\alpha=\theta(\phi)=\alpha+q$ and $q=\theta(x)\ \theta(y)$. Therefore we must have that $P_0(s^{-q}[y,\phi s^{-\alpha}])$ is a constant.

Sufficiency. If $P_0(s^{-q}[y\phi s^{-\alpha}])$ is a constant this means that $\theta(y)\leq q$ and since $\theta(\phi)=q+\alpha$ we must have that $\theta(x)=q$ which means that $x^{-1}y$ exists and is proper.

The above Theorem and Corollary speak about a particular solution. Suppose that we are interested in seeing whether there exists any acceptable ($x^{-1}y$ exists and is proper) solution to the equation $xd_p+yn_p=\phi$. We give an answer for the SISO case.

Let u,v be such that $ud_p+vn_p=1$. Then the general solution is given by

$$x = \phi u - nn_p$$
$$y = \phi v + nd_p \quad \text{for polynomial } n.$$

Let $\bar{x}$, $\bar{y}$ be the particular solution we obtain when $-\bar{n}$ is the quotient and $\bar{y}$ the remainder of the division $d_p | \phi v$.

Corollary 2.11. The equation $xd_p + yn_p = \phi$ will have an acceptable solution iff $\bar{x},\bar{y}$ is an acceptable solution.

proof:

If $\bar{x},\bar{y}$ is an acceptable solution then $xd_p + yn_p = \phi$ does have an acceptable solution.

Assume that x,y is an acceptable solution. From Corollary 2.1 this means that $P_0(s^{-q}y)$ is constant. Either $q \leq n-1$ or $q > n-1$.

If $q \leq n-1$, since the division ϕv by d_p gives a unique remainder of degree less than n-1 we must have $x = \bar{x}$, $y = \bar{y}$ and $\bar{x},\bar{y}$ is an acceptable solution.

If $q > n-1$, then $\theta(\bar{y}) \leq n-1$ and clearly $P_0(s^{-q}\bar{y})$ is a constant which means (Corollary 2.10) that $\bar{x},\bar{y}$ is an acceptable solution. This completes the proof.

Suppose now that we turn our attention to the Multivariable case and formulate methods that make a solution X,Y of $XD_{RP} + YN_{RP} = \Phi$ an acceptable one. We propose to do this by using Theorem 2.9 and by exploiting (as we did in the SISO case) the structure of the general solution

$$X = \Phi U - NN_{LP}$$
$$Y = \Phi V + ND_{LP}$$

where $UD_{RP} + VN_{RP} = I$.

Theorem 2.9 says that we just need to concentrate on Y and pick such a Φ so that condition 2.6 is satisfied. We now proceed to show two ways in which polynomial solutions are constructed which are"candidates" for acceptable solutions.

Method 1.

We use the SISO case as a basis. There we picked $\bar{y}$ as the remainder of the division $d_p | \phi v$.

Let $P = A_{LP}^{-1} B_{LP}$ be any left representation of P. Since A_{LP} is invertible we know that there exists a unimodular matrix M such that $\bar{A}_{LP} = MA_{LP}$ is row proper. Let β_1 be the largest row degree of $\bar{A}_{LP}$. We can then find a diagonal matrix Δ such that $A'_{LP} = \Delta \bar{A}_{LP}$ and all row degrees of A'_{LP} are β_1. Clearly A'_{LP} is regular. We can therefore assume that we pick $P = A_{LP}^{-1} B_{LP}$ such that A_{LP} is regular and $A_{LP} = A_{\beta_1} s^{\beta_1} + \ldots + A_0$.

From [15] we then have that right division by A_{LP} is possible. We then pick -N and Y as the right quotient and right remainder respective of the division $A_{LP} | \Phi V$. This is

$$\Phi V = -NA_{LP} + Y.$$

We then have that $\theta(Y) < \beta_1$.

Method 2.

Let $P = A_{LP}^{-1} B_{LP}$ in which A_{LP} is upper-triangular. Let $Q = \Phi V = (q_{ij})$. we pick N and Y by columns in the following manner. Let γ_i be the degrees of the diagonal elements c_{ii} of A_{LP}, $\gamma = \max (\gamma_1, \gamma_2, \ldots, \gamma_m)$. Let $-n_{i1}, y_{i1}$ be the quotient and remainder respectively of the divisions $c_{11} | q_{i1}$ $1 \leq i \leq \ell$. Let $-n_{i2}, y_{i2}$ be the quotient and remainder respectively of the divisions $c_{22} | q_{i2} + n_{i1} c_{12}$ $1 \leq i \leq \ell$. We continue in a similar fashion with the rest of the columns, and construct a Y such that $\theta(Y) < \gamma$.

It is interesting to note that if ΦV is some upper-triangular matrix then the Y chosen in this manner will have the property:

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_\ell \end{bmatrix} \qquad A_{LP} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1m} \\ 0 & c_{22} & c_{23} & \cdots & c_{2m} \\ & & & \vdots & \\ & 0 & & & \\ & & & & c_{mm} \end{bmatrix}$$

with y_i being the ith row of Y , $\theta(c_{ii}) = \gamma_i$ and

$$\theta(y_1) \leq \max(\gamma_1-1, \gamma_2-1, \ \ldots \ \gamma_m-1)$$

$$\theta(y_2) \leq \max(\gamma_2-1, \gamma_3-1, \ \ldots \ \gamma_m-1)$$

$$\vdots$$

$$\theta(y\ \ ) \leq \max(\gamma_\ell-1, \ \ldots \ \gamma_m-1).$$

Suppose that the γ_i's are a decreassing sequence. Using this method for constructing a Y may produce lower row degrees. This in turn will imply that a lower order compensator will be constructed.

In the SISO case we have seen (Corollary 2.11) that a solution x,y to $xd_p + yn_p = \phi$ is an acceptable solution iff $\bar{x},\bar{y}$ is an acceptable solution.A generalization of this result to the MIMO case is very desirable.

Corollary 2.12. Let P be a strictly proper transfer function with observability indecies equal to μ. Then there exists a left coprime representation $D_{LP}^{-1}N_{LP}$ with $D_{LP} = D_\mu s^\mu + \ \ldots \ + D_0$ and $\det D_\mu \neq 0$. Let $\bar{X},\bar{Y}$ be the polynomial solution obtained using method 1 and D_{LP}. Let Φ be a diagonal matrix with $\theta(\phi_i) = \alpha_i + \gamma$, where α_i are the controllability indecies.

Then $XD_{RP}' + YN_{RP} = \Phi$ has an acceptable solution X,Y iff $\overline{X},\overline{Y}$ is an acceptable solution.

<u>proof:</u>

If $\overline{X},\overline{Y}$ is an acceptable solution then $\overline{X}D_{RP} + \overline{Y}N_{RP} = \Phi$ has an acceptable solution.

Suppose that X,Y is an acceptable solution. From Lemma 2.7 and the proof of Theorem 2.9 (necessity) we have that $\theta(Y) \leq \gamma$ and $P_0(\mathrm{diag}(s^{-\gamma})Y)$ is a constant. We either have $\gamma \leq \mu-1$ or $\gamma > \mu-1$. If $\gamma \leq \mu-1$, since the division gives a unique remainder of degree less than μ we must have $X = \overline{X}$, $Y = \overline{Y}$ and $\overline{X},\overline{Y}$ is an acceptable solution.

If $\gamma > \mu-1$, then clearly $P_0(\mathrm{diag}(s^{-\gamma})\overline{Y})$ is a constant and $\overline{X},\overline{Y}$ is an acceptable solution.

2.4 Rosenbrock's State space Theorem.

Let $\dot{x} = Ax + Bu$ be a linear system with (A,B) a controllable pair. Suppose that the output of the system is the state (C=I). The input-output transfer function P is given by $P = (sI-A)^{-1}B$. Since (A,B) is a controllable pair then sI-A and B are left coprime. Let $D_{LP} = (sI-A)$, $N_{LP} = B$. If we now introduce state (output) feedback $u = Cy + v$, then the closed loop transfer function G can be written as

$$y = P(I+CP)^{-1}v$$
$$= (I+PC)^{-1}Pv$$

Let $G = P(I+CP)^{-1}$, $n = \theta(\det(sI-A))$, A (nxn) and B (nxℓ).

<u>Theorem 2.13 [24].</u> Let (A,B) be as above. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell \geq 0$ be the controllability indecies of P. Let ϕ_i be given polynomials such that $\phi_i | \phi_{i-1}$ with $\sum_{i-1}^{\ell} \theta(\phi_i) = n$.

Then there exists a constant C such that the invariant polynomials of G are the ϕ_i iff

$$\sum_{i-1}^{k} \theta(\phi_i) \geq \sum_{i-1}^{k} \lambda_i \qquad k = 1,2,\ldots \ell. \qquad (2.7)$$

with equality at k=ℓ.

We now prove the theorem using the results from section 2.3.

<u>proof:</u>

$$\text{Let} \quad \overline{\Phi} = \begin{bmatrix} \phi_1 & & & 0 \\ & \phi_2 & & \\ & 0 & \ddots & \\ & & & \phi_\ell \end{bmatrix}.$$

Suppose that $P = N_{RP}D_{RP}^{-1}$ is a right coprime representation of P with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell \geq 0$ being the column degrees of $\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix}$.

If $C = X^{-1}Y$ then $G = N_{RP}(XD_{RP} + YN_{RP})^{-1}$.

(sufficiency)

Suppose that condition 2.7 is satisfied. We will construct a constant compensator $C = X^{-1}Y$ such that

i) $XD_{RP} + YN_{RP} = \Phi'$ where $\Phi' \sim \overline{\Phi}$ and $X = I$.

ii) N_{RP} and Φ' right coprime.

Since $G = N_{RP}\Phi'^{-1}$ we will have that the ϕ_i are the invariant factors of G.

i) Using Lemma 2.6 with $\phi=\phi_1\phi_2 \ldots \phi_\ell$, $\beta_i=0$ and $\alpha_i=\lambda_i$, we can construct a matrix Φ such that $\det\Phi=\omega\phi$ and $P_0(\ \mathrm{diag}(s^{-\alpha_i}))=J$. From the proof of Lemma 2.6 we also see that $\overline{\Phi}\sim\Phi$.

We now show that there exists a constant acceptable solution to $XD_{RP} + YN_{RP} = \Phi$. Let $UD_{RP} + VN_{RP} = I$. Using method 1 we construct a Y, as the remainder of the right division $D_{LP}|\Phi V$. This means that $\theta(Y)=0$. From Theorem 2.9 we have that $P_0([Y\ \Phi \mathrm{diag}(s^{-\alpha_i})])$ is a constant and therefore that $X^{-1}Y$ exists and is proper. We also see that X is a constant.

Define $\overline{X}=I$, $C=\overline{Y}=X^{-1}Y$ $\Phi'=X^{-1}$. Then

$$D_{RP}+\overline{Y}N_{RP}=\Phi'$$

ii) Suppose that Δ is a g.c.r.d. of N_{RP}, Φ'.

$$N_{RP}=\overline{N}_{RP}\cdot\Delta \quad , \quad D_{RP}+\overline{Y}N_{RP}=\Phi_2\cdot\Delta$$

$\Rightarrow$ $D_{RP}=(\Phi_2-\overline{Y}N_{RP})\cdot\Delta$ which is a contradiction since N_{RP},D_{RP} are right coprime.

(Necessity)

Suppose that a constant C exists such that the invariant factors of G are the ϕ_i. Show that condition (2.7) is satisfied.

Now $G= N_{RP}(D_{RP}+CN_{RP})^{-1}$. Let $D_{RP}+CN_{RP}=\Psi$. We must have that N_{RP},Ψ are right coprime and that therefore $\Psi\sim\overline{\Phi}$.

Now

$$[I\ C]\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix} = \Psi$$

$$\Rightarrow \quad P_0\left([I\ C]\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix} \operatorname{diag}(s^{-\lambda_i})\right) = P_0(\Psi \operatorname{diag}(s^{-\lambda_i}))$$

$$\Rightarrow \quad [I\ C]\, P_o\left(\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix} \operatorname{diag}(^{-\lambda_i})\right) = P_0(\Psi \operatorname{diag}(s^{-\lambda_i}))$$

Let $P_0\left(\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix} \operatorname{diag}(s^{-\lambda_i})\right) = \begin{bmatrix} H \\ 0 \end{bmatrix}$ $\quad \det H \neq 0$

then $P_0(\Psi \operatorname{diag}(s^{-\lambda_i})) = H$.

With $\Psi' = H^{-1}\Psi$, then $P_0(\Psi' \operatorname{diag}(s^{-\lambda_i})) = I$.

Now $\Psi'\sim\overline{\Phi}$ and $P_0(\Psi' \operatorname{diag}(s^{-\lambda_i})) = I$. This means that Ψ' has the pxp bottom right minor of degree $\sum_{i=\ell-p+1}^{\ell} \lambda_i$. The greatest common divisor of all pxp minors (which is $\phi_{\ell-p+1}\phi_{\ell-p+2}\cdots\phi_\ell$) has degree less than or equal to this

$$\sum_{i=\ell-p+1}^{\ell} \theta(\phi_i) \leqslant \sum_{i=\ell-p+1}^{\ell} \lambda_i$$

Since we have equality for $p=\ell$ we can rearrange the inequalities obtaining

$$\sum_{i=1}^{k} \theta(\phi_i) \geqslant \sum_{i=1}^{k} \lambda_i \qquad k=1,2,\cdots \ell$$

with equality at $k=\ell$.

Remark: The proof presented here is much clearer than the comment concerning the proof found in [26]. The question of coprimeness of (N_{RP}, Φ) and $(\bar{X},\Phi)$ which ensures that $\Phi \sim \Psi$ is completely ignored in [26]. If C is a constant matrix this is automatically assured.

2.5 The Characteristic Polynomial Problem

We are now in a position to give an answer to the problem stated in section 2.1.

<u>Theorem 2.14</u> Let P be an $m\times\ell$ strictly proper plant and $N_{RP}D_{RP}^{-1}$ a right coprime representation which has the property that $\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix}$ is column proper with column degrees $\alpha_1 \geq \alpha_2 \geq \cdots \alpha_\ell \geq 0$.
Let $A_{LP}^{-1}B_{LP}$ be a left representation of P such that $\beta_1 \geq \beta_2 \geq \cdots \beta_m \geq 0$ are the row degrees of $[A_{LP}, B_{LP}]$, with A_{LP} regular. Let ϕ be a polynomial of degree $t = \sum_{i=1}^{\ell} \alpha_i + \ell(\beta_1 - 1)$.
Then there exists a proper compensator C such that the characteristic polynomial χ of the closed loop system is given by

$$\chi = \frac{\phi}{q_{XY}}$$

where q_{XY} is some polynomial with $0 \leq \theta(q_{XY}) \leq \ell(\beta_1 - 1)$.

<u>proof</u>:

Using Lemma 2.6 construct a matrix Φ which satisfies $P_0(\mathrm{diag}(s^{-(\beta_1-1)})\Phi\ \mathrm{diag}(s^{-\alpha_i})) = J$. If we now use method 1 for constructing a Y we have

$$\theta(Y) \leq \beta_1 - 1.$$

This implies that

$$P_0(\mathrm{diag}(s^{-(\beta_1-1)})[Y\ \Phi\mathrm{diag}(s^{-\alpha_i})]) \quad \text{is a constant.}$$

By Theorem 2.9 we then have that $X^{-1}Y$ exists and is proper and $\det \Phi = \alpha \cdot \phi$

This means that $\chi = \frac{\phi}{q_{XY}}$

If $X = \Delta\bar{X}$ and $Y=\Delta\bar{Y}$ with Δ a g.c.l.d., then $\Phi=\Delta\bar{\Phi}$, and $\bar{X}D_{RP}+\bar{Y}N_{RP}=\bar{\Phi}$. This means that

$$q_{XY}= \det \Delta$$

Now $P_0(\mathrm{diag}(s^{-(\beta_1-1)}X))$ is a constant (seen from the proof of Theorem 2.9) This means that

$$\theta(\det X)\leq \ell(\beta_1-1)$$

$$=> \quad \theta(\det \Delta)\leq \ell(\beta_1-1)$$

$$=> \quad 0\ \theta(q_{XY})\leq \ell(\beta_1-1)$$

As we can see if the acceptable solution (X,Y) so constructed has the additional property that X,Y are left coprime then q_{XY} is a constant we have $\chi=\alpha\cdot\phi$.

Remark We can easily see how the above theorem can be used when one wants to stabilize a strictly proper plant. If ϕ is taken to have all its zeros in the left hand plane then we are assured that the closed loop characteristic polynomial will have all its zeros in the left hand plane since $\chi|\phi$.

Remark It is quite reasonable to expect that if ϕ is a polynomial of degree $t=\sum_{i=1}^{\ell}\alpha_i+\ell(\beta_1-1)$ a proper compensator can be constructed such that $\chi|\phi$. By using Method 1 for constructing polynomial solutions to an equation $XD_{RP}+YN_{RP}=\Phi$ (Φ arbitrary) we have that a Y can be constructed with $\theta(Y)\leq\beta_1-1$. To use the sufficient condition of Theorem 2.9 we also need to construct a Φ with $q=\theta(\det\Phi)-\theta(\det D_{RP})=\ell(\beta_1-1)$ and $\det\Phi=\alpha\phi$ such that $P_0(\mathrm{diag}(s^{-(\beta_1-1)})\Phi\,\mathrm{diag}(s^{-\alpha_i}))$ is a constant. Condition 2.6 is then satisfied and $X^{-1}Y$ exists and is proper. But this is guaranteed by Lemma 2.6.

Remark Another interesting point is to investigate how small can t the degree of ϕ be, so that a proper compensator C will exist giving the relation $\chi = \frac{\phi}{q_{XY}}$. What we are able to say for this is as follows.

Since we have required that $P = N_{RP} D_{RP}^{-1}$ be a minimal right coprime representation, this means that the column degrees are the controllability indecies λ_i of P. We can also assure that we pick a left representation $A_{LP}^{-1} B_{LP}$ such that the largest row degree of $[A_{LP}, B_{LP}]$ is the largest observability index μ_1 of P. We can therefore let

$$t' = \sum_{i=1}^{\ell} \lambda_i + \ell(\mu_1 - 1)$$

For the SISO case t' becomes $t' = 2n-1$ $(n = \theta(d_p))$ and we have

Corollary 2.15 Let P be a strictly proper rational function and $n_p d_p^{-1}$ a (minimal) coprime representation. Let ϕ be a polynomial of degree $t' = 2n-1$.

Then there exists a proper compensator C such that the characteristic polynomial χ of the closed loop system is given by

$$\chi = \frac{\phi}{q_{xy}}.$$

where q_{xy} is some polynomial, $0 \leq \theta(q_{xy}) \leq n-1$.

We have shown in the SISO case that if ϕ is a polynomial of degree greater than or equal to 2n-1 there always exists an acceptable solution to the equation $x d_p + y n_p = \phi$. We have also said that even for ϕ such that $\theta(\phi) < 2n-1$ there may exist an acceptable solution. It would be quite interesting to see in the event that an acceptable solution does exist whether it is unique. If not then characterize all acceptable solutions. The idea behind such an investigation is to see whether any freedom afforded by a multitude of solutions can be used to attain other design objectives. The next two Lemmata address this question.

Remark Let ϕ be a polynomial with $\theta(\phi)=t\leq 2n-1$. If an acceptable solution x,y to $xd_p+yn_p=\phi$ exists then it is unique.

proof

Assume that x_1,y_1 is an acceptable solution to $xd_p+yn_p=\phi$. This means that

$$x_1d_p+y_1n_p=\phi \quad \text{and } \theta(x_1)=t-n \qquad \theta(y_1)\leq t-n$$

Suppose that x_2,y_2 is an acceptable solution. Then

$$x_2d_p+y_2n_p=\phi \quad \text{with } \theta(x_2)=t-n \qquad \theta(y_2)\leq t-n$$

$$\Rightarrow \quad \underbrace{(x_1-x_2)}_{\bar{x}}d_p+\underbrace{(y_1-y_2)}_{\bar{y}}n_p = 0$$

$\Rightarrow$ $\bar{x}$, $\bar{y}$ solutions to homogeneous equation

$\Rightarrow$ $\bar{x}=mn_p$ $\bar{y}=m\cdot d_p$

if $m\neq 0$ $\Rightarrow$ $\theta(\bar{y})\geq n$. But this cannot happen since

$\theta(\bar{y})\leq t-n\leq 2n-1-n=n-1$

therefore $x_1=x_2$, $y_1=y_2$.

Lemma 2.16 Let ϕ be a polynomial with $\theta(\phi)=2n-1+k$, $k\geq 1$. Let x_1,y_1 be an acceptable solution of $xd_p+yn_p=\phi$. All accetpable solutions are of the form

$$x_2 = x_1 - mn_p$$
$$y_2 = y_1 + md_p$$

where $\theta(m)\leq k-1$.

proof:

Since $\theta(\phi) = 2n-1+k$ an acceptable solution always exists.

$\Rightarrow$ $x_1 d_p + y_1 n_p = \phi, \theta(x_1) = 2n-1+k-n = n+k-1, \quad \theta(y_1) \leq n+k-1$.

I claim that any x_2, y_2 can be written as

$$x_2 = x_1 - mn_p$$
$$y_2 = y_1 + mn_p$$

with m polynomial such that $\theta(m) \leq k-1$, is an acceptable solution. Clearly $\theta(x_2) = n+k-1$, $\theta(y_2) \leq n+k-1$, which implies that $x_2^{-1} y_2$ exists and is proper.

On the other hand if $\theta(m) \geq k$ then no x_2, y_2 of the form $x_2 = x_1 - mn_p$, $y_2 = y_1 + md_p$ is an acceptable solution since $\theta(y_2) \geq n+k$ $\theta(x_2) \leq n+k-1$. Since we have taken into account all solutions the proof is complete.

The above Lemma shows that if we allow $\theta(\phi) \geq 2n-1+k$, $k \geq 1$ then we introduce k parameters (k degrees of freedom) which can be used to attain other design objectives such as to shape the sensitivity function or accomodate steady-state error specifications. As an example we now show how this freedom can be used in constructing stable proper compensators.

Example

Let $p = \frac{1}{s^2-1}$ and suppose that we want to construct a proper and stable compensator which makes the characteristic polynomial of the closed loop system equal to the stable polynomial $\phi = s^4+s^3+3s^2+s+1$.

The compensator $c_1 = \frac{5s^2+2s}{s^2+s-1} = \frac{y_1}{x_1}$ does meet the requirements but it is unstable, $x_1 d_p + y_1 n_p = \phi$.

From Lemma 2.16 we have that all acceptable solutions are of the form $x_2 = x_1 - mn_p$, $y_2 = y_1 + md_p$, where in this case m can be any constant. The polynomial $x_1 = s^2+s-1$ will become stable if its constant term becomes positive. Let m=-2. Therefore $x_2 = s^2+s+1$, $y_2 = 3s^2+2s+2$. Clearly $c_2 = x_2^{-1}y_2$ does meet all the requirements.

Motivated by the discussion in [5] suppose that we now consider the following problem.

We have a strictly proper plant P which we are allowed to modify by 1) using constant output feedback to obtain a system $\overline{P}$ and 2) by appropriately reducing the number of inputs of $\overline{P}$ to obtain $\overline{\overline{P}}$. Let ϕ be some polynomial. We then ask what are sufficient conditions for the existence of a proper compensator C (using output feedback) so that the closed loop system (comprised of $\overline{\overline{P}}$ and C) will have a characteristic polynomial χ which is a factor of ϕ.

The following two results are due to Kung [16].

<u>Theorem 2.17</u>. Let $D_{LP}^{-1} N_{LP}$ be a left coprime representation of P. Then $\overline{D}_{LP} = D_{LP} + N_{LP} K$ will be simple for almost all constant matrices K.

A polynomial matrix D is simple if it has only one non-unity invariant polynomial. We clearly have that N_{LP}, $\overline{D}_{LP}$ are left coprime. The effect of such a K on P is to modify P by applying constant output feedback K. Let $\overline{P} = (D_{LP}+N_{LP}K)^{-1}N_{LP}$ where $\overline{D}_{LP}$ is simple. The observability indecies remain the same as well as the order of the system.

Theorem 2.18. Let $\overline{D}_{LP}^{-1}N_{LP}$ be a left coprime representation of $\overline{P}$ where $\overline{D}_{LP}$ is simple. Then $\overline{D}_{LP}$ $N_{LP}c$ will be left coprime for almost all constant column vectors c.

Let c be such that $\overline{D}_{LP}$, $N_{LP}c$ are left coprime. This amounts to modifying $\overline{P}$ to obtain $\overline{\overline{P}}$, $\overline{\overline{P}} = \overline{P}\,c$. It is easy to see that $\overline{\overline{P}}$ is an mxl strictly proper system which has the same observability indecies as P and with only one controllability index n (the order of P).

We now apply Theorem 2.14.

Corollary 2.19. Let $\overline{\overline{P}}$ be the mxl strictly proper plant constructed above and $N_{RP}d_{RP}^{-1}$ a right coprime representation. The column degree of $\begin{bmatrix} d_{RP} \\ N_{RP} \end{bmatrix}$ is n. Let $A_{LP}^{-1}B_{LP}$ be a left representation of $\overline{\overline{P}}$ such that $\beta_1=\beta_2=\cdots\beta_m \geqslant 0$ are the row degrees of $[A_{LP}^{-1}B_{LP}]$ (A_{LP} regular) where $\beta_1=\mu_1$ is the largest observability index of P. Let ϕ be a polynomial of degree $\overline{t}=n+\beta_1-1$.

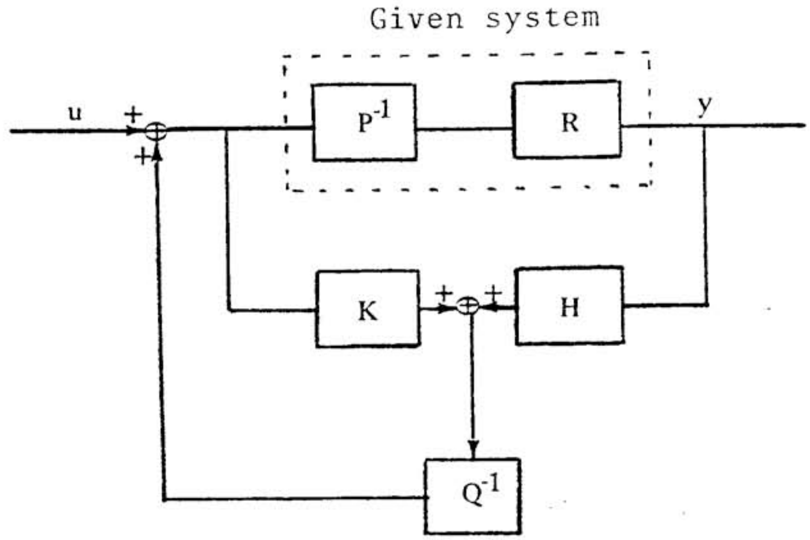
Then there exists a proper compensator C such that the characteristic polynomial χ of the closed loop system is given by

$$\chi = \frac{\phi}{q_{xy}}$$

where q_{xy} is some polynomial $0 \leq \theta(q_{xy}) \leq \beta_1 - 1$.

Remark: The difference with [5] is that in some cases, we may be able to add lesser order dynamics to accomplish a design objective.

The compensator technique presented in Wolovich [33] is as foolows.



The closed loop transfer function y(s)=G(s)u(s) has the form

$$G = R(s)[P(s)-F(s)]^{-1} Q^{-1}(s)\ Q(s)$$
$$= R(s)P_F^{-1}(s)\ ,$$

where F(s) arbitrary and Q(s) is stable cancellable portion. This along with the fact that the design uses input dynamics are among the main differences between the technique we follow and that of Wolovich. This idea of pole-zero cancellation is very reminiscent of the fact that in observer theory as well the "added dynamics" associated with the state estimator do not appear in the closed loop input-output transfer function.

In that situation:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}(t)-\dot{\tilde{x}}(t) \end{bmatrix} = \begin{bmatrix} A+BF & BF \\ 0 & A+KC \end{bmatrix} \begin{bmatrix} x(t) \\ x(t)-\tilde{x}(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} V(t)$$

and

$$G = C(sI-A-BF)^{-1}\ B.$$

2.6 The Invariant Factor Problem

In section 2.1 we defined two versions of the invariant factor problem. In this section we will be interested with the first version and will be giving sufficient conditions. Discussion on the second version will be differed to the next section when the notion of genericity is defined.

Let P be an $m\times\ell$ strictly proper plant and $\overline{\Phi}$ an $\ell\times\ell$ diagonal matrix in Smith form. If $P=N_{RP}D_{RP}^{-1}$ is then some right coprime representation of P for which there exists a polynomial solution X,Y to the equation of the form

$$XD_{RP} + YN_{RP} = \overline{\Phi}$$

with 1) Φ equivalent to $\overline{\Phi}$,

2) $X^{-1}Y$ existing and proper,

3) N_{RP}, Φ right coprime

4) Φ , X left coprime

then $C=X^{-1}Y$ is a proper compensator which makes the invariant factor matrix Ψ of G, equal to $\overline{\Phi}$. If conditions 3) and 4) are not met then this is not so. The following Lemma will help us see what happens in this case.

Lemma 2.22 [9] Let A_1, A_2 be $\ell\times\ell$ non-singular polynomial matrices with Smith normal forms $\{\gamma_1^{(1)},\cdots\gamma_\ell^{(1)}\}$, $\{\gamma_1^{(2)},\cdots\gamma_\ell^{(2)}\}$, $\gamma_{i-1}^{(j)} \mid \gamma_i^{(j)}$, $1\leq i\leq \ell$, $j=1,2$ respectively. If $A=A_1A_2$ has the Smith form $\{\gamma_1,\gamma_2,\cdots\gamma_\ell\}$ then $\gamma_i^{(1)}$, $\gamma_i^{(2)}$ each divide γ_i $1\leq i\leq\ell$.

Theorem 2.23

Let P be an $m\times\ell$ strictly proper plant and $\overline{\Phi} = \begin{bmatrix} \phi_1 & & 0 \\ & \ddots & \\ 0 & & \phi_\ell \end{bmatrix}$

be a matrix in Smith form. Let $P=N_{RP}D_{RP}^{-1}$ with $\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix}$ column proper and column degrees $\alpha_1 \geq \alpha_2 \geq \cdots \alpha_\ell \geq 0$. Let $A_{LP}^{-1}B_{LP}$ be a left representation of P with the row degrees of $[A_{LP}B_{LP}]$ being $\beta_1 \geq \beta_2 \geq \cdots \beta_m \geq 0$ and A_{LP} regular. Denote by Φ the matrix constructed using Lemma 2.6 with $\Phi_1 = \begin{bmatrix} \phi_1 & & 0 \\ & \ddots & \\ 0 & & \phi_\ell \end{bmatrix}$ such that

$$P_0(\text{diag}(s^{-(\beta_1-1)})\Phi\ \text{diag}(s^{-\alpha_i})\ ) = J.$$

If $\sum_{i=1}^{k} \theta(\phi_i) \geq \sum_{i=1}^{k} (\alpha_i+\beta_1-1)$ $k=1,2,\cdots\ell$ with equality at $k=\ell$, then there exists a proper compensator $C=X^{-1}Y$ with $XD_{RP}+YN_{RP}=\Phi$ and such that if Ψ is the closed loop invariant factor matrix then $\psi_i | \phi_i$.

<u>proof</u>:

We already have that $P_0(\text{diag}(s^{-(\beta_1-1)})\Phi\ \text{diag}(s^{-\alpha_i})) = J$. Now using method 1 for constructing Y we have that $\theta(Y)\ \beta_1-1$. But this means that

$$P_0(\text{diag}(s^{-(\beta_1-1)})[Y, \Phi\ \text{diag}(s^{-\alpha_i})]) \text{ is constant.}$$

By Theorem 2.9 we have that $X^{-1}Y$ exists and is proper with $XD_{RP}+YN_{RP}=\Phi$ and $\Phi \sim \overline{\Phi}$.

Now $G = N_{RP}\ \Phi^{-1}\ X$. Let K be the g.r.c.d. of N_{RP} and Φ. $(N_{RP}=\overline{N}_{RP}K$, $\Phi = \Psi_1 \cdot k$

$$\Rightarrow \quad G = \overline{N}_{RP}K(\Phi_1 \cdot K)^{-1}X$$

Let L be the g.c.l.d. of Φ_1 and X $(\Phi_1=L\Phi_2 \quad X=L\overline{X}$)

$$= \quad G = \overline{N}_{RP}K(L\ \Phi_2\ K)^{-1}\ L\overline{X}$$

$$= \overline{N}_{RP}\ \Phi_2^{-1}\ \overline{X}$$

where $\overline{N}_{RP},\Phi_2$ are right coprime and $\overline{X},\Phi_2$ are left coprime. If Ψ is the closed loop invariant factor matrix we have that Ψ is the Smith form of Φ_2. Now $\Phi=L\Phi_2K$. From the previous Lemma we have that

$$\psi_i \mid \phi_i \quad .$$

In the event that N_{RP},Φ are right coprime and X,Φ are left coprime then we do have that $\psi_i = \phi_i$.

Remark: We do have that $\alpha_i=\lambda_i$ where α_i are the controllability indecies of P. In the event that $\beta_i=\mu_i$ (μ_i the observability indecies) the condition in Theorem 2.23 becomes

$$\sum_{i-1}^{k} \theta(\phi_i) \geq \sum_{i-1}^{k} (\lambda_i+\mu_1-1) \qquad k=1,2,\ldots\ell \text{ with equality at } k=\ell.$$

In Theorem 2.23 we find that the degrees of the ϕ_i must satisfy $\sum_{i=1}^{k} \theta(\phi_i) \geq \sum_{i=1}^{k} \alpha_i+\mu_1-1$. It would be interesting to see whether this can be replaced by the more symmetrical one $\sum_{i=1}^{k} \theta(\phi_i) \geq \sum_{i=1}^{k} \alpha_i+\mu_i-1$. We have not been able to show that this second condition is sufficient in the general case. In the "Generic" case there is no difference between the two. At any rate here is a special case in which this second condition can be used.

Corollary 2.24. Let $P = \frac{n_{ij}}{d_{ij}}$ be an upper triangular with non-zero diagonal elements($n_{ii} \neq 0 \quad 1\leq i\leq \ell$) with $d_{ij}|d_{ii}$ $1\leq j\leq \ell$, $1\leq i \leq \ell$ and $d_{ij}|d_{jj}$ $1\leq j\leq \ell$, $1\leq i\leq j$, (n_{ii}, d_{ii} are coprime), and $\theta(d_{ii}) \geq \theta(d_{i+1i+1})$. Let $\lambda_1\geq\lambda_2\geq \cdots\lambda_\ell\geq 0$, $\mu_1\geq\mu_2\geq\cdots\mu_m\geq 0$ be the controllability and observability indecies of P.

Let $\overline{\Phi} = \begin{bmatrix} \phi_1 & & 0 \\ & \ddots & \\ 0 & & \phi_\ell \end{bmatrix}$ be a matrix in Smith form.

If $\theta(\phi_i) = \lambda_i + \mu_i - 1 \qquad 1 \leq i \leq \ell,$

then there exists a proper compensator C with $XD_{RP} + YN_{RP} = \Phi$ and such that if Ψ is the closed loop invariant factor matrix then $\psi_i | \phi_i$.

proof:

$$P = \underbrace{\begin{bmatrix} d_{11} & & & & & \\ & d_{22} & & & & \\ & & \ddots & & & \\ & & & d_{\ell\ell} & & \\ & & & & 1 & \\ & & & & & \ddots \; 1 \end{bmatrix}^{-1}}_{D_{LP}} \cdot \underbrace{\begin{bmatrix} n_{11} & \bar{n}_{12} & \cdots & \bar{n}_{1\ell} \\ 0 & n_{22} & & \vdots \\ & & \ddots & n_{\ell\ell} \\ & & 0 & \end{bmatrix}}_{N_{LP}}$$

$$= \underbrace{\begin{bmatrix} n_{11} & \bar{\bar{n}}_{12} & & \bar{\bar{n}}_{1\ell} \\ & n_{22} & & \vdots \\ 0 & & \ddots & n_{\ell\ell} \\ & 0 & & \end{bmatrix}}_{N_{RP}} \cdot \underbrace{\begin{bmatrix} d_{11} & & 0 \\ & \ddots & \\ 0 & & d_{\ell\ell} \end{bmatrix}}_{D_{RP}}^{-1}$$

Since $N_{RP}D_{RP}^{-1}$, $D_{LP}^{-1}N_{LP}$ are minimal $\lambda_i = \theta(d_{ii})$ and $\mu_i = \theta(d_{ii})$ $1 \leq i \leq \ell$ with $\mu_{\ell+1} = \mu_{\ell+2} = \cdots \mu_m = 0$.

We already have that $P_0(\text{diag}(s^{-(\mu_1 - 1)})\Phi \, \text{diag}(s^{-\lambda_i})) = I$.

We also have that there exist upper triangular U,V such that $UD_{RP} + VN_{RP} = I$.

Using method 2 for constructing Y we have that the degree of the i^{th} column of Y is less than $\mu_i - 1$. Since Y will be upper triangular and of the form

$$\begin{bmatrix} y_{11} & & y_{1\ell} & \\ 0 & \ddots & y_{\ell\ell} & 0 \end{bmatrix}$$

we will have that the degree of the i^{th} row of Y will be less than or equal to $\mu_i - 1$. This means that

$$P_0(\mathrm{diag}(s^{-(\mu_1-1)})[Y \quad \Phi\,\mathrm{diag}(s^{-\alpha_i})) \text{ is a constant.}$$

By Theorem 2.9 we have that $X^{-1}Y$ exists and is proper.

Now $G = N_{RP}\Phi^{-1}X$. If K is the g.c.r.d. of N_{RP} and Φ. ($N_{RP}=\overline{N}_{RP}K \quad \Phi=\Phi_1\cdot K$). Let L be the g.c.l.d. of Φ_1 and X ($\Phi_1=L\Phi_2 \quad X=L\overline{X}$).

$$\Rightarrow \quad G = \overline{N}_{RP}K(L\Phi_2K)^{-1}L\overline{X}$$
$$= \overline{N}_{RP}\Phi_2^{-1}\overline{X}$$

where $\overline{N}_{RP}$, Φ_2 are right coprime and $\overline{X}$, Φ_2 are left coprime. If Ψ is the closed loop invariant factor matrix we have Ψ being the Smith form of Φ_2. Now $\Phi = L\Phi_2K$. From Lemma 2.22 we have that $\psi_i|\phi_i$. In the event that L,K are unimodular then $\psi_i=\alpha_i\phi_i$, α_i constants.

<u>Remark.</u> In the beginning of this section we list four conditions that guarantee the existence of a proper compensator which makes the closed loop invariant factor matrix equal to the given $\overline{\Phi}$. Here is an example showing that these conditions are not necessary.

Let $p = \dfrac{s+1}{s^2-4} \qquad \psi = s^2-3.$

We have that the proper compensator $c = \dfrac{1}{s+1}$ makes the closed loop invariant polynomial equal to s^2-3 since

$$G = n_p(a_cd_p+b_cn_p)^{-1}a_c = \frac{s+1}{s^2-3}\,.$$

But for no coprime representation of p and no ϕ equivalent to ψ does there exist an acceptable solution to an equation of the form

$$xd_p + yn_p = \phi$$

(Use Corollary 2.11)

Any coprime representation of p can be written as $p=\dfrac{\alpha(s+1)}{\alpha(s^2-4)}$

$\alpha\neq 0$ a constant, and any polynomial ϕ equivalent to (s^2-3) is of the form $\phi=\beta(s^2-3)$ $\beta\neq 0$ a constant. So we ask whether an acceptable solution to any one of the set of equations

$$x\alpha(s^2-4)+y\alpha(s+1)=\beta(s^2-3)$$

exists.

From Corollary 2.11 we have that x,y is an acceptable solution iff $\overline{x},\overline{y}$ is an acceptable solution. Now u,v such that $ud_p+vn_p=1$ are

$$u=-\alpha(\tfrac{2}{3}s+1) \qquad v=\frac{1}{\alpha}(\tfrac{2}{3}s^2+\tfrac{1}{3}s-3)$$

and $\overline{y}=\frac{\beta}{\alpha}(\frac{1}{3}s+\frac{1}{3})$.

But $\theta(\overline{y})=1$ and $P_0([\overline{y}\quad (s^2-3)s^{-2}])$ is not a constant. Therefore for no $\alpha,\beta\neq 0$ does there exist an acceptable solution.

As we have repeatedly seen in the previous sections degree constraints on ϕ or the elements of Φ are not enough to ensure that the closed loop transfer function G will have the desired characteristics. Additional conditions like N_{RP},Φ being right coprime and X,Φ left coprime are required. It is very natural therefore to introduce the notion of Genericity. Roughly speaking given two arbitrary non-zero polynomials they "almost surely" are coprime. Said differently these two polynomials are generically coprime. We therefore hope that the notion of genericity will prove helpfull in two areas. On the one hand show that the sufficient conditions expressed will actually be enough generically. On the other assist in formulating necessary conditions as well.

2.7 Genericity

Let u be an ideal in the polynomial ring $R[x_1,x_2,\ldots x_q]$. The variety of u denoted by V(u), is the set of all points $x=(x_1,x_2,\ldots x_q)$, x_i in $\mathbb{C}$, such that $f(x) = 0$ for all f in u. It can be shown [35], that if the closed sets in $\mathbb{C}^q$ are defined to be the varieties in $\mathbb{C}^q$ then $\mathbb{C}^q$ becomes a topological space. This topology on $\mathbb{C}^q$ is called the Zariski topology. Let R^q have the subspace topology. This means that E is closed in R^q iff $E = R^q \cap V$ where V is closed in $\mathbb{C}^q$. We will strictly be dealing with this subspace topology on R^q so we also call it Zariski.

Definition. A set $S \subset R^q$ is called generic if it contains a non-empty Zariski open set of R^q.

It would be instructive to explain what is the consequence of the above terminology. Let $p(x_1,x_2,\ldots,x_q)$ be a polynomial in $R[x_1,x_2,\ldots,x_q]$ and let p be the ideal in $R[x_1,x_2,\ldots,x_q]$ generated by p(x). The set $E = R^q \cap V(p)$ can either be R^q or not. Suppose that $E \neq R^q$. Then E has Lebesque measure zero. This we see from the fact that points x in R^q such that $p(x) = 0$ is a set of lower dimension in R^q. And more generally if $E = R^q \cap V \neq R^q$ where V is some variety, since $V \subset V(g)$ for some $g \neq 0$ we again have that the measure of E is zero. If we then show that a set $S \subset R^q$ contains a non-empty Zariski open set in R^q it means that $\bar{S} \subset R^q$ is contained in a Zariski closed set not equal to R^q. A generic set then contains " almost all" of R^q.

We now want to use the notion of genericity in order to give answers to the Characteristic Polynomial Problem, the Invariant Factor Problem and the Denominator Matrix Problem.

Generic Characteristic Polynomial Problem.

Defenition. An $m \times \ell$ strictly proper system P given by $P = N_{RP}D_{RP}^{-1}$ has the generic characteristic polynomial assignability property if the monic polynomials ϕ in R^{n+q} for which there exists a proper compensator C making the closed loop characteristic polynomial equal to ϕ, is a generic subset of R^{n+q}.

Remark. The fact that the definition speaks about a specific coprime representation of P is not restrictive because one can see that:

If P given by $N_{RP}D_{RP}^{-1}$, does have the generic characteristic polynomial assignability property then P given by any right coprime representation $\overline{N}_{RP}\overline{D}_{RP}^{-1}$ has the generic characteristic polynomial assignability property.

If P given by $N_{RP}D_{RP}^{-1}$ does not have this property then no right coprime representation of P exists for which P does have this property.

Theorem 2.25. Let $P = n_p d_p^{-1}$ be a single-input, single-output strictly proper system. A necessary and sufficient condition for generic characteristic polynomial assignability property is $q \geq n-1$.

proof:

(necessity). Assume that the set S of monic polynomials ϕ in R^{n+q} for which there exists a proper compensator for which ϕ is the closed loop characteristic polynomial is generic. Suppose that $q < n-1$. Show a contradiction.

Since $q < n-1$ and $C = x^{-1}y$ with x,y coprime and $xd_p + yn_p = \phi$ $\theta(x) = q$, we must have $x = \overline{x}$, $y = \overline{y}$ where $\overline{y}$ is the remainder of $d_p | \phi v$ and $-\bar{n}$ the quotient. We then have $\bar{x} = \phi u - \bar{n}n_p$. By Corollary 2.10 we have that $\theta(\bar{y})<n-1$. This means that at least $\bar{y}_{n-1}=0$ ($\bar{y} = \bar{y}_{n-1}s^{n-1}+ \ldots + \bar{y}_0$). Now $\bar{y}_{n-1}$ as a polynomial in R^{n+q} cannot be the zero polynomial as we can show:

we have $ud_p+vn_p= 1$, $\theta(v)\leq n-1$.

a) if $\theta(v) = n-1$, let $\bar{n}$ be such that $\theta(\bar{n}) = q$, $\bar{n}d_p$ monic. Then define ϕ by

$$\phi = \bar{n}d_p + 1$$

$$\phi v = \bar{n}vd_p + v.$$

We have a ϕ which is monic of degree $n+q$ which gives $\theta(\bar{y}) = n-1$ and $\bar{y}_{n-1} \neq 0$.

b) if $\theta(v)<n-1$, let g be some polynomial with $\theta(gv) = n-1$, and let $\bar{n}$, $\theta(\bar{n}) = q$ be such that $\bar{n}d_p$ is monic. Then define ϕ by

$$\phi = \bar{n}d_p + g$$

$$\phi v = \bar{n}vd_p + gv.$$

We again have a ϕ which is monic of degree $n+q$ which gives $\theta(\bar{y}) = n-1$, $\bar{y}_{n-1} \neq 0$.

Therefore $\bar{y}_{n-1}$ is not the zero polynomial. To require then that $\bar{y}_{n-1}= 0$ we must have that S is contained in a Zariski closed set. But this is a contradiction. Therefore $q \geq n-1$.

(Note: The above can also be shown using Proposition 2.33.)

(sufficiency). Assume now that $q \geq n-1$. Let $t=n+q$ and define

$$S = \left\{ (\phi_0,\phi_1, \ldots ,\phi_{t-1}) \text{ in } R^t \;\middle|\; \begin{array}{l}\text{For which there exists}\\ \text{an acceptable solution}\\ x,y \;\; xd_p+yn_p=\phi \;\; x,y \text{ coprime.}\end{array} \right\}$$

Since $\theta(\phi)=n+q\geq 2n-1$ then from Corollaries 2.10, 2.11 we have that

$$\bar{x} = \phi u - \bar{n} n_p$$

$$\bar{y} = \phi v + \bar{n} d_p$$

with $\bar{y}$, $-\bar{n}$ being the unique remainder and quotient respectively of the division $d_p | \phi v$, is an acceptable solution of

$$\bar{x} d_p + \bar{y} n_p = \phi .$$

This is true for every ϕ in R^t. Let

$$g = \text{Res}(\bar{x},\bar{y}) .$$

Since $\theta(\bar{x}) = q$ we must have that $\bar{x}_q \neq 0$. Then $\bar{x},\bar{y}$ will be coprime iff $g \neq 0$. Let

$$V_g = \{(\phi_0,\phi_1,\ldots,\phi_{t-1}) \text{ in } R^t \mid g(\phi_0,\phi_1,\ldots,\phi_{t-1})=0\} .$$

It is clear that $S \supseteq \bar{V}_g$. We now want to show that $\bar{V}_g \neq \emptyset$. We do this by constructing a ϕ for which $\bar{x},\bar{y}$ are coprime. Let f be a polynomial in s with $\theta(f) = q$, and fd_p monic. Define

$$\phi = fd_p + n_p v .$$

$$\phi v = fvd_p + (1-ud_p)$$

$$= (fv - u)d_p + 1 .$$

This means that fv - u is the quotient and 1 the remainder of $d_p | \phi v$. Since $\bar{y}$ for this particular ϕ is 1 then $\bar{x},\bar{y}$ must be coprime.

Therefore $S(\supseteq \bar{V}_g)$ contains a non-empty Zariski open set making it generic.

Theorem 2.26. Let $P = N_{RP} d_{RP}^{-1}$ be an mxl strictly proper system with d_{i1}, d_{j1} $i \neq j$ (below) being coprime. A sufficient condition for generic characteristic polynomial assignability (when ϕ is in R^{n+q}) is $q \geq \mu_1 - 1$ where μ_1 is the largest observability index of P.

proof:

Assume that $q \geq \mu_1 - 1$. Let $t = n+q$ and

$$P = \begin{bmatrix} \frac{n_{11}}{d_{11}} \\ \frac{n_{21}}{d_{21}} \\ \vdots \\ \frac{n_{m1}}{d_{m1}} \end{bmatrix} = \underbrace{\begin{bmatrix} d_{11} & & \\ & d_{21} & \\ & & \ddots \\ & & & d_{m1} \end{bmatrix}}_{D_{LP}}^{-1} \cdot \underbrace{\begin{bmatrix} n_{11} \\ n_{21} \\ \vdots \\ n_{m1} \end{bmatrix}}_{N_{LP}} = \underbrace{\begin{bmatrix} n'_{11} \\ n'_{21} \\ \vdots \\ n'_{m1} \end{bmatrix}}_{N_{RP}} \underbrace{d^{-1}}_{d_{RP}}, \quad d = d_{11} \cdots d_{m1}.$$

Since n_{i1}, d_{i1} are coprime, $\mu_1 = \max(\theta(d_{i1}))$.

Define

$$S = \left\{ (\phi_0, \phi_1, \ \ldots \ , \phi_{t-1}) \text{ in } R^t \ \middle| \ \begin{array}{l} \text{For which there exists an ac-} \\ \text{ceptable solution } x,Y \text{ to} \\ xd+YN_{RP}=\phi, \ x,Y \text{ left coprime.} \end{array} \right\}.$$

Since $\theta(\phi) = n+q \geq n+\mu-1$, then from Theorem 2.9 we have that $\overline{x}, \overline{Y}$ obtained by

$$\overline{Y} = [\ \overline{y}_{11}, \overline{y}_{12}, \ \ldots \ \overline{y}_{1m}]$$

where $\overline{y}_{1i}$ is the remainder of the division $d_{1i} | \phi v_{1i}$ and $-\overline{n}_{1i}$ the quotient,

$$\overline{x} = \phi u - [\overline{n}_{11}, \overline{n}_{12}, \ldots, \overline{n}_{1m}] \cdot \begin{bmatrix} n_{11} \\ n_{21} \\ \vdots \\ n_{m1} \end{bmatrix}$$

is an acceptable solution of $\overline{x}d + \overline{Y}N_{RP} = \phi$. This is true for every ϕ in R^t. We also have that $\theta(\overline{x}) = q$, which means that $\overline{x} \neq 0$.

Let

$$g_1 = \mathrm{Res}(\bar{x},\bar{y}_{11})$$
$$g_2 = \mathrm{Res}(\bar{x},\bar{y}_{12})$$
$$\vdots$$
$$g_m = \mathrm{Res}(\bar{x},\bar{y}_{1m}).$$

Then $\bar{x},\bar{y}_{1i}$ iff $g_i \neq 0$.

Let

$$V_{g_i} = \{(\phi_0,\phi_1, \ldots ,\phi_{t-1}) \text{ in } R^t \mid g_i(\phi_0,\phi_1, \ldots ,\phi_{t-1})=0\}.$$

It is clear that $S \supseteq \bar{V}_{g_1} \cup \bar{V}_{g_2} \cup \ldots \cup \bar{V}_{g_m}$ since if $\bar{x},\bar{y}_{1i}$ are coprime then $[\bar{x}\ \bar{y}_{11}\ \bar{y}_{12}\ \cdots\ \bar{y}_{1m}]$ has rank 1 for all s making $\bar{x},\bar{Y}$ left coprime. We now have to show that some $\bar{V}_{g_i}$ is non-empty. We have u,V such that

$$ud + [v_{11}\ \underbrace{v_{12}\ \cdots}_{V}\ v_{1m}] \cdot \begin{bmatrix} n'_{11} \\ n'_{21} \\ \vdots \\ n'_{m1} \end{bmatrix} = 1.$$

This means that at least one $v_{1i} \neq 0$. Suppose that it is v_{11}.
Let f be a polynomial in s with $\theta(f) = q$ and fd monic.

Define $\phi = fd + n_{11}$.

$$\begin{aligned} \phi v_{11} &= fv_{11}d + n_{11} \\ &= (fv_{11}d_{21}\cdots d_{m1})d_{11} + (1-ud-n'_{21}v_{12}-\cdots-n'_{m1}v_{1m}) \\ &= (fv_{11}d_{21}\cdots d_{m1}\underbrace{-ud_{21}\cdots d_{m1}-n''_{21}v_{12}-\cdots-n''_{m1}v_{1m}}_{-\bar{n}})d_{11} + \underbrace{1}_{\bar{y}_{11}} \end{aligned}$$

This means that $-\bar{n}$ is the quotient and $1 = \bar{y}_{11}$ the remainder of the division $d_{11} \mid \phi v_{11}$. Since $\bar{y}_{11} = 1$ for this particular ϕ then $\bar{x}$, $\bar{y}_{11}$ must be coprime. Therefore $S\ (\supseteq(\bar{V}_{g_1} \cup \bar{V}_{g_2} \cup \ldots \cup \bar{V}_{g_m}))$ contains a non-empty Zariski open set making S generic.

Remark. In the last Theorem we see that $q \geq \mu_1 - 1$ is not solely sufficient to guarantee that S is generic. We also had to impose the condition that d_{i1}, d_{j1} $i \neq j$, are coprime. This in effect means that we have disregarded " some" mx1 systems. We will see in what follows that this will be the case in most of the results that follow, when we leave the single-input single-output case. If we look at the results of [30] we will see that they are stated for "almost all" systems. In effect what we are trying to show is that our results hold for any fixed but arbitrary system. This explains the difficulty we are experiencing. We overcome it by requiring more conditions on P.

Remark. Using Proposition 2.33 we see that in the case that $\mu_1 = \mu_2 = \mu_3 = \dots = \mu_m = \mu$, then the condition $q \geq \mu - 1$ is also necessary. It is not surprising to see that this condition coincides with the result of [30]. As a matter of fact we have shown that $m\ell \geq n$, $\ell = 1$ is a sufficient condition for the generic characteristic polynomial assignability property for the mx1 systems.

The Generic Invariant Factor Problem.

The work in this section deals with the Single-Input Single-Output (SISO) case.

Definition. A SISO strictly proper plant $P = n_p d_p^{-1}$, has the generic invariant factor assignability property if the monic polynomials ϕ in R^{n+q} for which there exists a proper compensator C making the invariant factor of the closed loop system ψ, equal to ϕ, is a generic subset of R^{n+q}.

Theorem 2.27. Let $P = n_p d_p^{-1}$ be a SISO strictly proper system. A necessary and sufficient condition for generic invariant factor assignability is $q \geq n-1$.

proof:

(necessity). Assume that the set S of monic polynomials ϕ in R^{n+q} for which a proper compensator C exists making ψ equal to ϕ, is generic. Suppose then that $q<n-1$. Show a contradiction.

The set E of ϕ for which n_p, ϕ are coprime is generic. Now if n_p and ϕ are coprime and $C = x^{-1}y$ is a compensator given in a coprime representation we must have

$$G = \frac{n_p x}{xd_p + yn_p} = \frac{n_p x}{q} = \frac{g}{\psi} .$$

Therefore $q = a\psi$, $n_p x = ag$. Because of the coprimeness assumptions $a = 1$.

$$xd_p + yn_p = \phi = \psi .$$

From Corollary 2.11 we have that $\bar{x}d_p + \bar{y}n_p = \phi$ and $x = \bar{x}$ $y = \bar{y}$. From Corollary 2.10 $P_0(s^{-q}\bar{y})$ is a constant. Which implies that for all the ϕ in E we must have $\bar{y}_{n-1} = 0$ $(\bar{y} = \bar{y}_{n-1}s^{n-1} + \dots + \bar{y}_0)$. From the proof of Theorem 2.25

we know that $\bar{y}_{n-1}$ cannot be the zero polynomial and that the elements ϕ in R^{n+q} for which it is zero must be contained in a Zariski closed set. This is a contradiction. Therefore $q \geq n-1$.

(sufficiency). The proof proceeds in a manner similar to the proof of Theorem 2.22. If we define

$$Q = \{(\phi_0, \phi_1, \ldots, \phi_{t-1}) \text{ in } R^t \mid \begin{array}{l}\text{For which there exists a proper}\\ \text{compensator making the closed loop}\\ \text{invariant polynomial equal to } \phi\end{array}\},$$

we have $Q \supset S$ (S is defined as in the last Theorem). We already know that S is generic, then so is Q.

The Generic Denominator Matrix Problem.

Definition. An $m \times \ell$ strictly proper system P given by $P=N_{RP}D_{RP}^{-1}$ has the denominator matrix assignability property if the $\ell \times \ell$ matrices Φ considereded as elements in R^{n+q}, for which there exists an acceptable solution X,Y to $XD_{RP}+YN_{RP} = \Phi$ with N_{RP}, Φ right coprime and X,Φ left coprime, is a generic subset of R^{n+q}.

Remark. Contrary to the generic characteristic polynomial problem the above definition for other than the SISO case is sensitive to the specific choice of right coprime representation. That is if P given by $N_{RP}D_{RP}^{-1}$ does have the generic denominator matrix assignability property we are not sure that if it were given by $\bar{N}_{RP}\bar{D}_{RP}^{-1}$ it would have it. And if P given by $N_{RP}D_{RP}^{-1}$ did not have this property we cannot conclude that no $\bar{N}_{RP}\bar{D}_{RP}^{-1}$ exists for which it does have it.

Theorem 2.28. Let $P = n_p d_p^{-1}$ be a SISO strictly proper system. Let ϕ be a monic polynomial of $\theta(\phi) = n+q$ (ie ϕ is in R^{n+q}). A necessary and sufficient condition for generic denominator matrix assignability is $q \geq n-1$.

proof:

(necessity). Assume that the set S of polynomials ϕ in R^{n+q} for which there exists an acceptable solution x,y to $xd_p+yn_p=\phi$ with n_p,ϕ coprime and x,ϕ coprime is generic. Suppose then that $q<n-1$. Show a contradiction.

Since $q<n-1$ and x,y is an acceptable solution with x,y coprime, (since n_p,ϕ are coprime and x,ϕ are coprime we must have x,y coprime), we do have that $y = \bar{y}$ and $x = \bar{x}$.

This means that $\bar{x},\bar{y}$ is an acceptable solution to $\bar{x}d_p+\bar{y}n_p = \phi$ and by Corollary 2.10 we have that $\theta(\bar{y})<n-1$. This means that at least $\bar{y}_{n-1} = 0$, ($\bar{y} = \bar{y}_{n-1}s^{n-1}+ \dots +\bar{y}_0$). Now $\bar{y}_{n-1}$ as a polynomial in R^{n+q} cannot be the zero polynomial as we see from the following:

We have $ud_p + vn_p = 1$, $\theta(v) \leqslant n-1$.

a) If $\theta(v) = n-1$, let $\bar{n}$ be such that $\theta(\bar{n}) = q$ and $\bar{n}d_p$ is monic.
Then define ϕ by

$$\phi = \bar{n}d_p + 1$$
$$\phi v = \bar{n}vd_p + v.$$

We have a ϕ which is monic of degree n+q which gives $\theta(\bar{y})=n-1$ and $\bar{y}_{n-1} \neq 0$.

b) If $\theta(v)<n-1$, let g be some polynomial with $\theta(gv) = n-1$, and let $\bar{n}$, $\theta(\bar{n}) = q$, be such that $\bar{n}d_p$ is monic.
Then define ϕ by

$$\phi = \bar{n}d_p + g$$
$$\phi v = \bar{n}vd_p + gv,$$

and we again have a ϕ which is monic of degree n+q which gives $\theta(\bar{y})=n-1$, $\bar{y}_{n-1} \neq 0$.

Therefore $\bar{y}_{n-1}$ is not the zero polynomial. To require then that $\bar{y}_{n-1} = 0$ we must have that S is contained in a Zariski closed set. But this is a contradiction. Therefore $q \geq n-1$.

(sufficiency). Assume that $q \geq n-1$. Let t=n+q and define S as

$$S = \{(\phi_0,\phi_1,\dots,\phi_{t-1}) \text{ in } R^t \mid \text{For which there exists an acceptable solution } x,y \text{ to } xd_p+yn_p=\phi \text{ with } n_p,\phi \text{ and } x,\phi \text{ coprime.}\}.$$

Show that S is a generic subset of R^t.

Since $\theta(\phi) \geq 2n-1$ we have from Corollary 2.10 that

$$\bar{x} = u - \bar{n}n_p$$
$$\bar{y} = v + \bar{n}d_p$$

with $\bar{y}$ and $-\bar{n}$ the remainder and quotient respectively of the division $d_p|\phi v$, is an acceptable solution to $\bar{x}d_p + \bar{y}n_p = \phi$. This is true for every ϕ in R^t.

Define

$$q_1 = \text{Res}(n_p,\phi)$$
$$q_2 = \text{Res}(\bar{x},\phi)$$

where q_1, q_2 are polynomials in R^{n+q}.

Let

$$V_{q_1} = \{(\phi_0,\phi_1,\ldots,\phi_{t-1}) \text{ in } R^t |\ q_1(\phi_0,\ldots,\phi_{t-1})=0\},$$
$$V_{q_2} = \{(\phi_0,\phi_1,\ldots,\phi_{t-1}) \text{ in } R^t |\ q_2(\phi_0,\ldots,\phi_{t-1})=0\}.$$

It is clear that $S \supset (\overline{V_{q_1} \cup V_{q_2}}) = \bar{V}_{q_1} \cap \bar{V}_{q_2}$, since if $\phi = (\phi_0,\phi_1,\ldots,\phi_{t-1})$ in R^t is such that n_p,ϕ and $\bar{x},\phi$ are coprime we must have ϕ in S. Therefore S contains a Zariski open set.

We now need to show that $(\overline{V_{q_1} \cup V_{q_2}})$ is non-empty, ie that there exists some ϕ in R^t for which both n_p,ϕ and $\phi,\bar{x}$ are coprime. We now define such a ϕ.

Let q be a polynomial in s with $\theta(q) = n-1$, q and n_p coprime and qd_p monic. Let $\phi = qd_p+n_p$. (We have $ud_p+vn_p= 1$). This implies that ϕ and n_p are coprime.

$$\begin{aligned}\phi v &= qd_pv + n_pv \\ &= qvd_p + (1-ud_p) \\ &= (qv-u)d_p + 1.\end{aligned}$$

Therefore qv - u is the quotient $(=-\bar{n})$ and 1 the remainder $(=\bar{y})$ of the division $d_p|\phi v$. Since $\bar{y} = 1$ we have that $\bar{y},\phi$ are coprime

$$\bar{x}d_p + \bar{y}n_p = \phi$$

$$\bar{x}d_p = \phi - n_p .$$

Suppose that $\bar{x}$ and ϕ had a non-trivial common factor k

$$x'kd_p = \phi'k - n_p \quad , \quad k|n_p .$$

But this is a contadiction since ϕ, n_p have been chosen coprime. This completes the proof.

We will now proceed by treating the Multi-Input Multi-Output (MIMO) case. In what we have just explored we found that the notion of the resultant of two polynomials proved to be extremely usefull. The notion of the resultant has been extended to the matrix case. One such exposition which we also follow is that contained in [4].

Let
$$D = D_t s^t + D_{t-1}s^{t-1} + \dots + D_0 \;, \quad (\ell x \ell),$$
$$N = N_t s^t + N_{t-1}s^{t-1} + \dots + N_0 \;, \quad (m x \ell),$$

with ND^{-1} existing and proper. We define the generalized resultant of D,N of order k, the $k(m+\ell)x\ell(t+k)$ matrix

$$S_k(D,N) = \begin{bmatrix} D_t & D_{t-1} & D_{t-2} & \dots & D_0 & 0 & \dots & 0 \\ N_t & N_{t-1} & N_{t-2} & \dots & N_0 & 0 & \dots & 0 \\ 0 & D_t & D_{t-1} & \dots & D_1 & D_0 & \dots & 0 \\ 0 & N_t & N_{t-1} & \dots & N_1 & N_0 & \dots & 0 \\ 0 & 0 & D_t & \dots & \cdot & \cdot & \dots & 0 \\ 0 & 0 & N_t & \dots & \cdot & \cdot & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & & & & \vdots \\ 0 & 0 & 0 & \dots & D_t & & \dots & D_0 \\ 0 & 0 & 0 & \dots & N_t & & \dots & N_0 \end{bmatrix} \begin{matrix} 2k \\ \text{block} \\ \text{rows} \end{matrix}$$

Lemma 2.29. [4]. Let μ_i be the observability indecies of ND^{-1} Then

$$\text{rank } S_k = (\ell + m)k - \sum_{i:\mu_i < k} (k - \mu_i).$$

Lemma 2.30. [4]. Let q be the least integer for which rank S_{q+1} - rank $S_q \leq \ell$. Then for n >q N,D are right coprime iff rank $S_n = \ell n + \theta(\det D)$.

This second Lemma translates in terms of minors to the statement: N,D are right coprime iff at least one $\ell n+\theta(\det D) \times \ell n+\theta(\det D)$ minor of $S_n(D,N)$ is not zero. Denote these minors by $m_i(N,D)$. By symmetry we can also say that Q,P will be left coprime if at least one minor of $S_{n'}(Q',P')$ (of appropriate order) is not zero. Denote these minors by $n_j(Q,P)$. We now state

Proposition 2.31. Let $P = N_{RP}D_{RP}^{-1}$ be a stricly proper plant with $\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix}$ column proper and column degrees $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \alpha_\ell \geqslant 0$. Let $A_{LP}^{-1}B_{LP}$ be a left representation of P with $\theta(A_{LP}) = \mu_1$, (μ_1 the largest observability index) and A_{LP} regular. Let R^t denote the set of $\ell \times \ell$ diagonal matrices $\Phi = (\phi_i)$ with ϕ_i monic and $\theta(\phi_i) = \alpha_i + t_i$, $t = \sum_{i=1}^{\ell} \alpha_i + t_i$.

If $t_i \geq \mu_1 - 1$, with at least one $m_i(N_{RP},\Phi) \neq 0$ and at least one $n_i(\Phi,\overline{X}) \neq 0$, then P has the generic denominator matrix assignability property.

proof:

We already know from Theorem 2.9 that for any such Φ in R^t we can find an acceptable solution X,Y of $XD_{RP} + YN_{RP} = \Phi$. This is because the $\overline{Y}$ obtained using method 1) of constructing polynomial solutions has the property that

$P_0(\mathrm{diag}(s^{-(\mu_1-1)})\overline{Y})$ is a constant. Then

$$P_0(\mathrm{diag}(s^{-t_i})[\overline{Y} \quad \Phi\mathrm{diag}(s^{-\alpha_i})]) \text{ is a constant.}$$

Now if $\overline{t} = \max(t_i)$

$$\Phi = \Phi_{\alpha_1+\overline{t}}\, s^{\alpha_1+\overline{t}} + \ldots + \Phi_0$$

$$\overline{N}_{RP} = \overline{N}_{\alpha_1-1}\, s^{\alpha_1-1} + \ldots + \overline{N}_0$$

$$\overline{X} = \overline{X}_{\mu_1-1}\, s^{\mu_1-1} + \ldots + \overline{X}_0 .$$

Clearly both $N_{RP}\Phi^{-1}$ and $\Phi^{-1}\overline{X}$ are proper so that lemmata 2.29, 2.30 both hold.

Now $m_i(N_{RP},\Phi)$, $n_i(\Phi,\overline{X})$ are polynomials in $R[x_1,x_2,\ldots,x_t]$.

Define

$$V_1 = \{(x_1,x_2,\ldots,x_t) \text{ in } R^t \mid m_1=m_2=m_3=\ldots=m_k=0\}$$

$$V_2 = \{(x_1,x_2,\ldots,x_t) \text{ in } R^t \mid n_1=n_2=n_3=\ldots=n_j=0\}$$

$$S = \left\{ (x_1,x_2,\ldots,x_t) \text{ in } R^t \;\middle|\; \begin{array}{l} \text{For which there exists an accepta-} \\ \text{ble solution } X,Y \text{ of } XD_{RP}+YN_{RP} = \Phi \\ \text{with } N_{RP},\Phi \text{ right coprime and} \\ \text{with } \Phi,\ X \text{ left coprime.} \end{array} \right\}.$$

Clearly V_1 and V_2 are Zariski closed sets in R^t. And from the above we have $S \supset (\overline{V_1 \cup V_2}) = \overline{V}_1 \cap \overline{V}_2$.

In order to show that S is generic we just need to show that $\overline{V}_1 \cap \overline{V}_2$ is non-empty. But we have assumed that some $m_j(N_{RP},\Phi) \neq 0$ and some $n_i(\Phi,\overline{X}) \neq 0$, $\overline{V}_1 \cap \overline{V}_2 \neq \emptyset$. Therefore S is generic and P does possess the generic denominator assignability property. This completes the proof.

Crollary 2.22. Let P be a stricly proper plant which is diagonal

$$P=\begin{bmatrix} \frac{n_1}{d_1} & & & \\ & \frac{n_2}{d_2} & & 0 \\ & 0 & \ddots & \\ & & & \frac{n_\ell}{d_\ell} \\ & 0 & & \end{bmatrix} = \underbrace{\begin{bmatrix} n_1 & & & \\ & n_2 & 0 & \\ 0 & & \ddots & \\ & & & n_\ell \\ & 0 & & \end{bmatrix}}_{N_{RP}} \underbrace{\begin{bmatrix} d_1 & & & \\ & d_2 & 0 & \\ 0 & & \ddots & \\ & & & d_\ell \end{bmatrix}}_{D_{RP}}^{-1}$$

with n_i, d_i coprime. This means that the controllability indecies $\lambda_i = \theta(d_i)$ and that the observability indecies are $\mu_i = \theta(d_i)$ $1 \leq i \leq \ell$ with $\mu_{\ell+1} = \dots = \mu_m = 0$.

Let R^t denote the set of $\ell x \ell$ diagonal matrices $\Phi = (\phi_i), \phi_i$ monic with $\theta(\phi_i) = \lambda_i + t_i$, $t = \sum_{i=1}^{\ell} \lambda_i + t_i$.

A sufficient condition for generic denominator matrix assignability is $t_i \geqslant \mu_i -1$.

proof:

Assume that $t_i \geqslant \mu_i -1$. Define $S \subset R^t$ to be the set

$$S = \left\{ (\phi_1, \phi_2, \dots, \phi_t) \text{ in } R^t \middle| \begin{array}{l} \text{For which there exists an acceptable} \\ \text{solution } X,Y \text{ to } XD_{RP}+YN_{RP}=\Phi \text{ with } N_{RP}, \\ \Phi \text{ right coprime}, \Phi, X \text{ left coprime.} \end{array} \right\}.$$

Show that S is a generic subset of R^t.

Since N_{RP}, D_{RP} are diagonal there exist diagonal U,V such that $UD_{RP}+VN_{RP} = I$. It also means that a diagonal Y_0 can be picked such that $\theta(y_i)<\theta(d_i)$ $1 \leq i \leq \ell$ and $y_{ij} = 0$, $i \neq j$. This is done by dividing $\phi_i v_i$ by d_i and picking y_i as the remainder. The associated quotiens will form a digonal matrix N_0. Therefore X is diagonal as well. This is true for all Φ in R^t.

Define

$$q_{1i} = \mathrm{Res}(n_i,\phi_i) \quad , \qquad q_{2i} = \mathrm{Res}(\phi_i,x_i) \qquad 1 \leq i \leq \ell$$

where q_{1i}, q_{2i} are polynomials in R^t.

Let

$$V_{q_{1i}} = \{(g_1,g_2,\ldots g_t) \text{ in } R^t | \; q_{1i}(g_1,g_2,\ldots,g_t)=0\},$$

$$V_{q_{2i}} = \{(g_1,g_2,\ldots,g_t) \text{ in } R^t | \; q_{2i}(g_1,g_2,\ldots,g_t)=0\} .$$

It is clear that $S \supset (\overline{V_{q_{11}} \cup V_{q_{12}} \cup \ldots \cup V_{q_{1\ell}} \cup V_{q_{21}} \cup \ldots \cup V_{q_{2\ell}}})$

$$= \sum_{i=1}^{\ell} (\overline{V}_{q_{1i}} \cap \overline{V}_{q_{2i}}) = E ,$$

since if $\Phi = (g_1,g_2,\ldots,g_t)$ in R^t is such that n_i, ϕ_i are coprime and x_i,ϕ_i are coprime then N_{RP},Φ are right coprime and X,Φ are left coprime which means that Φ is in S. Therefore S contains a Zariski open set.

We now need to show that E is non-empty, ie that there exists some Φ in R^t for which both N_{RP},Φ are right coprime and X,Φ left coprime.

Let f_i be a polynomial in s with $\theta(f_i) = \lambda_i - 1$, f_i and n_i coprime and $f_i d_i$ monic.

Let $\phi_i = f_i d_i + n_i$. This means that ϕ_i and n_i are coprime. With $u_i d_i + v_i n_i = 1$ we have

$$\begin{aligned} \phi_i v_i &= f_i d_i v_i + n_i v_i \\ &= f_i v_i d_i + (1 - u_i d_i) \\ &= (f_i v_i - u_i) d_i + 1. \end{aligned}$$

Therefore $f_i v_i - u_i$ is the quotient $(=-\overline{n}_i)$ and 1 is the remainder $(=y_i)$ of the division $d_i | \phi_i v_i$. Since $y_i = 1$ we have that y_i,ϕ_i are coprime.

This implies that

$$x_i d_i + y_i n_i = \phi_i$$

$$x_i d_i = \phi_i - n_i .$$

Suppose that x_i and ϕ_i had a non-trivial common factor k_i, then

$$x_i' k_i d_i = \phi_i' k_i - n_i$$

$$k_i | n_i .$$

But this is a contradiction since ϕ_i, n_i have been chosen coprime. Therefore for such a choice of Φ we have N_{RP}, Φ right coprime and Φ, X left coprime. This means that $S \neq \emptyset$. This completes the proof.

Remark. Under the assumptions of proposition 2.31 we have that a sufficient condition for the generic denominator matrix assignability property is the following:

a) $t_i \geqslant \mu_1 - 1$

b) at least one $m_i(N_{RP}, \Phi) \neq 0$ and
at least one $n_j(\Phi, \overline{X}) \neq 0$.

The objective of recent work was to eliminate condition b). To accomplish this one has to show that some polynomials m_i, n_j are both non-zero. This is done by showing that for some Φ_0 in R^t we have $m_i(\Phi_0) \neq 0$ and $n_j(\Phi_0) \neq 0$. From Corollary 2.32 we have that if P is diagonal then this can be accomplished. The idea is then the following:

Instead of looking at the space of all Φ look instead at $T x \Phi$, where T is an appropriate space of systems (which includes diagonal systems). Then attempt to show that $m_i \neq 0$, $n_j \neq 0$ ie that $m_i(T_0 x \Phi_0) \neq 0$, $n_j(T_0 x \Phi_0)$, where T_0 is some diagonal system and where m_i, n_j may be rational functions in T_0, Φ_0.

i) We now define T appropriately.

Suppose that $T = \{(A,B,C)\} = R^{n^2+mn+n\ell}$

where A is nxn, B is $n \times \ell$ C is mxn, $n=\lambda\ell$, $n=\mu m$.

Let $G(s) = C(sI - A)^{-1}B$.

The set Q of (A,B,C) for which (A,B,C) is minimal and the controllability indecies of G(s) are equal to λ and the observability indecies equal to μ is a generic subset of T.

ii) Obtain in a rational way a right coprime representation of G(s).

$$G(s) = C(sI-A)^{-1}B$$
$$= C(\frac{1}{|sI-A|} M)B$$

where M is the matrix of cofactors. If $\alpha = |sI-A|$

$$G(s) = CMB\,\frac{1}{\alpha}$$
$$= FH^{-1},$$

where F = CMB and $H = \text{diag}(\alpha)$, $(\ell \times \ell)$. Therefore F,H is some right fraction representation.

Using rational operations in the variable

$t = (c_{11}, c_{12}, \ldots c_{mn}, a_{11}, a_{12}, \ldots, a_{nn}, b_{11}, b_{12}, \ldots, b_{n\ell})$ one can write

$$E \begin{bmatrix} H \\ F \end{bmatrix} = \begin{bmatrix} D \\ 0 \end{bmatrix}$$

D uppertriangular,
E unimodular
D a g.c.r.d. of H,F.

This is done by working in $R_t[s]$, $R_t = R(t)$, and by extracting a g.c.r.d. of H,F considered as matrices over $R_t[s]$.

$$\overbrace{\begin{bmatrix} U & V \\ E_1 & E_2 \end{bmatrix}}^{E} \begin{bmatrix} H \\ F \end{bmatrix} \quad \begin{bmatrix} D \\ 0 \end{bmatrix}$$

$$UH + VF = D$$

$$\underbrace{UHD^{-1}}_{D_{RP}} + \underbrace{VFD^{-1}}_{N_{RP}} = I$$

$$UD_{RP} + VN_{RP} = I$$

where U,V,N_{RP},D_{RP} are all in $R_t[s]$ and N_{RP},D_{RP} are right coprime in $R_t[s]$.

This means that if we evaluate $U,V,N_{RP}D_{RP}$ at some t_0 which is not a pole of U,V,N_{RP},D_{RP} then $U_{t_0}D_{RP_{t_0}} + V_{t_0}N_{RP_{t_0}} = I$

where $N_{RP_{t_0}}D^{-1}_{RP_{t_0}} = G_{t_0}$ is a right coprime representation of G_{t_0}. This can be done for almost all t_0 in T.

iii) Construct $\overline{X}$.

From the above we see that we can do the same for a left representation of G and we can have

$$D_{LP} = Is^{\mu}+D_{\mu-1}s^{\mu-1}+\ldots+D_0$$

$$N_{LP} = N_{\mu-1}s^{\mu-1}+\ldots+N_0, \qquad N_i,D_i \text{ are in } R_t.$$

We divide V on the right by D_{LP} and obtain -N and Y as the right quotient and remainder respectively,

$$\Phi V = -ND_{LP} + \overline{Y}.$$

Then

$$\overline{X} = \Phi U - NN_{LP}.$$

We have therefore constructed an $\overline{X}$ which will be valid when evaluated at (t_0,Φ_0) for almost all (t,Φ) in $T\times\Phi$.

iv) We have now constructed the m_i,n_j which happen to be rational in (t,Φ). We now need to show that for some diagonal t_0 and some Φ_0 we have $m_i(t_0,\Phi_0)\neq 0$, $n_j(t_0,\Phi_0)\neq 0$. The problem that we now encounter is that we may have "thrown out" all diagonal t_0 in the previous steps.

Another idea is to show that <u>any</u> t_0 is contained in some Zariski open set,(possibly a different one for different t_0). This we have not been able to accomplish. End of Remark.

Proposition 2.33. Let $P = N_{RP}D_{RP}^{-1}$ be an $m\times\ell$ strictly proper plant with controllability indecies $\lambda_1=\lambda_2= \ldots =\lambda_\ell=\lambda$, and observability indecies $\mu_1=\mu_2= \ldots =\mu_m=\mu$, and D_{RP} N_{RP} of the form

$$D_{RP} = Is^{\lambda} + D_{\lambda-1}s^{\lambda-1}+\ldots+D_0\ ,\ N_{RP} = N_{\lambda-1}s^{\lambda-1}+\ldots+N_0.$$

Let R^t be the set of $\ell\times\ell$ matrices of the form

$$\Phi = Is^{\lambda+q}+\Phi_{\lambda+q-1}s^{\lambda+q-1}+ \ldots +\Phi_0.$$

Let $\quad Q = \{[X,Y]\ |X=Is^q+X_{q-1}s^{q-1}+\ldots+X_0,\ Y=Y_qs^q+\ldots+Y_0\}.$

A necessary and sufficient condition for the existence of a solution to $XD_{RP}+YN_{RP}=\Phi$ in the class Q, for generic Φ is $q\geq\mu-1$.

proof:

(necessity). Equation $XD_{RP} + YN_{RP} = \Phi$ with the conditions imposed can be written as:

$$[\ I\ Y_q\ X_{q-1}\ Y_{q-1}\ \ \ldots\ X_0\ Y_0]\ S_{q+1} = [I\ \Phi_{q+\lambda-1}\ \ldots\ \Phi_0].$$

S_{q+1} can be thought of as an operator

$$S_{q+1}\ :\ R^{(\ell+m)(q+1)} \longrightarrow R^{(\ell+q+1)\ell}\ .$$

From Lemma 2.29 we have that S_k (an $k(\ell+m)x(\lambda+k)\ell$ matrix) has rank

$$\text{rank } S_k = (\ell+m)k - \sum_{i,\mu_i<k}(k-\mu_i)$$

which under the special circumstances becomes

$$\begin{aligned}\text{rank } S_k &= (\ell+m)k \qquad \text{if} \quad 1\leq k\leq\mu\\ &= (\ell+m)k-m(k-\mu) \quad \text{if} \quad \mu<k.\end{aligned}$$

By examining the dimensions we see that:

a) $S_1, S_2, \ldots, S_{\mu-1}$ are not onto.

b) S_μ is both onto and one-one.

c) $S_{\mu+1}, S_{\mu+2}, \ldots$ are onto.

Assume now that $q<\mu-1$ and that $XD_{RP}+YN_{RP}=\Phi$ has a solution in Q for generic Φ. Show a contradiction.

If we think of $X = Is^q+X_{q-1}s^{q-1}+ \ldots +X_0$, $Y = Y_q s^q+ \ldots +Y_0$, as elements in $R^{\ell(\ell+m)q}$ and the Φ as elements in $R^{\ell(\lambda+q)\ell}$ we see that the Φ that can be reached from $[X,Y]$ in Q are a set of dimension less than $\ell(\lambda+q)\ell$ which means that the Φ that can be reached does not contain a Zariski open set which is non-empty. This is a contradiction.

It is therefore necessary that $q \geq \mu-1$.

Remark. By looking at the class Q we are in a sense attempting to impose a bound on the order of the compensator dynamics. As a matter of fact we are looking at a subset of compensators of order ℓq. This is actually the class of compensators of order $q\ell$ which have observability indecies equal to q (which are almost all such compensators).

This result can be used in the denominator matrix assignability problem as a necessary condition. (It will read as follows: A necessary condition for generic denominator matrix assignability in the class of proper compensators of order $q\ell$ and equal observability indecies is $q \geq \mu-1$.)

(sufficiency). Suppose that $q \geqslant \mu-1$ or equivalently that $q = \mu-1+k$, $k \geq 0$. We want to show that the set of Φ in R^t ($t = \ell(\lambda+b)\ell$), for which a solution in Q exists, is a generic subset of R^t.

We know that $S_{\mu+k}$ is an $(\ell+m)(\mu+k) x (\lambda+\mu+k)\ell$ matrix, $(\lambda+\mu+k)\ell = (\ell+m)\mu+\ell k)$, with

$$\text{rank } S_{\mu+k} = (\ell+m)\mu+\ell k.$$

This means that the operator

$$S_{\mu+k} : R^{(\ell+m)(\mu+k)} \longrightarrow R^{(\ell+m)\mu+\ell k}$$

is onto.

We want to show that $S_{\mu+k}(Q)$ is all of R^t. We know that $S_{\mu+k}$ is onto. Let Φ be an element in R^t. Now there exists some [X,Y]

$$X = X_{\mu+k-1}s^{\mu+k-1} + \dots + X_0$$
$$Y = Y_{\mu+k-1}s^{\mu+k-1} + \dots + Y_0$$

such that

$$[X_{\mu+k-1}\ Y_{\mu+k-1}\ \cdots\ X_0\ Y_0]\ S_{\mu+k} = [I\ \Phi_{\lambda+\mu+k-1}\ \cdots\ \Phi_0].$$

But this means that $X_{\mu+k-1} = I$. Which means that [X,Y] is in Q. This completes the proof.

Remark. In order to show that the above is a sufficient condition for the generic denominator matrix assignability property (which after combining it with the necessity part will become necessary and sufficient) we need to show generic coprimeness of N_{RP} and Φ, and X and Φ.

Remark. The necessary condition in Proposition 2.33 can be used in formulating nacessary conditions for the generic denominator matrix assignability problem, and the generic characteristic polynomial assignability problem (mxl case). For necessity (in the class of proper compensators of order q) they both require that (X,Y) exists which is an acceptable solution and (X,Y) left coprime for the equation $XD_{RP}+YN_{RP} = \Phi$. The necessity of Proposition 2.33 will then require that $q \geq \mu-1$ if proper compensators of order $q\ell$ and observability indecies equal to q, are to be used. This leaves out proper compensators with observability indecies not all equal to q. But such compensators cannot satisfy $XD_{RP}+YN_{RP} = \Phi$. This is because of the following:

Let C be of order $q\ell$ and of unequal observability indecies. Then there exists some index g>q. This means that $C = D_{LC}^{-1}N_{LC}$ D_{LC}, N_{LC} left coprime where

$$D_{LC} = D_{g_1}s^{g_1} + \ldots + D_0$$

$$N_{LC} = N_{g_1}s^{g_1} + \ldots + N_0 , \quad g_1>g>q.$$

This implies

$$D_{LC}D_{RP}+N_{LC}N_{RP} = \Phi.$$

But $D_{g_1} I \neq 0$, and therefore this cannot happen.

Therefore a necessary condition for the generic denominator matrix assignability property in the class of compensators of order q (for the class of Φ defined) is $q \geqslant \mu-1$.

<u>Corollary 2.34</u>. Let $P = N_{RP}D_{RP}^{-1}$ be a strictly proper plant with observability indecies $\mu_1=\mu_2= \ldots =\mu_m=\mu$. Let R^t be the set of $\ell x\ell$ diagonal matrices with ϕ_i monic and $\theta(\phi_i) = \alpha_i+\gamma$. Let $\overline{Y}$ be the unique remainder of the right division $D_{LP}|\Phi V$, that is $\Phi V = -ND_{LP}+\overline{Y}$ where $P = D_{LP}^{-1}N_{LP}$, $D_{LP} = D_\mu s^\mu+\ldots+D_0$, $\det D \neq 0$.

If $\overline{Y} \neq 0$, a necessary condition for generic denominator matrix assignability is $\gamma \geq \mu-1$. (α_i the controllability indecies)

proof:

Assume that the set $S \subset R^t$ ($t=\sum_i (\alpha_i+\gamma)$) for which there exists an acceptable solution (X,Y) to $XD_{RP}+YN_{RP}=\Phi$ with Φ, N_{RP} right coprime and Φ,X left coprime, is generic.

Suppose now that $\gamma<\mu-1$.

Since (X,Y) is an acceptable solution so ix $\bar{X},\bar{Y}$ (Corollary 2.12). This implies that $P_0(\mathrm{diag}(s^{-\gamma})\bar{Y})$ is a constant for almost all Φ in R^t. This in turn implies that $\bar{Y}_{\mu-1} = 0$ for almost all Φ in R^t, but this is a contradiction because $\bar{Y}_{\mu-1}$, by assumption, is zero for only a certain Zariski closed set in R^t.

Therefore $\gamma \geq \mu-1$.

Chapter 3

Linear Matrix Equations

3.1 Introduction

Linear equations play a very important role in System Theory. In this chapter we undertake the study of a family of linear matrix equations which take the form:

$$\sum_{i=0}^{s} \sum_{j=0}^{t} g_{ij} B^i P \, A^j = Q \tag{3.1}$$

where B (mxm), A (nxn) and Q (mxm) are given matrices over some field F and g_{ij} elements of F, using methods of modern algebra. The emphasis is on the use of finite algebraic proceedures which are easily implemented on a digital computer and which lead to an explicit solution to the problem.

Particular attention is given to equations PA + BP = Q and P - BPA = Q and their special sucases PA + A'P = Q the Lyapunov equation and P - A'PA = Q the Discrete Lyapunov equation. The Lyapunov equation appears in several areas of Control Theory such as stability theory, optimal control (evaluation of quadratic integrals), stochastic control (evaluation of covariance matrices) and in the solution of the algebraic Riccati equation using Newton's method.

The material in this chapter has been inspired by an important paper by Kalman [17]. Kalman's concern was the characterization of polynomials whose zeros lie in certain algebraic domains (and the unification of the ideas of Hermite and Lyapunov). In this chapter we show that the same ideas lead to finite algorithms for the solution of linear matrix equations of the form given above. The analysis is in terms of a module theoretic structure on matrices presented here is believed to be new.

The chapter is divided into 6 sections. In section 3.2 we define the action f_{BA} over an arbitrary commutative ring with identity and prove Proposition 3.3. In section 3.3 we consider equation (3.1) over some field F in great generality and prove Theorem 3.4. In section 3.4 we deal with the equation $PA + BP = Q$ and the Lyapunov equation $PA + A'P = Q$ for which we give algorithms for obtaining its solution and comment on the arithmetic complexity. In section 3.5 we deal with the equation $P - BPA = Q$ as well as with the Discrete Lyapunov equation $P - A'PA = Q$. In section 3.6 we look at equation (3.1) over an arbitrary commutative ring with identity and in section 3.7 we prove a stability result.

3.2 The action f_{BA}

Let A be an nxn matrix and B an mxm matrix both over E, a commutative ring with identity. Let E[x,y] be the ring of polynomials in two indeterminates x and y over E. Let $\Psi = (\phi_2(x),\psi_2(y))$ be the ideal in E[x,y] generated by $\phi_2(x),\psi_2(y)$ the characteristic polynomials of A,B. Elements of the quotient ring $E[x,y]/\Psi$ are cosets (equivalence clases) denoted by $\Psi+a(x,y)$. The Cayley-Hamilton Theorem holds [20] therefore $\phi_2(A) = 0$, $\psi_2(B) = 0$. Since $\phi_2(x)$ and $\psi_2(y)$ are monic polynomials division is possible and as a consequence we can state:

Lemma 3.1. Let g(x,y) be an element of E[x,y]. Then g(x,y) can be written uniquely as:

$$g(x,y) = t(x,y)\phi_2(x)\psi_2(y)+p(x,y)\phi_2(x)+q(x,y)\psi_2(y)+r(x,y)$$

where:

the degree of p(x,y) in y is less than m (it may be a polynomial in x) or p(x,y) is zero,

the degree of q(x,y) in x is less than n (it may be a polynomial in y) or q(x,y) is zero, (3.2)

the degree of r(x,y) in y is less than m, in x less than n, or r(x,y) is zero.

proof:

Division in x is possible therefore

$$g(x,y) = a(x,y)\phi_2(x) + b(x,y)$$

where

the degree of b(x,y) in x is less than n (b(x,y) may be a polynomial just in y) or b(x,y) is zero.

Division in y is possible therefore

$$a(x,y) = t(x,y)\psi_2(y) + p(x,y)$$

where

the degree of $p(x,y)$ in y is less than m ($p(x,y)$ may be a polynomial just in x) or $p(x,y)$ is zero.

Also

$$b(x,y) = q(x,y)\psi_2(y) + r(x,y)$$

where

the degree of $r(x,y)$ in y is less than m and the degree of $r(x,y)$ in x is less than n or $r(x,y)$ is zero.

Now then

$$g(x,y) = t(x,y)\phi_2(x)\psi_2(y) + p(x,y)\phi_2(x) + q(x,y)\psi_2(y) + r(x,y).$$

This representation is unique since suppose that

$$g(x,y) = t_1(x,y)\phi_2(x)\psi_2(y)+p_1(x,y)\phi_2(x)+q_1(x,y)\psi_2(y)+r_1(x,y)$$
$$= t_2(x,y)\phi_2(x)\psi_2(y)+p_2(x,y)\phi_2(x)+q_2(x,y)\psi_2(y)+r_2(x,y)$$

with p_1,p_2,q_1,q_2,r_1,r_2 satisfying requirements (3.2)

$$r_1(x,y) - r_2(x,y) = [\overbrace{t_1-t_2}^{\alpha}]\phi_2(x)\psi_2(y)+[\overbrace{p_1-p_2}^{\beta}]\phi_2(x)+[\overbrace{q_1-q_2}^{\gamma}]\psi_2(y)$$

Suppose that $\alpha \neq 0$. Then there exists a term on the r.h.s. say ax^iy^j $i \geq n$, $j \geq m$. This term cannot be cancelled by either β or γ. Therefore $\alpha = 0$. Suppose that $\beta \neq 0$. Then there exists a term on the r.h.s. say bx^iy^j $i \geq n$. This term cannot be cancelled by any term from γ. Therefore $\beta = 0$. But then $\gamma = 0$ as well and $r_1(x,y) = r_2(x,y)$.

From the above it is very easy to see

<u>Corollary 3.2</u>. Let $g_1 = t_1\phi_2(x)\psi_2(y) + p_1\phi_2(x) + q_1\psi_2(y) + r_1$ and $g_2 = t_2\phi_2(x)\psi_2(y) + p_2\phi_2(x) + q_2\psi_2(y) + r_2$ be in the same coset $\Psi + a(x,y)$. Then $r_1 = r_2$.

proof:

We have that

$$g_1-g_2 = c(x,y)\phi_2(x) + d(x,y)\psi_2(y).$$

Let $c(x,y)=c_1(x,y)\psi_2(y)+c_2(x,y)$

$d(x,y)=d_1(x,y)\phi_2(x)+d_2(x,y)$

with the degree of $c_2(x,y)$ in y is less than m , and the degree of $d_2(x,y)$ in x is less than n.

$$g_1-g_2=(c_1+d_1)\phi_2(x)\psi_2(y)+c_2\phi_2(x)+d_2\psi_2(y).$$

From the above Lemma it then follows that $r_1 = r_2$.

The above results allow us to pick a unique reprentative from each equivalence class $\Psi + g(x,y)$. If $g = t\phi_2(x)\psi_2(y)+p\phi_2(x)+q\psi_2(y)+r$ and g is an element in $\Psi + g(x,y)$ $r = g(x,y) \bmod \Psi$ is this unique representative.

Let MN be the set of all mxn matrices over E. Define the action $f_{BA} : E[x,y] \times MN \longrightarrow MN$ in the following manner:

$$f_{BA}(h(x,y),M) = \sum_{jk} h_{jk}B^j M A^k$$

where

$h(x,y) = \sum_{jk} h_{jk}y^j x^k$ is an element in E[x,y] and M an element in MN.

It can be shown that f_{BA} has the following properties:

i) $f_{BA}(u,m) = uM$ where u is a constant in E.

ii) $f_{BA}(g(x,y)+h(x,y),M) = f_{BA}(g(x,y),M)+f_{BA}(h(x,y),M)$

iii) $f_{BA}(g(x,y)h(x,y),M) = f_{BA}(g(x,y),f_{BA}(h(x,y),M))$

$= f_{BA}(h(x,y),f_{BA}(g(x,y),M))$

iv) $f_{BA}(g(x,y),M) = f_{BA}(g(x,y) \bmod \Psi \, , M)$

v) $f_{BA}(g(x,y),M+N) = f_{BA}(g(x,y),M) + f_{BA}(g(x,y),N)$

where $g(x,y), h(x,y)$ are any elements in $E[x,y]$ and M,N are any matrices in MN.

Properties i),ii),iii) and v) follow directly from the definition of f_{BA} [12]. Property iv) is true because of Lemma 3.1 and the Cayley-Hamilton Theorem.

The definition of f_{BA} allows for the interpretation of MN as an $E[x,y]/\Psi$ - module.

Proposition 3.3. The set MN of mxn matrices with elements in E is a module over the quotient ring $E[x,y]/\Psi$.

proof:

The set of mxn matrices under addition is an abelian group. Define multiplication (*) of cosets $\Psi + h(x,y)$ and mxn matrices M in the following manner.

$$(\Psi + h(x,y)) * M = f_{BA}(h(x,y) \bmod \Psi \, ,M).$$

The multiplication is well defined and satisfies the properties:

1) $(\Psi + h(x,y))*(M+N) = (\Psi + h(x,y))*M + (\Psi + h(x,y))*N$

2) $(\Psi + h(x,y))*[(\Psi + g(x,y))*M] = [(\Psi + h(x,y))(\Psi + g(x,y))]*M$

3) $[(\Psi + h(x,y))+(\Psi + g(x,y))]*M = (\Psi + h(x,y))*M + (\Psi + g(x,y))*M$

4) $(\Psi + 1)*M = M$

for all M,N in MN and all $\Psi + h(x,y)$, $\Psi + g(x,y)$ in $E[x,y]/\Psi$ with $\Psi + 1$ being the multiplicative identity in $E[x,y]/\Psi$.

Property v) of the action makes 1) true. Property iii) ensures 2), property ii) ensures 3) and property i) ensures 4).

3.3 The General Equation

Suppose that we restrict E to be some field F and let K be an algebraically closed extension of F. If $f(x,y)$ is an element of $F[x,y]$ we denote by $V(f)$ the variety of $f(x,y)$ in A_2^K [35]. Let $\lambda_1, \lambda_2, \ldots \lambda_n$ be the eigenvalues of A and $\mu_1, \mu_2, \ldots \mu_m$ the eigenvalues of B. suppose that $g(x,y)$ is a polynomial in $F[x,y]$ $g(x,y) = \quad g_{jk} y^j x^k$ then we define G_g the mnxmn matrix

$$G_g = \sum_{jk} g_{jk} B^j \otimes (A')^k \tag{3.3}$$

where $\otimes$ denotes tensor product ($A \otimes B = (a_{ij}B)$) and A' denotes transpose. The significance of the matrix G_g comes from the following:

Let $\underline{p}$ be the mnx1 column vector made up of the entries of matrix $P=(p_{ij})$ written as

$$\underline{p} = [\ p_{11}, p_{12}, \ldots\ p_{1n}, p_{21} \ldots\ p_{2n} \ldots\ p_{mn}\ ]'$$

Let $\underline{q}$ be the mnx1 column vector made up of the entries of Q. The equation (3.1) can simply be written as

$$G_g\ \underline{p} = \underline{q}\ . \tag{3.4}$$

We now state

Theorem 3.4. The following statements are equivalent.

1) Equation (3.1) has a unique solution for all Q.

2) G_g is invertible.

3) $g(\lambda_i, \mu_j) \neq 0$ for all $\mu_j, \lambda_i \quad 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m$

4) $V(g(x,y)) \cap V(\phi_2(x)) \cap V(\psi_2(y)) = \emptyset$

5) The coset $\Psi + g(x,y)$ is a unit in $F[x,y]/\Psi$.

proof:

We will show the equivalences in the order

1) → 2) → 3) → 4) → 5) → 1).

1) → 2).

Suppose then that equation (3.1) does have a unique solution for all Q. Well since equation (3.1) can equivalently be written as $G_g \underline{p} = \underline{q}$ then G_g is invertible.

2) → 3).

From [21] theorem 43.8 we have that the mn eigenvalues of G_g are $g(\lambda_i, \mu_j)$. Since $\det G_g = \Pi\, g(\lambda_i, \mu_j)$ and since $\det G_g \neq 0$ we have that

$$g(\lambda_i, \mu_j) \neq 0 \quad \text{for all } \mu_j, \lambda_i \quad 1 \leq i \leq n,\ 1 \leq j \leq m.$$

3) → 4).

If we look at 3) it really says the following: That the polynomials $g(x,y), \phi_2(x), \psi_2(y)$ have no common zero in A_2^K. But this is statement 4).

4) → 5).

Now $\Psi + g(x,y)$ is a unit iff there exists a $\Psi + f(x,y)$ such that

$$(\Psi + f(x,y))(\Psi + g(x,y)) = \Psi + 1.$$

Now

$\Psi + g(x,y)$ is a unit iff there exists $f(x,y)$ such that $\Psi + f(x,y)g(x,y) = \Psi + 1$,

iff there exists $f(x,y)$ such that $f(x,y)g(x,y) - 1 = q(x,y)$ an element in Ψ,

iff there exist $f(x,y), a_1(x,y), a_2(x,y)$ elements in $F[x,y]$ such that $f(x,y)g(x,y) + a_1\phi_2(x) + a_2\psi_2(y) = 1$.

Assume now that 4) holds (ie the polynomial h=1 vanishes at every common zero of $g(x,y)$, $\phi_2(x)$, $\psi_2(y)$). By the Hibert-Nullstellensatz [35] there exist polynomials $f(x,y)$, $a_1(x,y)$, $a_2(x,y)$ such that

$$f(x,y)g(x,y)+a_1(x,y)\phi_2(x)+a_2(x,y)\psi_2(y) = 1$$

this means that $\Psi+g(x,y)$ is a unit in $F[x,y]/\Psi$.

5) $\rightarrow$ 1) .

Suppose that $\Psi+g(x,y)$ is a unit in $F[x,y]/\Psi$, that is there exists a $\Psi+f(x,y)$ such that $(\Psi+f(x,y))(\Psi+g(x,y))=\Psi+1$.

Let $P = f_{BA}(f(x,y) \bmod \Psi, Q) = f_{BA}(f(x,y),Q)$. Show that this is a solution to (3.1).

$$\begin{aligned}\sum_{i=0}^{s}\sum_{j=0}^{t} g_{ij}B^iP\,A^j &= f_{BA}(g(x,y),P)\\ &= f_{BA}(g(x,y),f_{BA}(f(x,y),Q))\\ &= f_{BA}(g(x,y)f(x,y),Q)\\ &= f_{BA}(1,Q)\\ &= Q\end{aligned}$$

The P so defined is the unique solution to (3.1) since, if $P_1 \neq P_2$ are two distinct solutions of (3.1) this means

$$\sum_{i=0}^{s}\sum_{j=0}^{t} g_{ij}B^iP_1A^j = Q$$

$$\sum_{i=0}^{s}\sum_{j=0}^{t} g_{ij}B^iP_2A^j = Q$$

$$f_{BA}(g(x,y),P_1) = f_{BA}(g(x,y),P_2) = Q$$

$$f_{BA}(f(x,y),f_{BA}(g(x,y),P_1)) = f_{BA}(f(x,y),f_{BA}(g(x,y),P_2))$$

$P_1 = P_2$, which is a contradiction.

Therefore equation (3.1) has a unique solution for all Q. This completes the proof of Theorem 3.4.

Remark In the above proof we have an explicit expression for the solution of equation (3.1). A general method for constructing such a f(x,y) is through a constructive proof of the Hilbert-Nullstellensatz or using Resultant Theory [29]. As will be seen in later pages of this chapter for several important equations this generality is unnecessary and easier methods do exist.

Remark In our entire construction we have been using the ideal $\Psi = (\phi_2(x),\psi_2(y))$. Other ideals can be used, as an example the ideal $(\overline{\Phi}_2(x),\overline{\Psi}_2(y))$ where $\overline{\Phi}_2(x)$ and $\overline{\Psi}_2(y)$ are the minimal polynomials of A and B respectively. Since $\phi_2(x)=k(x)\overline{\Phi}_2(x)$ and $\psi_2(y)=l(y)\overline{\Psi}_2(y)$ we will be dealing with polynomials of smaller degree. This may have as an effect the reduction in the number of arithmetic operations performed.

Remark In the special case in which A is missing from equation (3.1),(ie suppose that it is of the form $\sum_{i=0}^{s} g_i B^i P = Q$, $A = I$), then it would seem that the analysis can take place in some quotient ring $F[y]/\Phi$. This actually is the case. Let Φ be the ideal in F[y] generated by $\psi_2(y)$. Then $\Phi+g(y)$ is a unit in $F[y]/\Phi$ iff $\Psi+g(y)$ is a unit in $F[x,y]/\Psi$. This follows from the fact that if there exist f(y), $a_2(y)$ elements of F[y] such that $f(y)g(y)+a_2(y)\psi_2(y) = 1$ then clearly there exist elements f(x,y) (=f(y)) and $a_1(x,y)$ (=0) and $a_2(x,y)$ ($=a_2(y)$) in F[x,y] such that $f(x,y)g(y)+a_1(x,y)\phi_2(x)+a_2(x,y)\psi_2(y) = 1$. On the other hand if there exist $f(x,y)$, $a_1(x,y)$, $a_2(x,y)$ elements of $F[x,y]$ such that $f(x,y)g(y)+a_1(x,y)(x-1)^n+a_2(x,y)\psi_2(y) = 1$ then evaluating at $x=1$ we get $f(1,y)g(y)+a_2(1,y)\psi_2(y) = 1$ which means that $\Phi+g(y)$ is a unit in $F[y]/\Phi$, ($\phi_2(x) = (x-1)^n$ since $A = I_n$). The action $f_B : F[y] \times MN \longrightarrow MN$ can similarly be defined as $f_B(h(y),M) = \sum_j h_j B^j M$ and MN becomes an $F[y]/\Phi$- module. The solution to $\sum_{i=0}^{s} g_i B^i P = Q$ is then given by $P = f_B(f(x) \bmod \Phi, Q)$. The situation is similar if $B = I_m$.

Remark Let us look at the very special case when we are dealing with the equation $B\,\underline{p} = \underline{q}$ where $\underline{p}$ and $\underline{q}$ are mx1 vectors. In this case $g(x,y) = y$. What we want to do is find $f(y)$ such that $f(y)y+a(y)\psi_2(y) = 1$. If $\psi_2(y) = y^m + k_{m-1}y^{m-1} + \ldots + k_0$ obvious choices become

$$f(y) = -\frac{1}{k_0}y^{m-1} - \frac{k_{m-1}}{k_0}y^{m-2} - \ldots - \frac{k_1}{k_0} \qquad a(y) = \frac{1}{k_0}$$

since $f(y)y + \frac{1}{k_0}\psi_2(y) = 1$.

Now $k_0 \neq 0$ since for a solution to exist $\det B = k_0 \neq 0$. The solution $\underline{p}$ is given by:

$$\underline{p} = f_B(f(y), \underline{q}).$$

Analysing this further we get

$$\underline{p} = -\frac{1}{k_0}\sum_{j=0}^{m-1} k_{j+1} B^j \underline{q}$$

$$= \left(-\frac{1}{k_0}\sum_{j=0}^{m-1} k_{j+1} B^j\right)\underline{q}.$$

As would be expected

$$B = -\frac{1}{k_0}\sum_{j=0}^{m-1} k_{j+1} B^j .$$

We will close this section by proving two Propositions which make clear the method of solution we have adopted. Let MN be the vector space of mxn matrices over the field F. Let M_n be the vector space of mnx1 vectors over F. We then have the obvious vector space isomorphism $f : M_n \longrightarrow M_n$ defined as:

$$f \; : \; \begin{bmatrix} p_{11} \\ p_{12} \\ \vdots \\ p_{1n} \\ p_{21} \\ \vdots \\ p_{mn} \end{bmatrix} \longmapsto \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ & \vdots & & \\ p_{m1} & p_{m2} & \cdots & p_{mn} \end{bmatrix}$$

Let G_g be as in (3.3). Let the polynomial $\pi(u)$ in F[u] be the characteristic polynomial of G_g, $\pi(u) = \det(I_{mn}u - G_g)$. Let $\Pi = (\pi(u))$ be the principal ideal in F[u]. Define the function $h : F[u]/\Pi \longrightarrow F[x,y]/\Psi$ in the following manner:

$$h : \; \Pi+a(u) \longmapsto \Psi+a(g(x,y))$$

<u>Proposition 3.5</u> The function h is a ring homomorphism.

<u>proof</u>:

We first show that h is well defined. Let $\Pi+a(u) = \Pi+b(u)$ (ie $a(u) - b(u) = k(u)\pi(u)$). Show that $\Psi+a(g(x,y)) = \Psi+b(g(x,y))$.

That is show that $a(g(x,y))-b(g(x,y))=c_1(x,y)\phi_2(x)+c_2(x,y)\psi_2(y)$. I claim that $\pi(g(x,y))=k(x,y)\phi_2(x)\psi_2(y)+k_1(x,y)\phi_2(x)+k_2(x,y)\psi_2(y)$ in the unique form (3.2). This is because of the following:

Let v_{ij} be the eigenvalues of G_g. We know that [21] these are the the mn values $g(\lambda_i,\mu_j)$. Assume that $v_{ij}=g(\lambda_i,\mu_j)$. Assume we are working over $K[x,y]$. Since $\pi(u) = \prod_{ij} (u-v_{ij})$ we have that

$$\pi(g(x,y)) = \prod_{ij} (g(x,y)-g(\lambda_i,\mu_j)) \qquad (3.5)$$

We now show that each factor $g(x,y)-g(\lambda_i,\mu_j)$ can be written in the form $g(x,y)-g(\lambda_i,\mu_j)=k_{ij}(x,y)(x-\lambda_i)+l_{ij}(y)(y-\mu_j)$. This can be seen easily from the fact that if

$$g(x,y)=g_t x^t+g_{t-1}x^{t-1}+\ldots+g_0 \qquad (g_t \text{ in } F[y])$$

then

$$\begin{aligned} g(x,y)-g(\lambda_i,\mu_j)=[&g_t x^{t-1}+(g_{t-1}+g_t\lambda_i)x^{t-2}+ \\ &+(g_{t-2}+g_{t-1}\lambda_i+g_t\lambda_i^2)x^{t-3}+\ldots \\ &+(g_1+g_2\lambda_i+\ldots+g_t\lambda_i^{t-1})](x-\lambda_i) \\ &+g(\lambda_i,y)-g(\lambda_i,\mu_j) \end{aligned}$$

(we divide $g(x,y)$ by $x-\lambda_i$ in x). Now $y-\mu_j$ does divide $g(\lambda_i,y)-g(\lambda_i,\mu_j)$. This means that (3.5) can be written as

$$\pi(g(x,y))= \prod_{ij} (k_{ij}(x,y))(x-\lambda_i)+l_{ij}(y)(y-\mu_j)).$$

In expanding this product we see that every term in the sum will be of either of the two forms $a(x,y)\phi_2(x)$ or $b(x,y)\psi_2(y)$. Now $\pi(g(x,y)) = a_1(x,y)\phi_2(x) + a_2(x,y)\psi_2(y)$ and over $K[x,y]$ we can write

$$\pi(g(x,y))=t_1(x,y)\phi_2(x)\psi_2(y)+p_1(x,y)\phi_2(x)+q_1(x,y)\psi_2(y)$$

which is in the form (3.2). Now we know that we can write

$$\pi(g(x,y))=t_2(x,y)\phi_2(x)\psi_2(y)+p_2(x,y)\phi_2(x)+q_2(x,y)\psi_2(y)+r(x,y)$$

which is in form (3.2).

But from Lemma 3.1 we have that this expression is unique. This means that $t_1(x,y)=t_2(x,y)$, $p_1(x,y)=p_2(x,y)$, $q_1(x,y)=q_2(x,y)$ are all elements in $F[x,y]$ and $r_2(x,y) = 0$. Therefore $\pi(g(x,y))$ is an element in Ψ and h is well defined.

Now h is a ring homomorphism since

$$\begin{aligned} h(\Pi+a(u)+\Pi+b(u)) &= h(\Pi+(a(u)+b(u))) \\ &= \Psi+(a(g(x,y))+b(g(x,y))) \\ &= \Psi+a(g(x,y))+\Psi+b(g(x,y)) \\ &= h(\Pi+a(u))+h(\Pi+b(u)) \end{aligned}$$

and

$$\begin{aligned} h((\Pi+a(u))(\Pi+b(u))) &= h(\Pi+a(u)b(u)) \\ &= \Psi+a(g(x,y))b(g(x,y)) \\ &= (\Psi+a(g(x,y)))(\Psi+b(g(x,y))) \\ &= h(\Pi+a(u))h(\Pi+b(u)). \end{aligned}$$

This completes the proof of Proposition 3.2.

Now since MN is an $F[x,y]/\Psi$-module and M_n an $F[u]/\Pi$-module and $h : F[u]/\Pi \rightarrow F[x,y]/\Psi$ a ring homomorphism, MN can be made into an $F[u]/\Pi$ -module in the natural way. Define multiplication $(\circ)$: $F[u]/\Pi \times MN \rightarrow MN$ by:

$$(\Pi+a(u))\circ P = h(\Pi+a(u))*P$$

We now have

<u>Proposition 3.6</u> The map f is an $F[u]/\Pi$-module isomorphism.

<u>proof:</u>

We already know that f is a vector space isomorphism. In order to show that f is an $F[u]/\Pi$-module isomorphism we just need to show that

$$f((\Pi+a(u))*\underline{p}) = (\Pi+a(u)).f(\underline{p}) = (\Pi+a(u))^\circ P = h(\Pi+a(u)*P$$

Let us show that

$$f((\Pi+u)*\underline{p}) = h(\Pi+u)*P.$$

Well

$$\begin{aligned} f((\Pi+u)*\underline{p}) &= f(G_g\underline{p}) \qquad\qquad (\text{with } G_g\underline{p} = q\_) \\ &= f(\underline{q})=Q \\ &= \sum_{jk} g_{jk}B^j P A^k \\ &= (\Psi+g(x,y))*P \\ &= h(\Pi+u)*P \end{aligned}$$

Now by induction we can show that

$$f((\Pi+u^i)*\underline{p}) = h((\Pi+u)^i)*P.$$

Suppose that $f((\Pi+u^{i-1})*\underline{p}) = h((\Pi+u^{i-1}))*P.$

Now

$$\begin{aligned} f((\Pi+u^i)*\underline{p}) &= f((\Pi+u)*[(\Pi+u^{i-1}))*\underline{p})]) \\ &= f((\Pi+u)*\underline{t}), \quad \underline{t} = (\Pi+u^{i-1})*\underline{p} \\ &= (\Pi+u)^\circ f(\underline{t}) \\ &= (\Pi+u)^\circ(h(\Pi+u^{i-1})*P) \\ &= h(\Pi+u)*[h(\Pi+u^{i-1})*P] \\ &= [h(\Pi+u).(h(\Pi+u^{i-1})]*P \\ &= h(\Pi+u^i)*P. \end{aligned}$$

This makes f an $F[u]/\Pi$-module isomorphism, and completes the proof.

Remark We are now in a position to explain our method of solution. Suppose that a unique solution exists. Then in order to construct it we either work with form (3.4) of the equation and invert G_g (or equivalently find the inverse of +u in F[u]/) or we work with form (3.1) of the equation and obtain the inverse of +g(x,y) in F[x,y]/ . Theoretically we have shown that the two methods to be the same. As will be seen later in some situations (the Lyapunov equation etc.) the second method leads to fewer arithmetic operations.

In the following sections we will be concerned with the problem of constructing the solution to several special cases of the general equation. It will be of course assumed that a unique solution does exist.

3.4 The Equation PA + BP = Q

As shown when proving Theorem 3.4, the solution to equation PA + BP = Q is given by

$$P = f_{BA}(\ f(x,y) \bmod \Psi,\ Q)$$

where $f(x,y)$ is in $F[x,y]$ such that $(\Psi + f(x,y))(\Psi + (x+y)) = +1$. It has also been mentioned that such an $f(x,y)$ can be found by using Resultant Theory [29] or from a constructive proof of the Hilbert-Nullstellensatz. But in simple cases like this we need not resort to such general theory.

In carrying out computations it may be advantageous instead of finding $f(x,y)$ such that $f(x,y)g(x,y) = k_1(x,y)\phi_2(x) + k_2(x,y)\psi_2(y) + 1$ to find $f_u(x,y)$ such that $f_u(x,y)g(x,y) = \bar{k}_1(x,y)\phi_2(x) + \bar{k}_2(x,y)\psi_2(y) + u$ where u is some non-zero element in F. The solution P is then given by

$$P = \frac{1}{u} f_{BA}(\ f_u(x,y) \bmod \Psi,\ Q).$$

We construct $f_u(x,y)$ in this manner:

We do have that

$$(x+y) \mid \phi_2(x)\psi_2(y) - \phi_1(y)\psi_1(x)$$

where $\phi_1(x) = \phi_2(-x)$, $\psi_1(x) = \psi_2(-x)$.

Let

$$p(x,y) = \frac{\phi_2(x)\psi_2(y) - \phi_1(y)\psi_1(x)}{x+y} \tag{3.6}$$

Since $\phi_2(x)$, $\psi_1(x)$ are coprime, (ie $\lambda_i + \mu_j \neq 0$ for all i,j) we have $\lambda_e(x)$, $\mu_e(x)$, $\lambda_e'(x)$, $\mu_e'(x)$ such that

$$\begin{aligned} \lambda_e(x)\psi_1(x) + \mu_e(x)\phi_2(x) &= e \\ \lambda_e'(x)\psi_2(x) + \mu_e'(x)\phi_1(x) &= e \end{aligned} \tag{3.7}$$

Let $\quad f_u(x,y) = \lambda_e(x)\mu_e'(y)p(x,y).$

$$f_u(x,y)(x+y)=\lambda_e(x)\mu_e'(y)p(x,y)(x+y)$$
$$=\lambda_e(x)\mu_e'(y)\phi_2(x)\psi_2(y)+e\lambda_e'(y)\psi_2(y)$$
$$+e\mu_e(x)\phi_2(x)-\mu_e(x)\lambda_e'(y)\phi_2(x)\psi_2(y)-e^2$$

With $u = -e^2$ we have that

$$(\Psi+f_u(x,y))(\Psi+(x+y)) = \Psi+u.$$

A different method for obtaining an $\bar{f}(x,y)$ such that $\bar{f}(x,y)(x+y) = k_1\phi_2(x) + k_2\psi_2(y) + 1$ is the following:

Divide $\phi_2(x)$ by $x+y$ in x

$$\phi_2(x) = h(x,y)(x+y) + h(y).$$

For $x = -y$ we have that $\phi_2(-y) = \phi_1(y) = h(y)$. Now since $\phi_1(x)$, $\psi_2(x)$ are coprime, there exist $\lambda(y)$, $\mu(y)$ such that

$$\lambda(y)\phi_1(y) + \mu(y)\psi_2(y) = 1$$
$$\lambda(y)[\phi_2(x) - h(x,y)(x+y)] + \mu(y)\psi_2(y) = 1$$
$$-\lambda(y)h(x,y)(x+y) + \lambda(y)\phi_2(x) + \mu(y)\psi_2(y) = 1.$$

Let $\bar{f}(x,y) = -\lambda(y)h(x,y)$.

The Lyapunov Equation $PA + A'P = Q$.

The Lyapunov equation is a special case of $PA + BP = Q$, $B=A'$, A stable. With the appropriate modifications to the first procedure for constructing the solution we have:

Let $\tau_e(x)$, $\lambda_e(x)$ be such that

$$\tau_e(x)\phi_1(x) + \lambda_e(x)\phi_2(x) = e \qquad e \neq 0$$

$$p(x,y) = \frac{\phi_2(x)\phi_2(y) - \phi_1(x)\phi_1(y)}{x+y}$$

$$f_u(x,y) = \tau_e(x)\tau_e(y)p(x,y) \tag{3.8}$$

Algorithms for solving the Lyapunov equation ca be found in [12]. Here we comment on the Integer case.

Suppose matrices A and Q contained integer entries. The polynomials $\phi_2(x)$, $p(x,y)$ then have integer coefficients.

Let

$$\phi_2(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$$
$$\phi_1(x) = d_n x^n + d_{n-1} x^{n-1} + \dots + d_0.$$

Define the 2nx2n resultant matrix S

$$S = \begin{bmatrix} a_n & 0 & \cdots & 0 & d_n & 0 & \cdots & 0 \\ a_{n-1} & a_n & \cdots & 0 & d_{n-1} & d_{n-1} \cdots & & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ & & & a_n & & & & d_n \\ a_0 & & & a_{n-1} & d_0 & & & d_{n-1} \\ 0 & a_0 & & & 0 & d_0 & & \\ 0 & 0 & & & 0 & 0 & & \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & & a_0 & 0 & 0 & & d_0 \end{bmatrix}$$

We know that $\det S \neq 0$ since $\phi_1(x), \phi_2(x)$ are coprime. Let $e = \det S$. The linear system of equations

$$S \begin{bmatrix} \lambda_{n-1} \\ \lambda_{n-2} \\ \cdot \\ \cdot \\ \cdot \\ \lambda_0 \\ \tau_{n-1} \\ \cdot \\ \cdot \\ \cdot \\ \tau_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ \\ \\ 0 \\ 0 \\ \\ \\ \\ e \end{bmatrix}$$

has an integer solution and we have integer polynomials

$\tau_e(x) = \tau_{n-1}x^{n-1}+\ldots+\tau_0, \quad \lambda_e(x) = \lambda_{n-1}x^{n-1}+\ldots+\lambda_0$

which satisfy

$$\tau_e(x)\phi_1(x) + \lambda_e(x)\phi_2(x) = e.$$

This means that $f_u(x,y)$ in (3.8) has integer coefficients and so does $f_u(x,y)\bmod\Psi$, which implies that $P_u=f_{BA}(f_u(x,y)\bmod\Psi,Q)$ has integer entries. The solution is given by $P=\frac{1}{u}P_u$, $(u=-e^2)$.

The algorithm proceeds as follows:

I_1) Find $\phi_2(x)$.

I_2) Set $p(x,y) = \dfrac{\phi_2(x)\phi_2(y) - \phi_1(x)\phi_1(y)}{x+y}$.

I_3) Find $\tau_e(x)$ and e.

I_4) Form $f_u(x,y) = \tau_e(x)\tau_e(y)p(x,y)$.

I_5) Find $P_u = f_{BA}(x,y)\bmod\Psi,\ Q)$.

I_6) Set $P = \frac{1}{u}P_u \quad u = -e^2$.

We now want to comment on the number of integer operations (addition, subtraction, multiplication, division) involved in running the Integer algoritm, when A and Q are nxn matrices using classical operations.

Step I_1. There are several methods for obtaining the characteristic polynomial $\phi_2(x)$ of a stable matrix [19]. Evaluating

$\phi_2(x)$ at n distinct points and then solving for the coefficients requires $O(n^4)$ operations. If n is small (say $n \leqslant 20$) evaluating $\phi_2(x)$ at x=1 where $\phi_2(1) = \Lambda$, $\lambda = \lceil \log_{10}\Lambda \rceil$ and then at $x = 10^\lambda$ allows one to "read off" the coefficients of $\phi_2(x)$ from a large integer. This proceedure requires only $O(n^3)$ operations.

Step I_2. This step can be done in $O(n^2)$ operations.

Step I_3. Solving a linear set of 2n equations simultaneously is an $O(n^3)$ operation.

Step I_4. Performing the multiplication as $\tau_e(x)[\tau_e(y)p(x,y)]$ requires $O(n^3)$ operations.

Step I_5. Obtaining $f_u(x,y) \bmod \Psi$ requires two polynomial divisions and can be done in $O(n^3)$ operations. To form $f_{BA}(f_u(x,y) \bmod \Psi, Q)$ we use $O(n^4)$ operations. In the event that the matrix Q is a product of vectors this calculation can be done in $O(n^3)$ operations.

Step I_6. It can be done in $O(n^2)$ operations.

It can therefore be seen that the overall calculation requires $O(n^4)$ operations in the general case and $O(n^3)$ operations in the special cases mentioned.

3.5 The equation $P - BPA = Q$.

We again wish to construct an $f(x,y)$ such that

$$(\Psi+f(x,y))(\Psi+(1-xy)) = \Psi+1.$$

Let

$$\phi_2(x) = \det(Ix-A) = a_n x^n + a_{n-1}x^{n-1}+\ldots+a_0$$
$$\psi_2(x) = \det(Ix-B) = b_m x^m + b_{m-1}x^{m-1}+\ldots+b_0$$
$$\phi_3(x) = a_0x^n+a_1x^{n-1}+\ldots+a_n$$
$$\psi_3(x) = b_0x^m+b_1x^{m-1}+\ldots+b_m$$

From the above definition we can see that the roots of $\phi_3(x)$ are the values $\frac{1}{\lambda_i}$ where $\lambda_i \neq 0$. Since we assume that a unique solution exists we must have that $\phi_2(x),\psi_3(x)$ are coprime, because if they have a nontrivial factor $k(x)$ they must also have at least one common root (ie $\lambda_i = \frac{1}{\mu_j}$ for at least some i,j).

On the other hand we also have that

a) if $n \geqslant m$ then $1-xy \mid y^{n-m}\phi_2(x)\psi_2(y)-\phi_3(y)\psi_3(x)$

b) if $n < m$ then $1-xy \mid x^{m-n}\phi_2(x)\psi_2(y)-\phi_3(y)\psi_3(x)$.

We comment on the validity of a). Let

$$p_1 = y^{n-m}\phi_2(x)\psi_2(y) = y^{n-m}(a_nx^n+a_{n-1}x^{n-1}+\ldots a_0)(b_my^m+\ldots+b_0)$$

$$p_2 = \phi_3(x)\psi_3(y) = (a_0y^n+a_1x^{n-1}+\ldots+a_n)(b_0x^m+b_1x^{m-1}+\ldots+b_m).$$

In forming p_1-p_2 combine terms from p_1 and p_2 of like coefficients (ie $a_nb_mx^ny^my^{n-m}$ and a_nb_m, $a_{n-1}b_mx^{n-1}y^my^{n-m}$ and $a_{n-1}b_my$ ... in general $a_kb_1x^ky^1y^{n-m}$ and $a_kb_1y^{n-k}x^{m-1}$ $0 \leqslant k \leqslant n$, $0 \leqslant 1 \leqslant m$).

We then can see that

$$a_kb_1(x^ky^1y^{n-m} - y^{n-k}x^{m-1}) = k(x,y)(1-x^iy^j) \quad \text{for some } i$$

and consequently

$$1-xy \mid a_kb_1(x^ky^1y^{n-m} - y^{n-k}x^{m-1}).$$

Therefore $1-xy \mid y^{n-m}\phi_2(x)\psi_2(y) - \phi_3(y)\psi_3(x)$.

We are now ready to construct $f(x,y)$.

Since $\phi_3(x)$, $\psi_2(x)$ are coprime we have $\lambda(x)$, $\mu(x)$, $\lambda'(x)$, $\mu'(x)$ such that

$$\lambda(x)\psi_3(x) + \mu(x)\phi_2(x) = 1$$

$$\lambda'(x)\psi_2(x) + \mu'(x)\phi_3(x) = 1.$$

If $n \geqslant m$ let

$$p(x,y) = \frac{y^{n-m}\phi_2(x)\psi_2(y) - \phi_3(y)\psi_3(x)}{1-xy}.$$

If $n < m$ let

$$p(x,y) = \frac{x^{n-m}\phi_2(x)\psi_2(y) - \phi_3(y)\psi_3(x)}{1-xy}.$$

Then $f(x,y) = \lambda(x)\mu'(y)p(x,y)$.

The Discrete Lyapunov equation $P - A'PA = Q$ is a special case.

3.6 Over Integral Domains.

Suppose now that we are investigating equation (3.1) over E some integral domain. The next Proposition gives a necessary and sufficient condition for the existence of a unique solution to (3.1) for all Q.

Proposition 3.7. Equation (3.1) has a unique solution over E for each Q iff $\psi + g(x,y)$ is a unit in $E[x,y]/\Psi$.

proof:

Let $P = f_{BA}(f(x,y) \bmod \psi, Q)$

where $f(x,y)g(x,y) = k_1(x,y)\phi_2(x) + k_2(x,y)\psi_2(y) + 1$.

$$\sum_{i=0}^{s}\sum_{j=0}^{t} g_{ij}B^iPA^j = f_{BA}(g(x,y),P)$$
$$= f_{BA}(g(x,y), f_{BA}(f(x,y),Q))$$
$$= f_{BA}(1,Q)$$
$$= Q.$$

The solution P is unique. This follows in the same manner as in the proof of Theorem 3.4.

Suppose that equation (3.1) does have a unique solution for each Q. This means that G_g in (3.4) is invertible. We have that $\pi(u) = \det(Iu-G_g)$ is the characteristic polynomial of G_g. From the Cayley-Hamilton theorem if $\pi(u) = \pi_t u^t + \pi_{t-1}u^{t-1} + \ldots + \pi_0$ we have

$$\pi(G_g) = \pi_t G_g^t + \pi_{t-1}G_g^{t-1} + \ldots + \pi_0 I = 0$$
$$G_g^{-1} = -\frac{\pi_t}{\pi_0}G_g^{t-1} - \frac{\pi_{t-1}}{\pi_0}G_g^{t-2} - \ldots - \frac{\pi_1}{\pi_0}$$

Let $f(u) = -\frac{\pi_t}{\pi_0}u^{t-1} - \frac{\pi_{t-1}}{\pi_0} - \ldots - \frac{\pi_1}{\pi_0}$

then $f(u)u + \frac{1}{\pi_0}\pi(u) = 1$. Therefore $\Pi + u$ is a unit in $E[u]/\Pi$.

Now we can see that Proposition 3.5 remains valid when F(=E) is an integral domain, (E can be imbedded in its field of quotients k and let K be the algebraically closed extension of k).

We can now see that

$$(\Pi+f(u))(\Pi+u) = \Pi+1$$

$$h((\Pi+f(u))(\Pi+u)) = h(\Pi+1)$$

$$h(\Pi+f(u))h(\Pi+u) = \Psi+1$$

$$(\Psi+f(g(x,y)))(\Psi+g(x,y)) = \Psi+1$$

which means that $\Psi+g(x,y)$ is a unit in $E[x,y]/\Psi$.

3.7 A Stability Result

Using the explicit expression for the solution of the Lyapunov equation $A'P + PA = Q$, we now prove the following result.

Theorem 3.8. Let A be an nxn matrix. A is stable iff for any c 1xn such that (A,c) is observable there exists a unique solution to the equation $A'P + PA = -c'c$ which is positive definite.

proof:

Suppose that A is stable and that (A,c) is an observable pair. Then the equation $A'P + PA = -c'c$ does have a unique solution which is expressed as

$$P = \sum_{ij} f_{ij} A'^{i} c'cA^{j}$$
$$= [\,c', (cA)', \ldots, (cA^{n-1})'\,] F \underbrace{\begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix}}_{L}$$

where $F=(f_{ij})$ and $\det L \neq 0$.

Since A is stable the Bezoutian matrix B is positive definite and so is the polynomial $p(x,y) = \dfrac{\phi_2(x)\phi_2(y) - \phi_1(x)\phi_1(y)}{x+y}$.

From Lemma 3 of [17] we have that f(x,y) is positive and that $F > 0$. Therefore $P > 0$. On the other hand if P is the unique positive definite (symmetric) solution of $A'P+PA = -c'c$ then $F > 0$ and so is the polynomial f(x,y). This means that p(x,y) is positive and that $B > 0$. Therefore A is stable.

It would be interesting to investigate whether the explicit expression of the solution P, can be used to prove necessary and sufficient conditions that some matrix has all its roots in a certain region of the complex plane (more general than the left half plane) which is described by the polynomial $g(x,y)$ in (3.1).

Chapter 4

Conclusions and Areas for future Research

4.1 Conclusions

In Chapter 2 by exploiting the notion of matrix fraction representations we were able to formulate several problems in the area of General Pole Assignment by dynamic output feedback. Specifically the Characteristic Polynomial Problem, the Invariant Factor Problem and the Denominator Matrix Problem. As it has been demostrated the common basis for the investigation of these problems was the equation $XD_{RP} + YN_{RP} = \Phi$. The major difficulty in constructing acceptable solutions to such equations was the requirement that $X^{-1}Y$ be a proper transfer function. This requirement was imposed because of the desire to construct proper compensators to accomplish the required tasks. The manner in which acceptable solutions were constructed was to use the general expression for polynomial solutions $X = \Phi V - NN_{LP}$, $Y = \Phi U + ND_{LP}$ and appropriately choose N so that $X^{-1}Y$ will exist and be proper.

Using these ideas we were able to give sufficient conditions in both the Characteristic Polynomial and Invariant Factor Problems. We were able to give a constructive proof to Rosenbrock's State Space result, rederive results due to Brasch and Pearson using algebraic ideas and suggest ways of using the freedom afforded by the technique to achieve other design objectives.

In recognizing the importance of the coprimeness conditions the idea of Genericity was introduced and with it the formulation of the Generic counterparts of the Characteristic Polynomial, Invariant Factor and Denominator Matrix Problems. The hopes of introducing this idea to formulate necessary conditions have been justified in several situations. Of fundamental importance was the notion of the Generalized Sylvester Resultants S_k and the interpretation of the equation $XD_{RP} + YN_{RP} = \Phi$ as an operator taking the (X,Y) into Φ, as expressed in terms of these Resultants. In some of the results (as in Proposition 2.31) additional requirements to degree constraints are imposed (such as $n_i(\Phi,\overline{X}) \neq 0$, $m_j(N_{RP},\Phi) \neq 0$). We conjecture that these can be removed. Such and other difficulties which have been pointed out are matters of continuing investigation.

One last comment is the fact that all of the procedures outlined in this thesis can be turned into constructive algorithms, which can be programmed on a digital computer. It will be especially interesting to implement these on MACSYMA as was done with some of the equations in Chapter 3 [12].

In Chapter 3 we undertake the study of a family of linear equations and give necessary and sufficient conditions for the existence of a unique solution. Emphasis is also given in suggesting algorithms for constructing the solution, which make use of finite algebraic procedures, which are easily implemented on a digital computer. This work is a continuation of research carried out in my Masters Thesis.

The basic idea is that the set of mxn matrices over an arbitrary field F can be thought of as an $F[x,y]/\Psi$- module. This is done by employing the action f_{BA}. Of fundamental importance is the realization that there is a connection between the existence of a unique solution to a linear set of equations and the Hilbert-Nullstellensatz Theorem.

Special attention is given to equations $PA + BP = Q$ and $P - BPA = Q$. Results are also extended in the case of Integral domains. Of great importance is the fact that one method of solution does not involve the computation of eigenvalues, something essential in the Theory of Systems over Rings and allows for parametric studies. We also show how the theory can be used in proving stability theorems.

4.2 Areas for Future Research

1. In constructing polynomial solutions to the equation $XD_{RP} + YN_{RP} = \Phi$ we are able to construct a Y with $\theta(Y) \leq \beta_1 - 1$. The best we can do so far is to bound the row degrees by $\mu_1 - 1$ where μ_1 is the largest observability index. An improvement, in terms of reducing the order of additional dynamics, would be achieved if a method is found so that the row degrees of Y are bounded by d_i, $d_i < \mu_1 - 1$.

2. As has been mentioned the technique affords freedom that can be used for achieving other design objectives. Examples for the single input single output case have been given. Surely much more can be done, especially in the Multivariable case.

3. Several results have been shown in the section on Genericity. The work is not complete. It is conjectured that the remark concerning Proposition 2.31 can be shown to be true.

4. The resultants S_k, thought of as operators, were essential in expressing several results in section 2.6. This certainly suggests a method for obtaining sufficient conditions for the Characteristic Polynomial Problem. This can be done by looking more closely in the structure of these operators. A closer examination might give better sufficient conditions than the ones presently in existence.

5. All of our work here has been conducted in the frequency domain. It would certainly be interesting to show connections with the state space approach. It would also be worthwhile to apply the ideas presented here to other problems encoutered in System Theory (Regulator Problem, Servo Probem, etc.)

6. In chapter 3 we give an explicit expression for the solution of a general equation (3.1). It would be quite interesting to investigate whether this can be used, as in section 3.7, to prove necessary and sufficient conditions that some matrix has all its roots in a certain region of the complex plain, (more general that the left half plane), which is described by the polynomial $g(x,y)$.

Appendix A

In section 2.1 we introduced the idea of non-commutative localization and stated two Propositions without proof. We now proceed with the proofs.

Let $\underline{A} = F^{nxn}[s]$ and $\underline{T}$ be the set $\underline{T} = \{G \text{ in } \underline{A} \mid \det G \neq 0\}$. $\underline{T}$ is a multiplicatively closed set. We first want to show that S_1) is satisfied.

Let S and A be elements of $\underline{T}$ and $\underline{A}$ respectively. Then clearly S is invertible in $\underline{A}_1 = (F(s))^{nxn}$.

Let $G = S^{-1}A$.

$$G = \begin{bmatrix} \frac{a_{11}}{b_{11}} & \frac{a_{12}}{b_{12}} & \cdots & \frac{a_{1n}}{b_{1n}} \\ \frac{a_{21}}{b_{21}} & \frac{a_{22}}{b_{22}} & \cdots & \frac{a_{2n}}{b_{2n}} \\ \vdots & \vdots & & \vdots \\ \frac{a_{n1}}{b_{n1}} & \frac{a_{n2}}{b_{n2}} & \cdots & \frac{a_{nn}}{b_{nn}} \end{bmatrix}$$

We can write G as

$$G = \underbrace{\begin{bmatrix} n_{11} & n_{12} & \cdots & n_{1n} \\ n_{21} & n_{22} & \cdots & n_{22} \\ \vdots & \vdots & & \vdots \\ n_{n1} & n_{n2} & \cdots & n_{nn} \end{bmatrix}}_{B} \cdot \underbrace{\begin{bmatrix} d_1 & & & \\ & d_2 & & 0 \\ 0 & & \ddots & \\ & & & d_n \end{bmatrix}}_{T}{}^{-1}$$

We then have that $S^{-1}A = BT^{-1}$ which implies that $AT = SB$, where B is in $\underline{A}$ and T is in $\underline{T}$.

Let us now show that S_2) is satisfied.

If S is in $\underline{T}$ and $SA = 0$ (A in $\underline{A}$), then we must have $A = 0$ because otherwise we will contradict the fact that the n columns of S are linearly independent. But then $AS = 0$. Therefore S_2) is trivially satisfied in our case.

We now have that $F^{nxn}[s][\underline{T}^{-1}]$ exists. For the case when $n = 1$ $F^{1x1}[s][\underline{T}^{-1}]$ is actually the field of quotients of F[s] and we clearly have that $F(s) = F[s][\underline{T}^{-1}]$.

<u>Proposition 2.1</u> The rings $\underline{A}_1 = (F(s))^{nxn}$ and $F^{nxn}[s][\underline{T}^{-1}]=\underline{A}[\underline{T}^{-1}]$ are isomorphic.

<u>proof</u>:

Define $f : \underline{A}_1 \quad \underline{A}[\underline{T}^{-1}]$ in the following manner.

Let G be an element of $\underline{A}_1$.

$$G = \begin{bmatrix} \frac{a_{11}}{b_{11}} & \frac{a_{12}}{b_{12}} & \cdots & \frac{a_{1n}}{b_{1n}} \\ \frac{a_{21}}{b_{21}} & \frac{a_{22}}{b_{22}} & \cdots & \frac{a_{2n}}{b_{2n}} \\ \vdots & & & \\ \frac{a_{n1}}{b_{n1}} & \frac{a_{n2}}{b_{n2}} & \cdots & \frac{a_{nn}}{b_{nn}} \end{bmatrix} = \begin{bmatrix} \frac{n_{11}}{d} & \frac{n_{12}}{d} & \cdots & \frac{n_{1n}}{d} \\ \frac{n_{21}}{d} & \frac{n_{22}}{d} & \cdots & \frac{n_{2n}}{d} \\ \vdots & & & \\ \frac{n_{n1}}{d} & \frac{n_{n2}}{d} & \cdots & \frac{n_{nn}}{d} \end{bmatrix}$$

$$G = \underbrace{\begin{bmatrix} n_{11} & n_{12} & \cdots & n_{1n} \\ n_{21} & n_{22} & \cdots & n_{2n} \\ \vdots & & & \\ n_{n1} & n_{n2} & \cdots & n_{nn} \end{bmatrix}}_{N} \cdot \underbrace{\begin{bmatrix} d & & & 0 \\ & d & & \\ & & \ddots & \\ 0 & & & d \end{bmatrix}^{-1}}_{D}$$

where $d = \prod_{ij} b_{ij}$.

Then $f(G) = [N,D]$,

where the notation $[N,D]$ indicates the equivalence class in which the pair lies.

a) We first show that f is well defined. Let $E = G$ with

$$E = \begin{bmatrix} \frac{u_{11}}{v_{11}} & \frac{u_{12}}{v_{12}} & \cdots & \frac{u_{1n}}{v_{1n}} \\ \frac{u_{21}}{v_{21}} & \frac{u_{22}}{v_{22}} & \cdots & \frac{u_{2n}}{v_{2n}} \\ \vdots & & & \\ \frac{u_{n1}}{v_{n1}} & \frac{u_{n2}}{v_{n2}} & \cdots & \frac{u_{nn}}{v_{nn}} \end{bmatrix} = \begin{bmatrix} \frac{t}{g}11 & \frac{t}{g}12 & \cdots & \frac{t}{g}1n \\ \frac{t}{g}21 & \frac{t}{g}22 & \cdots & \frac{t}{g}2n \\ \vdots & & & \\ \frac{t}{g}n1 & \frac{t}{g}n2 & \cdots & \frac{t}{g}nn \end{bmatrix}$$

$$E = \underbrace{\begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & & & \\ t_{n1} & t_{n2} & \cdots & t_{nn} \end{bmatrix}}_{H} \cdot \underbrace{\begin{bmatrix} g & & & \\ & g & & 0 \\ 0 & & \ddots & \\ & & & g \end{bmatrix}^{-1}}_{J}$$

Now $\frac{t}{g}ij = \frac{n}{d}ij$ ie $t_{ij}d = n_{ij}g$.

Show that $f(E) = f(G)$, ie $[H,J] = [N,D]$, that is show that there exist X,Y in $\underline{A}$ such that

$$NX = HY$$

$$DX = JY \text{ an element in } \underline{T}.$$

Well let $X = J$ and $Y = D$. Then clearly $DJ = JD$, an element in $\underline{T}$, and $NJ = HD$. Therefore f is well defined.

b) Show that f is a ring homomorphism. We do have that $f(I) = [I,I]$, the identity element in $F^{nxn}[s][\underline{T}^{-1}]$.

i) Show that $f(G+E) = f(G) + f(E)$.

$$G+E = \begin{bmatrix} \frac{n_{11}g + t_{11}d}{gd} & \frac{n_{12} + t_{12}d}{gd} & \cdots & \frac{n_{1n}g + t_{1n}d}{gd} \\ \frac{n_{21}g + t_{21}d}{gd} & & \cdots & \frac{n_{2n}g + t_{2n}d}{gd} \\ \vdots & & & \vdots \\ \frac{n_{n1}g + t_{n1}d}{gd} & & \cdots & \frac{n_{nn}g + t_{nn}d}{gd} \end{bmatrix}$$

$$G+E = \underbrace{\begin{bmatrix} n_{11}g+t_{11}d & n_{12}g+t_{12}d & \cdots & n_{1n}g+t_{1n}d \\ \vdots & \vdots & & \vdots \\ n_{n1}g+t_{n1}d & n_{n2}g+t_{n2}d & \cdots & n_{nn}g+t_{nn}d \end{bmatrix}}_{P} \cdot \underbrace{\begin{bmatrix} gd & & 0 \\ & \ddots & \\ 0 & & gd \end{bmatrix}^{-1}}_{Q}$$

Now $f(G+E) = [P,Q]$, and we also have $f(G)=[N,D]$, $f(E)=[H,J]$.

We want to show that $[N,D] + [H,J] = [P,Q]$.

Now $[N,D] + [H,J] = [NC+HD, DJ]$ where $K=DC=JF$ an element of $\underline{T}$.

Let $C = J$ and $F = D$ then $[N,D]+[H,J] = [NJ+HD, DJ]$.

Show that tere exist X,Y elements in $\underline{A}$ such that

$$(NJ+HD) = PY$$

$$DJX = QY \text{ element of } \underline{T}.$$

Let $X = Q$ and $Y = DJ$. Then clearly $DJQ = QDJ$ (all are diagonal) and $(NJ + HD)Q = PDJ$.

ii) Show that $f(GE) = f(G)f(E)$.

$$GE = \begin{bmatrix} \frac{n_{11}}{d} & \frac{n_{12}}{d} & \cdots & \frac{n_{1n}}{d} & \frac{t_{11}}{g} & \frac{t_{12}}{g} & \cdots & \frac{t_{1n}}{g} \\ \frac{n_{21}}{d} & & \cdots & & \frac{t_{21}}{g} & & \cdots & \\ & & \vdots & & & & & \\ \frac{n_{n1}}{d} & & \cdots & \frac{n_{nn}}{d} & \frac{t_{n1}}{g} & & \cdots & \frac{t_{nn}}{g} \end{bmatrix}$$

$$GE = \begin{bmatrix} \frac{n_{11}t_{11}+n_{12}t_{21}+\cdots+n_{1n}t_{n1}}{gd} & \cdots & \frac{n_{11}t_{1n}+\cdots+n_{1n}t_{nn}}{gd} \\ \frac{n_{21}t_{11}+n_{22}t_{21}+\cdots+n_{2n}t_{n1}}{gd} & \cdots & \\ \vdots & & \\ \frac{n_{n1}t_{11}+n_{n2}t_{21}+\cdots+n_{nn}t_{n1}}{gd} & \cdots & \frac{n_{n1}t_{1n}+\cdots+n_{nn}t_{nn}}{gd} \end{bmatrix}$$

$$GE = \underbrace{\begin{bmatrix} n_{11}t_{11}+\cdots n_{1n}t_{n1} & \cdots & n_{11}t_{1n}+\cdots n_{1n}t_{nn} \\ & \vdots & \\ n_{n1}t_{11}+\cdots n_{nn}t_{n1} & \cdots & n_{n1}t_{1n}+\cdots n_{nn}t_{nn} \end{bmatrix}}_{Q} \cdot \underbrace{\begin{bmatrix} gd & & 0 \\ & \ddots & \\ 0 & & gd \end{bmatrix}^{-1}}_{P}$$

Now $f(GE) = [Q,P]$, $f(G) = [N,D]$, $f(E) = [H,J]$ and $[N,D][H,J] = [NC,JU]$ where $DC = HU$ U in $\underline{T}$. Let $C = H$ and $U = D$, then $DH = HD$. Therefore $[N,D][H,J] = [NH,JD]$ and we clearly see that $[Q,P] = [NH,JD]$ since $Q = NH$ and $P = JD$.

Therefore $f(GE) = f(G)f(E)$. This shows that f is a ring homomorphism.

c) Show that f is one-one.

Suppose that $f(G) = [N,D] = [H,J] = f(E)$. does this imply that $G = E$?

Now $G=ND^{-1}$, $E=HJ^{-1}$ and there exist X,Y such that $NX = HY$, $DX = JY$ an element of $\underline{T}$. Since DX,JY are elements of $\underline{T}$ this means that X,Y are elements of $\underline{T}$ which implies that $(DX)^{-1}$, $(JY)^{-1}$ exist in $\underline{A}_1$.

Since
$$NX = HY$$
$$NX(DX)^{-1} = HY(JY)^{-1}$$
$$ND^{-1} = HJ^{-1}$$
$$G = E.$$

d) We finally show that f is onto.

Let $[N,D]$ an element of $\underline{A}[\underline{T}^{-1}]$ and $G = ND^{-1}$ an element in A_1. Now $G = \overline{N}\overline{D}^{-1}$ where $\overline{D}$ is diagonal of the form

$$\overline{D} = \begin{bmatrix} \overline{d} & 0 \\ 0 & \overline{d} \end{bmatrix}.$$

Do we have that $f(G) = [N,D]$?

Well we do have that $f(G) = [\overline{N},\overline{D}]$. So we just need to show that $[\overline{N},\overline{D}] = [N,D]$. From condition S_1), we know that there exist X,Y such that $DX = \overline{D}Y$ an element in $\underline{T}$. Does this also mean that $NX = \overline{N}Y$? Suppose not, ie $NX \neq \overline{N}Y$.

$$NX(DX)^{-1} \neq \overline{N}Y(\overline{D}Y)^{-1}$$
$$ND^{-1} \neq \overline{N}\overline{D}^{-1}$$

which is a contradiction. Therefore f is onto.

We have demonstrated that f is a ring isomorphism.

In a similar manner we could also define the left ring of fractions of $\underline{A}$ with respect to $\underline{T}$. We would then have

$$[\underline{T}^{-1}]F^{nxn}[s] \simeq (F(s))^{nxn}$$

Proposition 2.2. The right $\underline{A}_1$-modules $\underline{A}_2 = (F(s))^{mxn}$ and $F^{mxn}[s][\underline{T}^{-1}] = \underline{M}[\underline{T}^{-1}]$ are isomorphic.

proof:

Define $q : \underline{A}_2 \qquad \underline{M}[\underline{T}^{-1}]$ in the following manner: if G is given by

$$G = \begin{bmatrix} \frac{a_{11}}{b_{21}} & \frac{a_{12}}{b_{21}} & \cdots & \frac{a_{1n}}{b_{21}} \\ \frac{a_{21}}{b_{21}} & & \cdots & \\ & & \vdots & \\ \frac{a_{m1}}{b_{m1}} & & & \frac{a_{mn}}{b_{mn}} \end{bmatrix}$$

$$G = \underbrace{\begin{bmatrix} n_{11} & n_{12} & \cdots & n_{1n} \\ n_{21} & n_{22} & \cdots & n_{2n} \\ & \vdots & & \\ n_{m1} & n_{m2} & \cdots & n_{mn} \end{bmatrix}}_{N} \cdot \underbrace{\begin{bmatrix} d & & & 0 \\ & d & & \\ & & \ddots & \\ 0 & & & d \end{bmatrix}^{-1}}_{D}$$

where $d = \prod_{ij} b_{ij}$.

Then $q(G) = [N,D]$.

In a manner similar to the proof of Proposition 2.1 one can show that q is well defined and that $q(G+E) = q(G)+q(E)$.

Because of Proposition 2.1 we can think of $F^{mxn}[s][\underline{T}^{-1}]$ as a right $(F(s))^{nxn}$-module.

Let L be:

$$L = \begin{bmatrix} \frac{t}{g}11 & \frac{t}{g}12 & \cdots & \frac{t}{g}1n \\ \frac{t}{g}21 & \cdots & & \\ & \vdots & & \\ \frac{t}{g}n1 & \cdots & & \frac{t}{g}nn \end{bmatrix}$$

$$L = \underbrace{\begin{bmatrix} t_{11} & \cdots & t_{1n} \\ & \vdots & \\ t_{n1} & \cdots & t_{nn} \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} d & & 0 \\ & \ddots & \\ 0 & & d \end{bmatrix}}_{J}^{-1}.$$

Now L is thought of as an element in $(F(s))^{nxn}$ and $f(L)=[H,J]$. Suppose that $[A,B]$ is some element in $F^{nxn}[s][\underline{T}^{-1}]$, then $[A,B]L$ is noting but $[A,B][H,J]$.

We now want to show that

$$q(GL) = q(G)f(L).$$

Let

$$G = \underbrace{\begin{bmatrix} n_{11} & n_{12} & \cdots & n_{1n} \\ n_{21} & n_{22} & \cdots & n_{2n} \\ & \vdots & & \\ n_{m1} & n_{m2} & \cdots & n_{mn} \end{bmatrix}}_{N} \cdot \underbrace{\begin{bmatrix} d & & & 0 \\ & d & & \\ 0 & & \ddots & \\ & & & d \end{bmatrix}}_{D}^{-1}$$

Then

$$GL=\begin{bmatrix} \dfrac{n_{11}t_{11}+n_{12}t_{21}+\cdots+n_{1n}t_{n1}}{gd} & \cdots & \dfrac{n_{11}t_{1n}+n_{12}t_{2n}+\cdots+n_{1n}t_{nn}}{gd} \\ & \vdots & \\ \dfrac{n_{m1}t_{11}+n_{m2}t_{21}+\cdots+n_{mn}t_{n1}}{gd} & \cdots & \dfrac{n_{m1}t_{1n}+n_{m2}t_{2n}+\cdots+n_{mn}t_{nn}}{gd} \end{bmatrix}$$

$$=\underbrace{\begin{bmatrix} n_{11}t_{11}+\cdots+n_{1n}t_{n1} & \cdots & n_{11}t_{1n}+\cdots+n_{1n}t_{nn} \\ & \vdots & \\ n_{m1}t_{11}+\cdots+n_{m1}t_{1n} & \cdots & n_{m1}t_{1n}+\cdots+n_{mn}t_{nn} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} gd & & 0 \\ & \ddots & \\ 0 & & gd \end{bmatrix}^{-1}}_{P}$$

Therefore $q(GL) = [Q,P]$.

Now $q(GL) = [Q,P]$

$q(G) = [N,D]$

$f(L) = [H,J]$.

We have that $q(G)f(L) = [N,D][H,J] = [NC,JU]$ with $DC = HU$ U an element in $\underline{T}$. Let $C=H$ and $U=D$. Clearly $DH = HD$. Therefore

$$q(G)f(L)=\left[\begin{bmatrix} n_{11}t_{11}+\cdots+n_{1n}t_{n1} & \cdots & n_{11}t_{1n}+\cdots+n_{1n}t_{nn} \\ & \vdots & \\ n_{m1}t_{11}+\cdots+n_{mn}t_{n1} & \cdots & n_{m1}t_{1n}+\cdots+n_{mn}t_{nn} \end{bmatrix}, \begin{bmatrix} dg & & 0 \\ & \ddots & \\ 0 & & dg \end{bmatrix}\right]$$

which is nothing but $[Q,P]$.

Therefore q is an $\underline{A}_1$-module homomorphism.

In a manner similar to the proof of Proposition 2.1 one can show that q is one-one and onto. This completes the proof of Proposition 2.2.

What we have done in these two Propositions is to show that even in the matrix case the familiar ideas of localization remain valid. In doing this we have also given a mathematical interpretation of the notion of matrix fraction representation.

It was hoped that the ideas presented here would lead to some group theoretic formulation of the feedback problem. At the present stage we are not able to do this. These ideas remain for the future.

References

[1] B. D. O. ANDERSON, E. I. JURY, Generalized Bezoutian and Sylvester Matrices in Multivariate Linear Control, IEEE Trans. on AC, Aug., 1976.

[2] M. F. ATIYAH, I. G. MACDONALD, Introduction to Commutative Algebra, Addison-Wesley, Reading Mass., 1969.

[3] G. BENGTSSON, Output Regulation and Internal Models a Frequency Domain Approach, Automatica, Vol. 13 pp 333-345, 1977.

[4] R. R. BITMEAD, S.-Y. KUNG, B. D. O. ANDERSON, T. KAILATH, Greatest Common Divisors via Generalized Sylvester and Bezout Matrices, IEEE Trans. on AC, Vol. AC-23, No. 6, Dec. 1978.

[5] F. M. BRACSH, J. B. PEARSON, Pole Placement Using Dynamic Compensators, IEEE Trans. on AC, Vol. AC-15, No. 1 Feb. 1970.

[6] R. BROCKETT, Finite Dimensional Linear Systems, Wiley, New-york, 1970.

[7] C. T. CHEN, C. H. HSU, A proof of the Stability of Multi-Variable Systems, Proc. IEEE 56 (11) 2061-2062, 1968.

[8] L. CHENG, J. B. PEARSON, Frequency Domain Synthesis of Multivariable Linear Regulators, IEEE Trans. on AC,Vol. AC-23, No. 1, Feb. 1978.

[9] W. A. COPPEL, Bulletin Australian Mathematical Society, 1974.

[10] C. A. DESOER, M. VIDYASAGAR, Feedback Systems: Input-Output Properties, Academic Press, New-york, 1975.

[11] C. A. DESOER, J. D. SCHULMAN, Zeros and Poles of Matrix Transfer Functions and Their Dynamical Interpretation, IEEE Trans. on CAS, Vol. CAS-21, No. 1, Jan. 1974.

[12] T. E. DJAFERIS, Exact Solution to Lyapunov's Equation using Algebraic Methods, M. S. Thesis, Jan. 1977.

[13] G. D. FORNEY JR., Minimal Bases of Rational Vector Spaces with Applications to Multivariable Linear Systems, SIAM J. Control, Vol. 13, No. 3, May 1975.

[14] P. A. FUHRMAN, On strict system equivalence and similarity, Int. J. Control, Vol. 25, No. 1, 1977.

[15] F. R. GANTMACHER, The Theory of Matrices, Vol. I, Chelsea, New-York, 1960.

[16] T. KAILATH, A Course in Linear System Theory, a book which is to appear.

[17] R. E. KALMAN, Algebraic Characterization of Polynomials whose Zeroes Lie in Certain Algebraic Domains, Proc. N.A.S. Mathematics, Vol.64, No.3, Nov. 1969.

[18] D. L. KLEINMAN, On an Iterative Technique for Riccati Equation Computations, IEEE Trans. on AC, Vol. AC-13, No. 1, Feb. 1968.

[19] D. E. KNUTH, Seminumerical Algorithms, Addison Wesley, Reading Mass. 1969.

[20] S. LANG, Algebra, Addison Wesley, Reading Mass., 1965.

[21] C. C. MACDUFFEE, The Theory of Matrices, Chelsea, New-York, 1946.

[22] A. S. MORSE, System Invariants Under Feedback and Cascade Control, Proc. of Intern. Symp. Udine, Italy, 1975.

[23] L. PERNEBO, Algebraic Control Theory for Linear Multivariable Systems, Tekn. Dr. Thesis, May 1978.

[24] H. H. ROSENBROCK, State Space and Multivariable Theory, Wiley, New-York, 1970.

[25] H. H. ROSENBROCK, The transformation of strict system equivalence, Int. J. Control, Vol. 25, No.1, 1977.

[26] H. H. ROSENBROCK, G. E. HAYTON, The general problem of pole assingment, Int. J. Control, Vol. 27, No.6, 1978.

[27] R. W. SCOTT, B. D. O. ANDERSON, Least Order Stable Solution of the Exact Model Matching Problem, Automatica, Vol. 14, pp 481-492, 1978.

[28] B. STENSTROM, Rings of Quotients, Springer-Verlag, New-York, 1975.

[29] A. L. VAN DER WARDEN, Modern Algebra, Vol. II, Ungar, 1966.

[30] J. C. WILLEMS, W. H. HESSELINK, Generic Properties of the Pole Placement Problem, Presented at 1978 IFAC Congress, Helsinki, Finland.

[31] W. M. WONHAM, On Pole Assignment in Multi-Input Controllable Systems, IEEE Trans. on AC, Vol. AC-12, No.6, Dec. 1967.

[32] W. M. WONHAM, A. S. MORSE, Feedback Invariants of Linear Multivariable Systems, Automatica, Vol. 8, 1972.

[33] W. A. WOLOVICH, Linear Multivariable Systems, Applied Mathematical Sciences Vol. 11, Springer-Verlag, New-York 1974.

[34] D. YOULA, H. JABR, J. BONGIORNO, Modern Wiener-Hopf Design of Optimal Controllers- Part II: The Multivariable Case, IEEE Trans. on AC, Vol. AC-21, No. 3, June 1976.

[35] O. ZARISKI, P. SAMUEL, Commutative Algebra, Vol I, Van Nostrand, Princeton, 1958.