INFORMATION AND DISTORTION IN FILTERING THEORY

by

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ABSTRACT

The objective of this thesis is to apply Shannon Information Theory to Modern Filtering (estimation) theory. The thesis begins with a comparison of the information theory and filtering problems. From this comparison an information theoretic formulation of the filtering problem (linear and nonlinear) is synthesized that realistically imbeds the filtering problem in an information framework. Subject to a basic but reasonable assumption this formulation explains the useful relation between information and distortion (e.g., MSE) in filtering and makes possible optimum and suboptimum filter design, evaluation and comparison. The theory developed is applied to the linear Gaussian Reduced Order Filter problem where, since computability is elementary, all concepts can be clearly illustrated in a simple example.

Lower bounds are developed for optimum and suboptimum filters based on Shannon's lower bound on the rate distortion function. First application is made to the Reduced Order Filter problem where the easy to compute lower bound gives an indication of how good a suboptimal filter can perform given its information. Secondly a formula for a lower bound on optimum MSE applicable to both the discrete and continuous nonlinear filtering problem is obtained based on the Bucy representation theorem. The formula is then applied to the Phase Locked Loop where it is shown that in the nonlinear, high noise to signal ratio, region the bound performs significantly better than other rate distortion lower bounds.

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TABLE OF SYMBOLS AND ABBREVIATIONS

Symbol	Meaning
a.s.	almost surely
C^n [o,t]	Space of R^n valued continuous functions on [o,t]
$dP_{X\cdot Y}/dP_{X \times Y}$	Radon Nikodym derivative
ε	is an element of
$E\{\cdot\}$, $E\{\cdot\|\cdot\}$	Expectation, Conditional expectation
E_{P_X} or E_x	Expectation with respect to the indicated probability measure
E_{P_Y} or E_y	
$E_{P_{X\cdot Y}}$ or E_{xy}	
E.O.L.	End of Lemma
E.O.T.	End of Theorem
I(x), H(x)	Self information density, Entropy of random variable (random entity) x
$i(x\|y)$, $H(x\|y)$	Conditional self information density, Conditional entropy of random variable (random entity) x given random variable (entity) y
i(x;y), I(x;y)	Information density, Average mutual information of random variables (random entities) x,y.
ITP	Information Theory Problem
ITPWD	Information Theory Problem With Distortion

Symbol	Meaning
MSE	Mean Squared Error
$N(a,B)$	Random variable or vector normal with mean a, covariance B
NLFP	Nonlinear Filtering Problem
$P_{X \cdot Y}$	"Joint" probability measure
$P_{X \times Y}$	Product probability measure
PDF	Probability Density Function
PI	Performance Index
pr	probability
PLLP	Phase Locked Loop Problem
$p(x)$, $p_x(\cdot)$	Probability Density Function of random variable x
$p(x\|y)$, $p_{x\|y}(\cdot\|\cdot)$ $q(x\|y)$, $q_{x\|y}(\cdot\|\cdot)$	Conditional probability density function of x given y
$P(x,y)$, $P_{xy}(.,.)$	Joint probability density of x and y
R^n	Euclidean n dimensional space
R^n_k	Space of sequences with k entires of R^n
ROFE	Reduced Order Filter Example
ROFP	Reduced Order Filter Problem
rv('s)	random variable(s)
$\sigma\{\cdot\}$	σ-field(50) induced by the quantity within the brackets

TABLE OF SYMBOLS AND ABBREVIATIONS (Continued)

' (apostrophe)	Transpose of vector matrix
$\subset, \supset$	Set inclusion in the sense of $\subseteq$, $\supseteq$
(Ω, B_t, P)	Process noise quantities in continuous time (see Section 2.1)
B_t, $Q(t)$	
$(\tilde{\Omega}, \tilde{B}_t, \tilde{P})$	Observation noise quantities in continuous time (see Section 2.1)
B_t, $R(t)$	
$(\underline{\Omega}, \underline{B}_t, \underline{P})$	Probability space formed from Cartesian product of the product of the process and observation noise probability spaces. (see Section 2.1)
P_X, P_Y, P_{xy}	Measures induced by Eqs. (2.1), (2.2) (see Section 2.1)
$P_X^{\mathcal{Y}_t}$	Conditional probability conditioned by σ-field $\mathcal{Y}_t$ (see Section 2.1)
x_a^b	x_s, $a \leq s \leq b$
s_m^n	x_k, $m \leq k \leq n$
$\triangleq$	definition
BMD	Bucy-Mortensen-Duncan representation (see Section 2.1)
x_k^*, x_t^*	Optimal estimate $E(x_t \vert \mathcal{Y}_o^t)$
ε_k^*, ε_t^*	MSE associated with optimal estimate

CHAPTER 1

INTRODUCTION

1.1 Motivation and Objectives

The objectives of this thesis is to apply Shannon Information Theory to Modern Filtering (estimation) theory.

The technological requirements of the second world war brought about the development of two mathematical theories for solving the problem of extracting desirable information out of signals corrupted with undesirable noise. These two approaches, Shannon's Information Theory and Wiener's Filtering Theory, not only shared a common objective but utilized probability theory as a basis for modeling the attendant physical phenomenon. Nevertheless, these two disciplines started and have continued to develop along different paths guided by seemingly different applications, communications and control.

The relation between these two bodies of knowledge in terms of their mutual relevancy has remained to a great extent an unanswered question. The opinions usually given range from "they are essentially the same" to "they are essentially different" to an uncommital "there must be some connection." The question of how precisely do the filtering problem and the information theory problem compare has not been adequately addressed. And in particular the question of how can information theory be profitably used in filtering has not been given careful consideration.

These questions are certainly more than an intelectual curiosity. Thanks to the celebrated works of Kalman and Bucy and that of Kushner and Stratonovich modern filtering theory, liberated from the difficulties of Wiener's formulation, has emerged as the area of modern system theory with perhaps the greatest practical importance. Its versatility and applicability has been proven time and again in diverse physical situations too many to mention in the present context. New applications arise every day.

Success in filtering applications however has been invariably accompanied with considerable difficulties in going from theory to practice which continue to make filter design, evaluation, and comparison an uncertain art. At the heart of these obstacles is a persistent trend in filtering research towards development in the two separate areas of theory and practice. Thus in nonlinear filtering a number of representation theories that characterize the "solution" to the problem have been derived but in practice the optimal estimator cannot be realized nor is it possible to compute the error associated with the optimal solution or that associated with that of the various ad hoc suboptimal filtering schemes (except perhaps through expensive and not always reliable Monte Carlo simulation). In linear filtering an analogous situation exists for hile in theory the Kalman filter provides the solution to the problem, in practice the optimal estimator cannot be realized due to excessive computational requirements or poor sensitivity to modeling errors. The dichotomy between theory and practice can therefore be attributed to the lack of a general framework into which optimal and suboptimal filtering can be imbedded and within which suboptimal filter design, evaluation, and comparison can be straightforwardly executed.

The objectives of this study are (1) to analyze the difference between the information theory and filtering problem; and (2) based on this analysis, to apply information theory to practical filter design, evaluation, and comparison. We shall see that the central question is the relation between information and distortion in the filtering context and that information theory, when properly used, can provide a general framework for optimum and suboptimum, linear and nonlinear filtering.

1.2 Chapter by Chapter Summary

The specific ways in which the study of information and distortion in filtering is conducted is as follows.

Chapter 2 begins with a summary of the main concepts of information theory followed by a description of the various information theory problems. The object of the presentation is to precisely delineate the features that characterize the information theory problem and thus lay the groundwork for a critical comparison in Chapter 3 of the information and filtering problem. Also included in this chapter is a short survey of the work that has appeared in the literature on the subject of information and filtering.

Chapter 3 contains the basic results of this study upon which subsequent chapters elaborate. First, based on the conclusions of Chapter 2, the differences between the information theory and filtering problems are first presented. From this comparison an information theoretic formulation of the filtering problem* is synthesized that realistically imbeds the filtering problem in an information framework. Subject to a basic but reasonable assumption this formulation explains the useful relation between information and distortion (e.g. MSE) in filtering and makes possible filter design, evaluation, and comparison based on information. Also included in this chapter are lowerbounds on distortion (e.g. MSE) for optimum and suboptimum filters based on Shannon's lowerbound on the rate distortion function.

Chapter 4 considers the linear gaussian reduced order filter problem where computation of the various information quantities is elementary. Most of the development is carried out in terms of a simple numerical example for which it is easy to show that the necessary assumptions hold and which clearly

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*"Filtering problem" is taken to include both linear and nonlinear filtering.

illustrates the relation between information and distortion in filtering and thus the concepts and methods of filter design, evaluation, and comparison developed in Chapter 3. A general lowerbound on MSE for the reduced order filter problem is also derived based on the bounds derived in Chapter 3.

In Chapter 5 a specific formula is derived for the lowerbound on optimum filtering MSE of Chapter 3 applicable to both the continuous and discrete time nonlinear filtering problem. This formula offers several advantages over the Zakai-Ziv bound (38) as discussed in Section 5.2.

Finally in Chapter 6 the results of Chapter 5 are applied to the phase locked loop and it is seen that in the nonlinear high noise to signal ratio, region the performance of the lowerbound of Chapter 5 is superior to that of the Zakai-Ziv bound.

1.3 General Remarks

Historically this study was motivated by the paper by Zakai and Ziv "Lower and Upper Bounds on Optimal Filtering Error of Certain Diffusion Processes" (38) which is summarized in Section 5.1. Originally the object was to generalize the bound in (38) to produce an improved version of the rate distortion bound. This in turn led to the question of what could be a meaningful way to relate information and error in filter design. Special mention must also be made of the Ph.D. Thesis by T. Duncan (3) where the Bucy representation is given a careful and complete analysis and which influenced the development of Chapter 5.

In regard to mathematical sophistication, rigor has been sacrificed in places where it would stand in the way of concepts and make the presentation lengthy with irrelevant technicalities. On the other hand, advanced mathematics has been used whenever necessary.

A special effort has been made to clearly label all results taken from the literature with the name of the relevant source. In particular lemmas and theorems available in the literature are accordingly referenced (e.g. "Lemma 1 (Bucy-Mortesen-Duncan)").

Finally we call attention to the existence of the Table of Symbols and Abbreviations as well as the Table of Lemmas, Theorems, Corollaries, and the Bibliography.

CHAPTER 2

INFORMATION AND FILTERING: A SURVEY

This chapter is basically a survey of relevant concepts in filtering and information that we shall need, and of the work that has appeared in the literature relating these two disciplines. It is written primarily for the filtering specialist who has had little acquaintance with information theory.

In Section 2.1 the filtering problem is precisely defined and as a byproduct the notation that will be used in the sequel is introduced. Section 2.2 contains a brief exposition of the elements of Shannon's Theory of Information. The presentation includes both intuitive and formal definitions of the basic information and the rate distortion function as well as a summary of the central results in information theory -- the various coding theorems. Finally a survey of the various contributions that have appeared in the literature on the subject of information and filtering is given in Section 2.3 together with a discussion of the difficulties common to these studies. In the next chapter we shall show how these difficulties can be overcome by properly formulating the filtering problem.

2.1 The Filtering Problem

In this section we first define precisely the nonlinear filtering problem (NLFP) in continuous and discrete time and then give their "solution" in terms of the Bucy-Mortensen-Duncan representation theorem. While there are other continuous time representation results available (e.g. Kushner (1), Frost-Kailath (2) we choose to present the Bucy-Mortensen-Duncan representation because we will use it later in Chapters 3 and 5. Finally what we will refer to as "the filtering problem" is defined. Unless otherwise stated states, observations, etc. are vectors of arbitrary and compatible dimensions.

The NLFP: Continuous Time (See, e.g., (3))

(1) Let $C^n[0,t]$ be the space of R^n valued continuous functions. Let

$$\underline{\Omega} = \Omega \times \tilde{\Omega} = C^n[0,t] \times C^n[0,t] \ .$$

Let

$$\{B_s(\omega) = \omega_s : s \varepsilon [0,t]\} \quad \text{on} \quad (\Omega,\ B_t,\ P)$$

$$\{\tilde{B}_s(\tilde{\omega}) = \tilde{\omega}_s : s \varepsilon [0,t]\} \quad \text{on} \quad (\tilde{\Omega},\ \tilde{B}_t,\ \tilde{P})$$

be independent Brownian motions with parameters $Q(t)$, $R(t)$ respectively and take as the basic probability space

$$(\underline{\Omega},\ \underline{B}_t,\ \underline{P}) = (\Omega \times \tilde{\Omega},\ B_t \times \tilde{B}_t,\ P \cdot \tilde{P})$$

(2) Consider the Ito equations

$$dx_t = a(x_t,\ t)\ dt + b(x_t,\ t)\ dB_t \qquad (2.1)$$

$$dy_t = g(x_t,\ t)\ dt + d\tilde{B}_t \qquad (2.2)$$

where the usual conditions required for the processes x_t, y_t to exist, to be unique, and to be diffusion processes are assumed: a, b, g are measurable in both arguments and satisfy a global Lipshitz condition on the first argument.

(3) The problem then is to find the functional $\hat{x}_t$ of the observation process $\{y_s, s\varepsilon[0,t]\}$ that minimizes the mean-squared error (MSE)

$$\varepsilon_t = E_{\underline{P}} \{(x_t - x_t)(x_t - x_t)'\}.$$

The discrete time nonlinear filtering problem is analogous to the continuous time problem except that the various conditions can be relaxed since existence and uniqueness do not present a problem.

The NLFP: Discrete Time (See e.g.(4))

(1) Let R_k^n be the space of sequences with k entries in R^n. Let

$$\underline{\Omega} = \Omega \times \tilde{\Omega} = R_k^n \times R_k^n$$

Let

$$\{B_j(\omega) = \omega_j \mid j\varepsilon\ [0,k]\} \quad \text{on } (\Omega,\ B_k,\ P)$$

$$\{\tilde{B}_j(\tilde{\omega}) = \tilde{\omega}_j \mid j\varepsilon\ [0,k]\} \quad \text{on } (\tilde{\Omega},\ \tilde{B}_k,\ \tilde{P})$$

be sequences of independent Gaussian random variables such that for i, $j\varepsilon[0,k]$,

$$E\{B_i B_j\} = Q_i \ \delta(i-j), \quad E\{\tilde{B}_i \tilde{B}_j\} = R_i \delta(i-j)$$

$$E\{B_i \tilde{B}_j\} = 0 \quad \text{for all } i, j$$

where

$$\delta(i-j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

Take as the basic probability space

$$(\underline{\Omega}, \underline{B}^k, \underline{P}) = (\Omega \times \tilde{\Omega}, B^k \times \tilde{B}^k, P \cdot \tilde{P})$$

(2) Consider the difference equations

$$x_{j+1} = a(x_j, j) + b(x_j, j)B_j \tag{2.3}$$

$$y_{j+1} = g(x_{j+1}, j+1) + \tilde{B}_{j+1} \tag{2.4}$$

where a, b, g are assumed to be measurable in both arguments.

(3) The problem is to find the functional $\hat{x}_k$ of the observation sequence $\{y_j, j \varepsilon [0,k]\}$ that minimizes the MSE

$$\varepsilon = E_{\underline{P}}(x_k - \hat{x}_k)(x_k - \hat{x}_k)'$$

As is well known, the solution to the NLFP in both continuous and discrete time can be expressed in terms of the conditional expectations

$$\hat{x}_t = E(x_t | \mathcal{Y}_o^t) \quad , \quad \hat{x}_k = E(x_k | \mathcal{Y}_o^k)$$

where $\mathcal{Y}_o^t$ and $\mathcal{Y}_o^k$ are the sub-σ-fields of $\underline{B}^t$, $\underline{B}^k$ induced by the observation processes $\{y_s, s\varepsilon[0,t]\}$, $\{y_j, j\varepsilon[0,k]\}$ respectively. These conditional expectations can be expressed (leaving all technical difficulties aside) in terms of the conditional densities $p(x_t|y_o^k)$, $p(x_k|y_o^k)$ and these densities can in turn be obtained from a representation theorem first proposed by Bucy(5) in terms of a discrete to continuous time limit. The theorem was rigorously proved by Mortensen(6) without recourse to "discrete-continuous arguments" but under severe restrictions (which, for example, excluded linear systems). Duncan (3) removed these restrictions. The theorem has also been proved by Wonham (7) for the case of Markov chains and by Kaliampur, Striebel, and Fujisaki (8, 9, 10, 11, 12).

The Bucy-Mortensen-Duncan representation theorem which we now present is central to both the information theoretic formulation of the filtering problem that we introduce in Chapter 3 as well as for the rate distortion lower bound on filtering MSE that we derive in Chapter 5. The continuous time version (Lemma 1, below) is based on Duncan (3) while the discrete time version (Lemma 2) is taken from Bucy (4). The proofs of these theorems can be found in the indicated references -- the discrete time proof follows almost immediately from Bayes rule while the continuous time version is considerably more difficult (see Jazwinski (13) for an intuitive demonstration).

Lemma 1 (Bucy-Mortensen-Duncan, (5), (6), (3)). Continuous time NLFP defined above. Let P_X be the measure induced on B_t by Eq. 2.1; P_y be the measure induced on $\tilde{\underline{B}}_t$ bt Eqs. 2.1, 2.2. Let $y_t = \sigma\{y_s, s\varepsilon[0,t]\} \subset \underline{B}_t$. Then,

$$\frac{dP_X^{y_t}}{dP_X} = \frac{E_{P_X}\{\psi_t \mid x_t = x\}}{E_{P_X}\{\psi_t\}} = \frac{p(x,t\ a,0,y_s,s\varepsilon[0,t])}{p(x,t|\alpha,o)} \tag{2.5}$$

where:

$$\psi_t(x_s, y_s, s\varepsilon[0,t]) = \exp\{\zeta_t(x_s,y_s,s\varepsilon[0,t])\}$$

$$\zeta_t(x_s,y_s\ s[0,t]) = \{\int_0^t g(s,x_s)'R^{-1}(s)dy_s$$
$$-\tfrac{1}{2}\int_0^t g(s,x_s)'R^{-1}(s)g(s,x_s)ds\}$$

$$P_X^{y_t}(A) = \int_B p(x,\ t|\alpha,0,y_s,s\varepsilon[0,t])dx$$

$$P_X\ (A) = \int_B p(x,t|\alpha,0)dx$$

for A εB_t, B ε the Borel field and the derivative on the left hand side of (2.5) is a Radon-Nikodym derivative (50,78,93). E.O.L.

Lemma 2 (Bucy, (4)). Consider the discrete time NLFP defined above. Let P_X be the measure indiced on B_k by Eq. (2.3): P_y be the measure induced on $\tilde{B}_k$ induced by Eqs. (2.3),(2.4). Let

$\mathcal{Y}_k = \sigma\{y_j, j \; \varepsilon[0,k]\}$. Then

$$\frac{dP_X^{\mathcal{Y}_k}}{dP_X} = \frac{E_{P_X}\{\psi_k \mid x_k = x\}}{E_{P_X}\{\psi_k\}} = \frac{p(x,k \mid \alpha,0,y_j,j\varepsilon[0,k])}{p(x,k|\alpha,0)}$$

where

$$\psi_k\,(x_j,y_j,\; j\;\varepsilon[0,k]) = \exp\,\{\zeta_k(x_j,y_j,j\varepsilon[0,k])\}$$

$$\zeta_k(x_j,y_j,j\varepsilon[0,k]) = \sum_{j=0}^{k} y_j' R^{-1}(j) g(x_j,j)$$

$$-\tfrac{1}{2}\sum_{j=0}^{k} g(x_j,j)' R^{-1}(j) g(x_j,j)$$

$$P_X^{\mathcal{Y}_k}(A) = \int_B p(x,k|\alpha,0,y_j,j\varepsilon[0,k])\;dx$$

$$P_X\;(A) = \int_B p(x,k|\alpha,0)\;dx$$

for $A\varepsilon B_k$, $B\varepsilon$ the Borel field. E.O.L.

Except in very simple cases (14,15), the Bucy-Mortensen-Duncan representation theorems, cannot be used to compute the optimal estimate. However, the characterization it offers is, as will become apparent in later chapters, particularly suited to the information study of both the NLFP and the Filtering Problem which we now define.

The Filtering Problem. The term "Filtering Problem" will be used here to denote the situation where the functional of the data that defines the estimate $\hat{x}_t$ or $\hat{x}_n$ may be constrained to a given class (e.g., when the filter is constrained to have a specific structure such as being recursive or of a certain dimension). Compare with the NLFP where virtually no constraint is placed on the class to which the filter belongs.

Having introduced the pertinent definitions and results of filtering theory that we shall need in the sequel we now proceed to do likewise with information theory.

2.2 Elements of Shannon Information Theory

In this section we give a brief introduction to the main concepts of Shannon information and rate distortion theory. The presentation starts with a simple and intuitive definition of information in subsection 2.2.1 followed by more general and rigorous definitions in subsection 2.2.2. In subsection 2.2.3 the basic information theory problem is defined and its central result (the various coding theorems) is presented. Next in subsection 2.2.4 we introduce the rate distortion function and describe its role in information theory. Finally in subsection 2.2.5 a modification of the information theory problem which arises when the are "remote" sources and users is presented.

It is important to emphasize that the choice in the aspects of information theory presented in this section is determined by what is needed in the remainder of this study.

In particular the concepts introduced in this section play a key role in the logical development that leads to the synthesis in Chapter 3 of a realistic and useful information theoretic formulation of the filtering problem. Naturally, then, no attempt is made at presenting a comprehensive treatise of information theory, this being readily available in standard texts (16,17,18,19,20) and the original work of Shannon (21,22).

2.2.1 Intuitive Definition of Information

This section borrows heavily from Carlson (23) and also from Gallagher (16).

Shannon Information Theory is based on a precise yet intuitively appealing quantizing of uncertainty. That is, the information associated with an event is intended to be a measure of the uncertainty of the event, where in turn the uncertainty of the event is measured by its probability of occurrence.

Consider a set of events {A,B,C,...} with which we have associated corresponding probabilities of occurrence $\{P_A, P_B, P_C, \ldots\}$ and suppose we wish to find a function f defined on these probabilities that will give us a measure of the information $\{I_A = f(P_A),\ I_B = f\ (P_B),\ I_C = f(P_C)\ldots\}$ contained in these events -- i.e., the self-information of these events. It seems "reasonable" to require that f have the following properties:

(1) That f be always positive.

(2) That if $P_A = 1$ (no uncertainty associated with event A) then $I_A = 0$ (there is no information contained in the event)

(3) That if $P_A > P_B$ (event B more uncertain that A) then $I_B > I_A$ (event B has more self-information than event A)

(4) If events A and B are independent ($P(A,B)=P(A)\cdot P(B)$) then the information contained in the event (A,B) is simply the sum $I_A + I_B$ of the informations associated with events A and B. The only function f that satisfies the above properties is

$$I_A = f(P_A) = -\log P_A = \frac{1}{P_A}$$

and I_A is called the self-information associated with event A.

Generalizing the concept of self-information a bit, consider the random variable $x(\omega)$ which takes values in the alphabet set say that the self-information contained in the event $\{x(\omega)=j\}$, $j\varepsilon J$ is given by

$$I_x(x(\omega)=j) = -\log P_x(x(\omega)=j)$$

and we can define the average self-information of the alphabet J (corresponding to the probability assignment P_x) or associated with the random variable x as

$$H(J) \text{ or } H(x) = \sum_{j\varepsilon J} [-\log P_x(x(\omega)=j)] \quad P(x(\omega)=j)$$

$H(x)$ or $H(J)$ is called the entropy of x or J and the random variable $I(x(\omega))$ is called the self-information of x.

Suppose now that jointly defined with x we have another random variable y taking up values in the alphabet set $K = \{1,2 \ldots m\}$ with marginal probability assignment P_y, joint probability assignment $P_{x \cdot y}$, and conditional assignment $P_{x|y}/P_y$. We can then define the conditional self-information and conditional entropy (average conditional self-information) of x given y as:

$$I(x(\omega)|y(\omega)) = -\log P_{x|y}(x(\omega)|y(\omega))$$

$$H(x|y) = -\sum_{j \varepsilon J} \sum_{k \varepsilon K} \log P_{x|y}(x(\omega)=j|y(\omega)=k)$$

$$P_{x \cdot y}(x(\omega)=j, y(\omega)=k)$$

Just as the self-informationof x was a measure of the uncertainty contained in x, the self-information of x given y is a measure of the uncertainty about x remaining after y is given or observed.

Consider now the difference $\Delta = H(x) - H(x|y)$. This is the difference between the apriori (average) uncertainty in x and the remaining uncertainty in x once y has been observed -- i.e., Δ is a measure of the reduction in uncertainty in x caused by the observation or a measure of the information about x contained in y. It is called the average mutual information between x and y,

$$I(x;y) = H(x) - H(x|y)$$

and

$$i(x(\omega);\ y(\omega)) = I(x(\omega)) - I(x(\omega)|y(\omega))$$

is called the mutual information or information density. A little arithmetic shows that

$$i(x(\omega);y(\omega)) = \log \frac{P_{x|y}(x(\omega)|y(\omega))}{P_x(x(\omega))} = \log \frac{P_{x\cdot y}(x(\omega),y(\omega))}{P_x(x(\omega))P_y(y(\omega))}$$

and similarly

$$I(x;y) = \sum_{j\varepsilon J} \sum_{k\varepsilon k} \left[\log \frac{P_{x|y}(x=j|y=k)}{P_x(x=j)}\right] P_{x\cdot y}(x=j,y=k)$$

$$= \sum_{j\varepsilon J} \sum_{k\varepsilon k} \left[\log \frac{P_{x\cdot y}(x=j,\ y=k)}{P_x(x=j)\ P_y(y=k)}\right] \cdot P_{x\cdot y}(x=j,\ y=k)$$

The average mutual information I(x;y) (often called the mutual information or just information for short) is the fundamental quantity in information theory. In fact the other quantities can be expressed in terms of mutual information e.g., H(x)≡Ix;x)) or in terms of conditional mutual information

$$i(x(\omega);y(\omega)|z(\omega)) = \log \frac{P_{x\cdot y\ z}(x(\omega),y(\omega)|z(\omega))}{P_{x|z}(x(\omega)|z(\omega))P_{y|z}(y(\omega)|z(\omega))}$$

$$I(x;y|z(\omega)) = \sum_{j\varepsilon J} \sum_{k\varepsilon k} \log \frac{P(x=j,y=k|z(\omega))}{P(x=j|z(\omega))\ P(y=k|z(\omega))} \cdot$$

$$P(x=j\cdot y=k|z(\omega))$$

$$I(x;y|z) = \sum_{\ell \varepsilon L} I(x;y|z=\ell)\ P_z(z=\ell)$$

(e.g. $H(x|y)=I(x;x|y)$) where z is another random variable jointly defined with x, y and taking values in the alphabet set L.

Mutual information has the following properties:

$I(x;y)=I(y;x)$

$i(x;y) \geq 0$ with probability one

$I(x;y) \geq 0$

$I(x;y) \leq H(x)$

Similar properties can be derived for conditional mutual information.

The purpose of this section has been to give an intuitive understanding of what the quantities arising in Shannon Information Theory represent. We may summarize the preceding discussion as follows:

$H(x)$ = [Average Self-Information of x] = [Total information required to eliminate uncertainty about x]

$I(x;y)$ = [Average mutual info between x and y] = [Info provided about x by y]

$H(x|y)$ = [Average conditional entropy of x given y] = [Info <u>not</u> provided about x by y]

= [Uncertainty about x remaining after y has been observed]

A more extensive introduction to Shannon information can be found in many places in the literature, for example;

Gallagher (15) Chapter

Carlson (23) Chapter 8

Sakrison (17) Chapter 6

Toby Berger (20) Chapter 2

2.2.2 Formal Definition of Information

This section is based on Sakrison (17) and Pinsker (19). In the preceding section we defined the various information quantities associated with random variables x, y, z -- i.e., with real valued measurable functions (a.s. finite) on some probability space. We now wish to extend these definitions to random variables taking values in arbitrary spaces; more precisely to random entities of countable dimension (Sakrison (17)) such as a random vector, a random sequence, or a separable stochastic process. Thus, taking $(\Omega, S_\omega, P_\omega)$ as our basic probability space we will speak of the measurable maps x, y, z, etc.,

$$x: (\Omega, S_\omega, P_\omega) \longrightarrow (X, S_x, P_x)$$

$$y: (\Omega, S_\omega, P_\omega) \longrightarrow (Y, S_y, P_y)$$

$$z: (\Omega, S_\omega, P_\omega) \longrightarrow (Z, S_z, P_z)$$

where X, Y, Z are the spaces where x, y, z take sample values, S_x, S_y, S_z are associated σ-fields, and P_x, P_y, P_z are the induced measures (e.g., $P_x = P_\omega \cdot x^{-1}$). We will also speak of maps such as

$$(x,\ y):(\Omega,\ S_\omega,\ P_\omega) \longrightarrow (X \times Y,\ S_x \times S_y,\ P_{x \cdot y})$$

One way to extend the definition of I(x:y) from random variables to arbitrary random entities is to consider a sequence of partitions, say Π_n, composed of rectangles of X × Y, ordered by refinement as done by Gelfand-Yaglom(24). We thus have associated with partition Π^n

$$I^n(x;y) = \sum_i \sum_j \left[\log \frac{P_{xy}(A_i,\ B_j)}{P_x(A_i)P_y(B_j)} \right] P_{xy}(A_i,\ B_j) \tag{2.6}$$

I(x;y) can be defined in terms of this sequence (which is monotonically increasing) as

$$I(x;y) = \sup_{\text{all } \Pi^n} I^n(x;y) \tag{2.7}$$

whenever the sup exists.

This definition can be shown to be equivalent with one that makes the concept of mutual information considerably clearer thanks to the theorem of Dobrushin. Dobrushin's thorem states that we need not look at rectangles but only at the measurable sets so that in effects mutual information is a property of the probability spaces involved:

Lemma 3 (Dobrushin, 19) Let $\mathcal{L}$ be a field (50) that generates $S_x \times S_y$ and let R be a family of partitions of X × Y whose elements belong to $\mathcal{L}$. The I(x;y) as defined by Eqs. (2.6), (2,7) is given by

$$I(x;y) = \sup_{\{E_i\}\varepsilon R} \sum_i \left[\log \frac{P_{xy}(E_i)}{P_{x \times y}(E_i)}\right] P_{xy}(E_i)$$

provided every partition of sets of $\mathcal{L}$ has a subpartition that belongs to R (where $P_{x \times y}$ is the product measure).

E.O.L.

<u>Lemma 4</u> (Gelfand-Yaglom (24), Perez (25)). If P_{xy} is absolutely continuous with respect to $P_{x \times y}$ then

$$I(x;y) = \int_{X \times Y} \log \frac{dP_{x \cdot y}}{dP_{x \times y}} (x,y)\ P_{x \cdot y}\ P_{x \cdot y}(dx,dy) \qquad (2.8)$$

Otherwise $I(x;y) = \infty$.

E.O.L.

A similar development can be made for conditional mutual information resulting in the following analogue to Lemma 4.

<u>Lemma 5</u> (Dobrushin, (19)). If $P_{x \cdot y|z}$ is absolutely continuous with respect to $P_{x \cdot y|z}$ then

$$I(x;y|z) = \int_{X \times Y \times Z} \log \frac{dP_{x \cdot y|z}(x,y,z)}{dP_{x \times y|z}} P_{x \cdot y \cdot z}(dx,dy,dz) \qquad (2.9)$$

provided the conditional pr's are regular (see Translation remarks to Pinsker(19) Chapter 3). Otherwise $I(x;y|z) = \infty$.

E.O.L.

There are many interesting and useful relations that can be derived in terms of mutual information and conditional mutual information for abstract valued random variables. (See for example Pinsker.) Among the properties, two that we shall need in what follows are Kolmogorov's formula and a result on the invariance of information due to Gelfand-Yaglom.

Lemma 6 (Kolmogorov). Let x, y, z be random entities of countable dimension as defined above. Then

$$i(x;\, y,\, z) = i(x;\, y|z) + i(x;z) \quad \text{a.s.}$$
$$I(x;\, y,\, z) = I(x;\, y|z) + I(x;z)$$

E.O.L.

Lemma 7 (Gelfand-Yaglom, (24)). Let x, y, z be random entities of countable dimension as defined above. Further suppose that $z=A \cdot y$ where $A\colon (Y,S_y) \longrightarrow (Z,S_z)$ is measurable. Then

$$I(x;y) \geq I(x;z) \equiv I(x;A \cdot y) \tag{2.10}$$

and the equality holds if A is one-one.
E.O.L.

The inequality in this last lemma makes sense in the light of Dobrushin's Theorem (Lemma 3, above) since clearly $S_y \supset A^{-1}.\ S_z$. The sufficiency of one-oneness is equality clear. The question naturally arises as to whether one-oneness is also necessary for the equality to hold. It turns out that one-oneness can be relaxed by introducing the concepts of subordination and everywhere dense. These concepts are more natural in information than the concept of measurability and we shall find them useful in later sections.

Consider the random entities x, y as defined above*. y is subordinate(19) to x if $\sigma\{x\} \subset s_\omega$ is P_ω-everywhere dense in $\sigma\{y\}$. y is everywhere dense (19) in x if (1) y is a measurable function of x and (2) is subordinate to y.

Observe that

$$[y \text{ e.d. in } x] \Longrightarrow [y \text{ is a meas. function of } x] \Longrightarrow [y \text{ subordinate to } x]$$

so that subordination and denseness are weaker and stronger respectively than measurability. Also observe that "y everywhere dense in x" is stronger than y and x mutually subordinate. Mutual subordination in fact allows us to relax the condition for equality in Lemma 7 as evident from the following lemma.

Lemma 8 (Pinsker 19). As far as information relations are concerned, mutually subordinate random entities may be considered equivalent in the sense that replacing one of them by

*Recall that the notation $\sigma\{x\}$ means sigma field induced by x. For a more detailed exposition of the concepts of subordination and everywhere dense see Pinsker (19).

the other effects no change in such a relation.
E.O.L.

Thus in Lemma 7 we can replace "A measurable and one-one" with the weaker "y and $z=A\cdot y$ mutually subordinate". This is still a sufficient condition, however, and it seems that a necessary and sufficient condition for equality in Eq. (2.9), if one exists, must involve as well the random entity x.

Intuitively, y subordinate to x implies that with knowledge of x, y can almost be determined so that if x is given, y provides almost no information. In precise terms, then, we have the following lemma which we shall need later.

Lemma 9 (Pinsker, 19). Let x, y, z be random entities as defined above and suppose y is subordinate to x. We then have

$$i(y,z|x) = 0 \quad \text{a.s.}$$

$$I(y,z|x) = 0 \tag{2.11}$$

E.O.L.

Having considered in this section the basic information quantities and some of their properties we now proceed to outline how they cna be used in the definition and solution of the information theory problem.

2.2.3 Main Results of Information Theory (16,23)

Since Shannon's formulation in the 1940's, a considerable amount of scientific work has been done in developing and applying information theory. As in the previous sections, our objective here is not to give a complete survey of important results in information theory, but rather to bring out the main results that characterize information theory and that will be useful later on in our study. In particular we wish to bring out the basic features of information theory that would permit a meaningful comparison with filtering and allow a formulation of the filtering problem in an information theoretic context. As it was done in Section 2.1 for the filtering problem, we begin in this section with a formulation of the Information Theory Problem (ITP) and then follow it by the problem's "solution."

The information theory problem is illustrated in Fig. 2.1 in its simplest form and can be stated as follows:

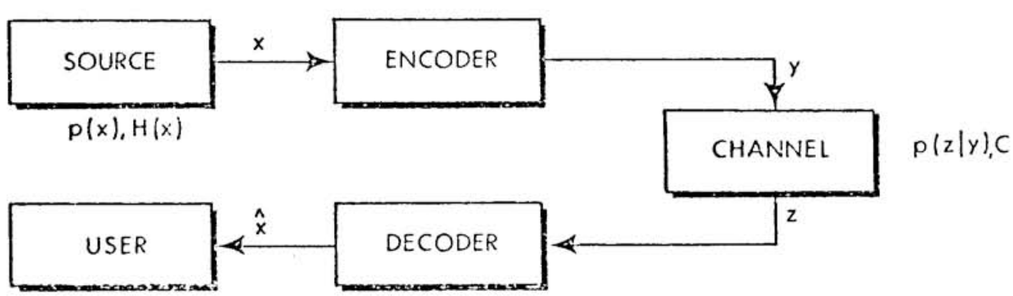

Figure 2.1 Simplified Information Theory Problem

Given: A probabilistic source whose output is the random variable x with* PDF p(x).

A probabilistic transition called the channel described in terms of the conditionsal PDF $p(z|y)$.

Find: The encoder and decoder which, if possible, will achieve zero (i.e., arbitrarily small) probability of error in reproducing x as $\hat{x}$ to the user.

It turns out that the solution to the information theory problem (which will be given below after introducing a model more realistic than that of Fig. 2.1) can esentially be expressed in terms of the information quantities defined in the previous section. Specifically for the purposes of the solution to the theory problem the source can be characterized by its entropy H(x) and the channel can be characterized by its capacity C which in the elementary case is given by (the maximization is over all possible channel inputs as characterized by their PDF's p(y))

$$C = \max_{p(y)} (y;z)$$

$$= \max_{p(y)} \int_y \int_z \left[\log \frac{p(z|y)}{p(z)}\right] p(z|y)p(y)\ dy\ dz$$

where p(z) is clearly a funciton of p(y) (Bayes rule). A similar definition can be given for the case when y, z are vectors

*PDF stands for probability density function.

and for the case when y, z are random processes in terms of the appropriate Radon-Nikodym derivative.

A more realistic version of the information theory problem is illustrated in Fig. 2.2 where time enters into the picture and the encoder and decoder have been divided into source and channel encoders and decoders with respective buffer memories. The source produces for simplicity independent indentically distributed random variables with entropy H(x) at the rate of r=1 letter per second. A block of L of these letters are stored in the buffer and then this block gets passed to the source encoder which converts this sequence into binary data. That this conversion can be accomplished without distortion is guaranteed by the important coding theorem.

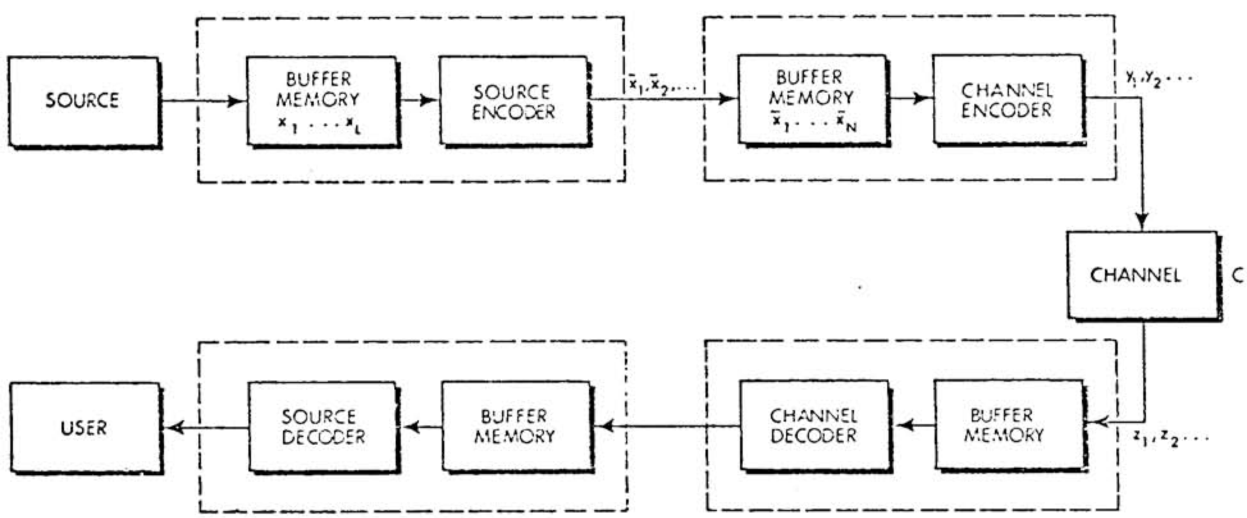

Figure 2.2 More realistic version of Information Theory Problem

Shannon Source Coding Theorem for discrete memoryless sources (21, 16). The block of source letters $x_1 \ldots x_L$ (Fig. 2.2) can be converted by an appropriate source encoder into a sequence of $L' = L \cdot H(x)$ binary digits with zero probability of error provided L is sufficiently large.

Next consider the channel encoder. The buffer stores a block of N binary letters and then passes it to the channel encoder whose purpose is to code this sequence into signals to be transmitted through a possibly noisy channel, in such a way that they can be decoded and delivered to the user with zero probability of error. That this is possible is assured by the celebrated Shannon Channel Coding Theorem which is the fundamental result of information theory when no distortion is allowed.

Main Result of Information Theory (no distortion case): Shannon Channel Coding Theorem. There exists a channel encoder such that transmission through the possibly noisy channel can be achieved with arbitrarily small probability of error provided $H(x) \leq C$ and the block length N is sufficiently large. More specifically the probability of error P_e is bounded by

$$P_e \leq \exp\,[-N\, f(H,C)]$$

where f(H,C) is a function only of the source entropy* and the channel and has the general properties illustrated in Fig. 2.3, so that for $H \leq C$ it is possible to make $P_e \rightarrow 0$ by choosing N large enough.

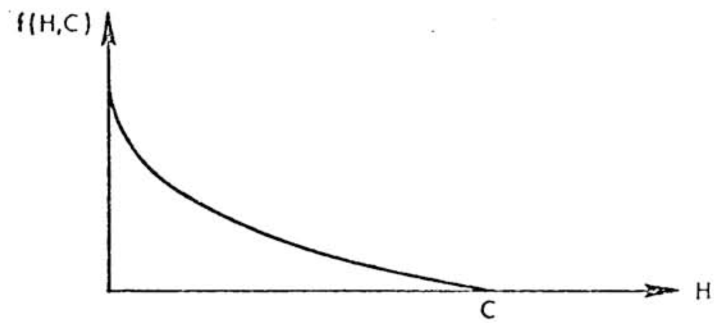

Figure 2.3 Sketch of f(H,C) as a Function of H

The channel coding theorem is a remarkable result since, when taken together with the source coding theorem, it given conditions in terms of information quantities under which it is possible to have error free communication even in the presence of noise by choosing an appropriate coder-decoder combination and using large enough sequences. In fact, the starting point of information theory problems is to prove a specialized version of the coding theorem directly applicable to the particular situation in order to assure the feasibility of approximately solving the problem by the various coding techniques.

<u>The purpose of the above presentation has been to bring out five specific characteristics for our future use of</u>

*Actually a function of the entropy rate rH but here r=1 letter/sec.

the information theory problem (ITP) which can be summarized as follows:

1. In the ITP both the encoder and decoder are to be designed.
2. The performance index in the ITP is an interval* performance index.
3. The solution to the ITP is asymptotic in nature in the sense that it holds for block lengths approaching infinity.
4. The solution to the ITP is an existence result (and no technique presently exists of synthesizing this solution).
5. Perfect communication is the object of the ITP considered in this section.

In the next section we consider the ITP problem where less than perfect communication is allowed -- i.e. a certain amount of distoriton is tolerable. In order to study this situation we introduce Shannon's Rate Distortion Theory. We will find that analogous "coding theorems" can be formulated and that conclusions analogous to 1, 2, 3, and 4 in the previous paragraph will be reached.

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*For a discussion of interval vs. point performance indices see Van Trees (42).

2.2.4 Rate Distoriton Theory (20, 16, 17, 22)

The most comprehensive treatment of rate distortion theory available is Berger(20). This section is based on this as well as on Sakrison(17) and Gallagher(16). Shannon's fundamental paper is reference (22).

Rate Distortion Theory treats the case of the ITP where a certain level of distortion is tolerable. Naturally the starting point is to define distortion and in order to do this consider Fig. 2.4. The distortion measure between x_1 and its reproduction

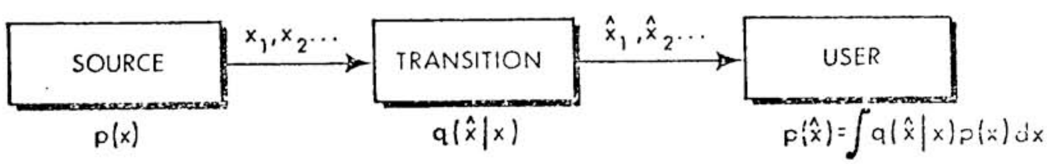

Figure 2.4

$\hat{x}_1$ is simply defined as a function $d_1(x_1, \hat{x}_1)$ taking up values in the nonnegative reals. The particular distortion measure we will be interested in is the squared error,

$$d_1(x_1,\hat{x}_1) = (x_1 - \hat{x}_1)^2$$

but there are many other possible choices (see subsection 3.3.3).

In a more general context we need a measure of distortion between words $\underline{x} = \{x_1,...,x_n\}$, $\hat{\underline{x}} = \{\hat{x}_1,...,\hat{x}_n\}$, say $d_n(\underline{x},\hat{\underline{x}})$. The set F_d of all possible $d_n(\underline{x}, \hat{\underline{x}})$,

$$F_d = \{d_n(\underline{x}, \underline{\hat{x}}), \qquad 1 \leq n \leq \infty\}$$

is called a fidelity criterion. One possible way to define d_n is in terms of a function $d(x_k, \hat{x}_k)$,

$$d_n = \frac{1}{n} \sum_{k=1}^{n} d(x_k, \hat{x}_k)$$

in which case F_d is called a single letter fidelity criterion. Since d and d_n are random variables, we can form their averages,

$$\varepsilon = \int \int d(x_k, \hat{x}_k)\, p(x_k, \hat{x}_k)\, dx_k\, d\hat{x}_k$$

$$\varepsilon_n = \int \int d(\underline{x}, \underline{\hat{x}})\, p(\underline{x}, \underline{\hat{x}})\, d\underline{x}\, d\underline{\hat{x}}$$

Observe that for a given source since $p(x_k, \hat{x}_k) = q(\hat{x}_k | x_k)\, p(x_k)$, ε and ε_n are functions of the conditional PDF q:

$$\varepsilon = \varepsilon(q), \qquad \varepsilon_n = \varepsilon_n(q)$$

The <u>information theory problem with distortion (ITPWD)</u> can be defined as follows:

Given: A source and a fidelity criterion

Find: A communication system that achieves average distortion (ε or ε_n) D.

Thus we are interested in delivering to the user the $\underline{\hat{x}}$ sequence of Fig. 2.4 with no more distortion than, say, D. The question is then how much information need $\underline{\hat{x}}$ have about $\underline{x}$ in order that the distortion does not surpass D. The answer can be given in terms of the rate distortion function R(D). In the case where the sequence x_1, x_2,... is composed of independent identically distributed random variables we can define R(D), the rate distortion of the source relative to fidelity criterion F_d, in terms of the PDF of one single random variable:

$$R(D) = \min_{q \varepsilon Q_D} I(x;\hat{x}) \tag{2.12}$$

$$Q_D = \{q(x|\hat{x}) \ \varepsilon \text{ conditional PDF } ; \ \varepsilon(q) \leq D\}$$

In the case that the sequence x_1, x_2,... is not formed of independent identically distributed rv's

$$R_n = \min_{q \varepsilon Q_D} \frac{1}{n} I\ (\underline{x};\underline{\hat{x}}) \tag{2.13}$$

$$Q_D = \{q(\underline{x}|\underline{\hat{x}}) \ : \ \varepsilon_n(q) \leq D\}$$

where $\underline{x}$ and $\underline{\hat{x}}$ have n letters each and the rate distortion function becomes

$$R(D) = \lim_{n \to \infty} R_n(D)$$

if the limit exists. More generally if x_1, x_2,... is a sequence independent random entities of countable dimension with identical probability law we can define the rate distortion function as in Eq. (2.12) (since in section 2.2.2 we have defined $I(x;\hat{x})$ in general) and for the case where the x_k are not independent and with equal pr. law we can define it as in Eq. (2.13) provided some regularity conditions are met (see Berger (26), (20), Sakrison (17)). The rate distortion function is convex in D as shown in Fig. 2.5.

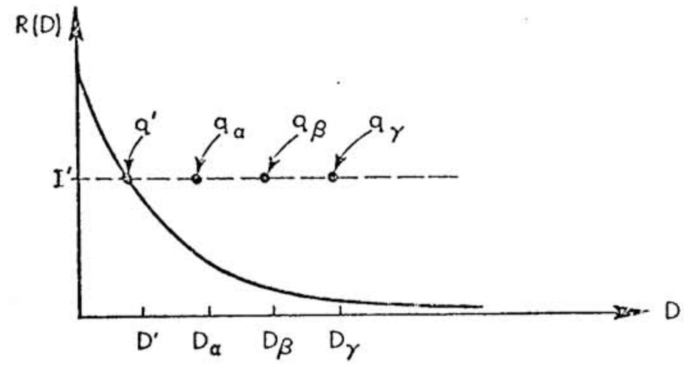

Figure 2.5 Typical Rate Distortion Function

An alternate and perhaps more intuitively pleasing way of defining R(D) is as a distortion-rate curve D(R). Refering to Figs. 2.4 and 2.5 consider all transitions $q', q_\alpha, q_\beta, q_\gamma, \ldots$ which give mutual information $I(x;\hat{x})=I'$ at corresponding distortion levles $D', D_\alpha, D_\beta, D_\gamma, \ldots$ thus specifying the points illustrated in Fig. 2.5. Among all these "iso-information" transitions choose the one (in this case q') that achieves the minimum distortion. A point on the rate distortion curve (I',D') (or on the distortion-rate curve (D', I')) would then be specified.

While arguments may shed some light on the somewhat obscure definition of R(D) of Eqs. (2.12,13) it is important to realize that <u>in information theory the raison d'etre for the</u>

rate distortion function is that as defined it can be used in proving a coding theorem. Specifically the solution to the ITPWD can be given in terms of the rate distortion function. We now proceed to present this solution.

Observe that, as illustrated in Fig. 2.4, the rate distortion function and the formulation of the ITPWD are concerned with the overall performance obtained and not on what is inside the transition defined by $q(\hat{x}|x)$ (i.e., whether inside the transition there is a "poor coder" and a "good channel," viceversa, etc.). In order to produce the solution to the ITPWD, however, we model the transition as illustrated in Fig. 2.6. We observe that if we can code the sequence $x_1, x_2, \ldots$ into a binary sequence $\bar{x}_1, \bar{x}_2, \ldots$ of finite entropy H' while preserving fidelity $\varepsilon \leq D$ then by the Channel Coding Theorem of the previous section we can recover this sequence at the output of the decoder (provided $H' < C$, proper channel coding, long sequences, etc.) with no further distortion. Thus the solution to the ITPWD reduces to proving a source coding theorem and only the Reduced Transition of Fig. 2.6 need be considered.

Main Result of Rate Distortion Theory:

Shannon Source Coding Theorem with distortion. For any $D>0$ and $\delta>0$ there exists a source encoder such that for the block length N sufficiently large the $x_1, x_2, \ldots$ sequence can be reproduced by the $\bar{x}_1, \bar{x}_2, \ldots$ with average distortion $\varepsilon \leq D + \delta$

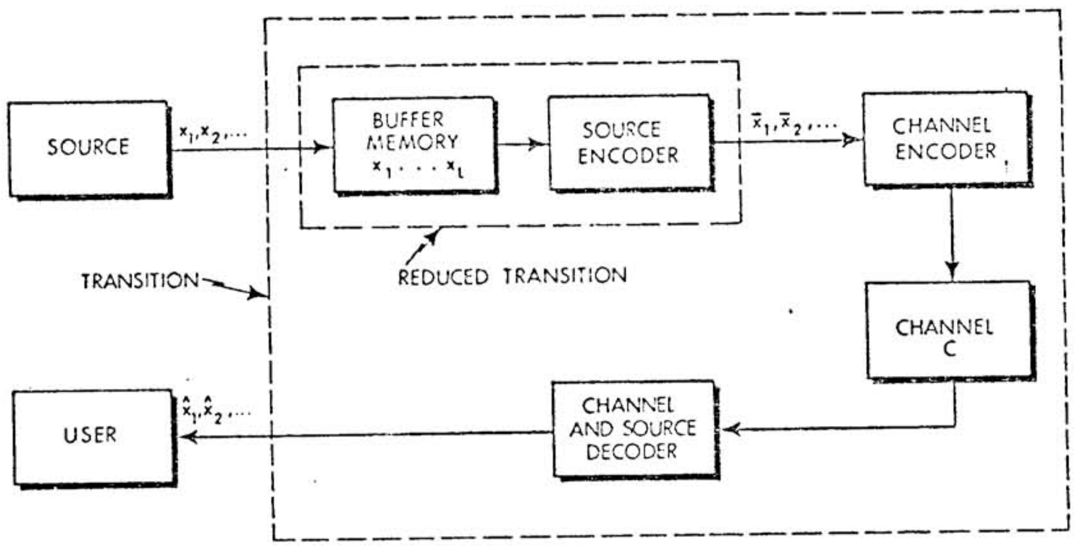

Fig. 2.6 Transition model for solution of ITPWD

It can also be shown that the channel capacity required by the channel such that the average distortion of the overall system of Fig. 2.6 is arbitrarily close to D is R(D). Thus R(D) is the effective entropy of the source when distortion D is allowed. As in the case of the ITP, the starting point of problems where distortion is allowed is to prove a specialized version of the coding theorem directly applicable to the particular situation in order to be able to ascertain the feasibility of approximately solving the problem by various coding techniques.

Summarizing the features of the ITPWD which are relevant for our study we observe that they are analogous to those of the ITP of the previous section, namely:

1. In the ITPWD both the encoder and decoder are to be designed.

2. The performance index in the ITPWD (the fidelity criterion F_d) is an interval performance index.

3. The solution to the ITPWD is asymptotic in nature in the sense that it holds for block lengths approaching infinity.

4. The solution to the ITPWD is an existence result (and no technique presently exists of synthesizing this solution).

2.2.5 Remote Sources and Users

We now introduce a modification of the basic ITPWD presented in the last section first considered by Dobrushin and Tsybakov (41) and more recently by Wolf and Ziv (40) and Berger (20). The concepts involved in this modified ITPWD (in particular that of a random map) will be useful in the next chapter. We concentrate on the work of Wolf and Ziv, indicating at the end the ways in which the results of Dobrushin, Tsybakov, and Berger differ.

The basic situation of remote sources and users is illustrated in Fig. 2.7 where a source puts out a signal x_o^T (or x_o^N for the discrete time case) denoted by x. The source and encoder are separated by a random map* over which the designer has no control and similarly the decoder and the user are separated by another random map so that the source and the

*See (40), (20).

user are thought of being at locations "remote" to the communication system composed of encoder-channel-decoder. The random maps, blocks 2,3,6, are specified by conditional probability measures q_1, q_2, q_3 whereas the encoder and decoder are described by deterministic maps f, g which are to be chosen. The cascade of maps formed by blocks 1 through 7 are defined to be such that random entities x, s, y, $\hat{y}$, $\hat{s}$, $\hat{x}$ form a Markov chain. The distortion measure is taken to be

$$d(x_o^T,\hat{x}_o^T) = \frac{1}{T}\int_0^T E(x_t-\hat{x}_t)^2 \quad \text{or} \quad d(x_o^N,\hat{x}_o^N)=\frac{1}{N}\sum_{n=0}^{N} E(x_n-\hat{x}_n)^2$$

respectively for the continuous and discrete parameter case.

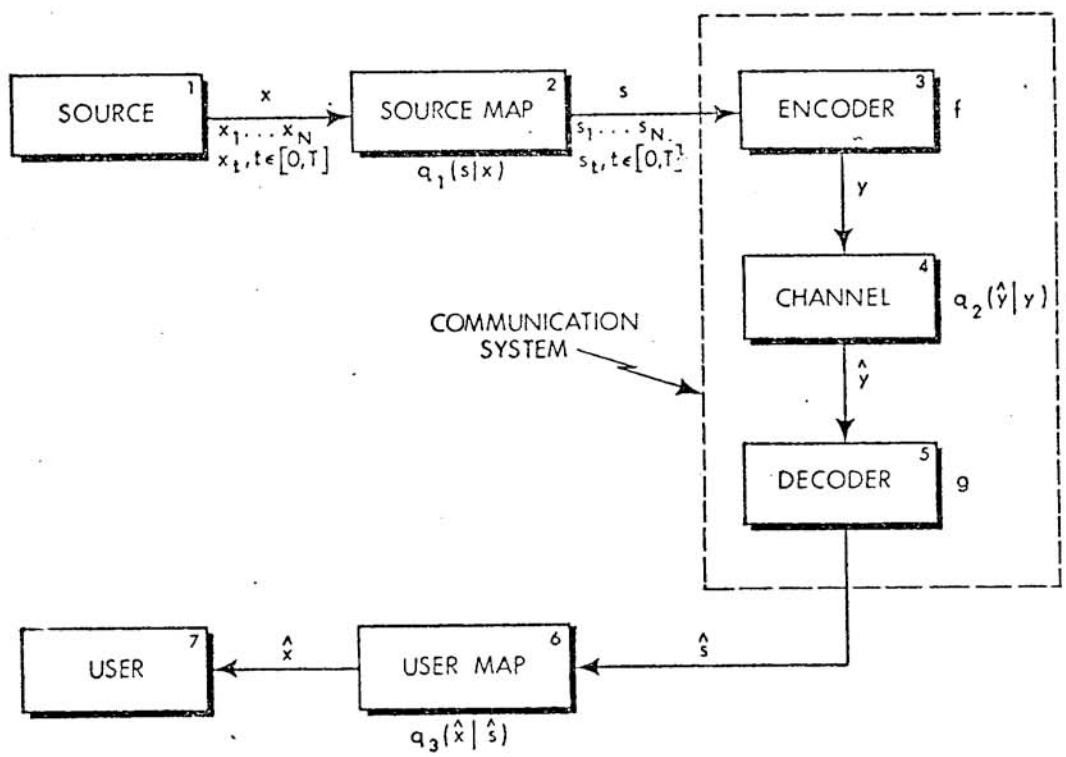

Figure 2.7 ITPWD For Remote Sources and Users

The main result of Wolf-Ziv is that the average distortion for this problem is given by (e.g. in the continuous case)

$$d(x_o^T, \hat{x}_o^T) = E\frac{1}{T}||x-E(x|s)||^2 + E\frac{1}{T}||u-E(u|\hat{y})||^2$$

$$+ E\frac{1}{T}||x-E(u|y)||^2 \quad (2.14)$$

where

$$||\cdot||^2 = \int_0^T (\cdot)^2 dt$$

$$u = E(x|s)$$

$$v = E(u|y)$$

so that the optimum distortion is given by

$$d(x_o^T, x_o^T) = E\frac{1}{T} ||x-E(x|s)||^2 +$$

$$+ \inf_{f,g} \{E\frac{1}{T} ||u-E(u|\hat{y})||^2 + E\frac{1}{T}||\hat{x}-E(u|\hat{y})\}^2 \quad (2.15)$$

These equations in effect say that the structure of the optimum encoder and decoder must be indicated in Fig. 2.8. Both the encoder and decoder must include a conditional mean computer, the distortion incurred in this extimation step being an unavoidable component of the overall distortion.

The above development does not in principle solve the problem since it does not assure that reliable (up to a certain distortion level) communication is possible. That is, a coding

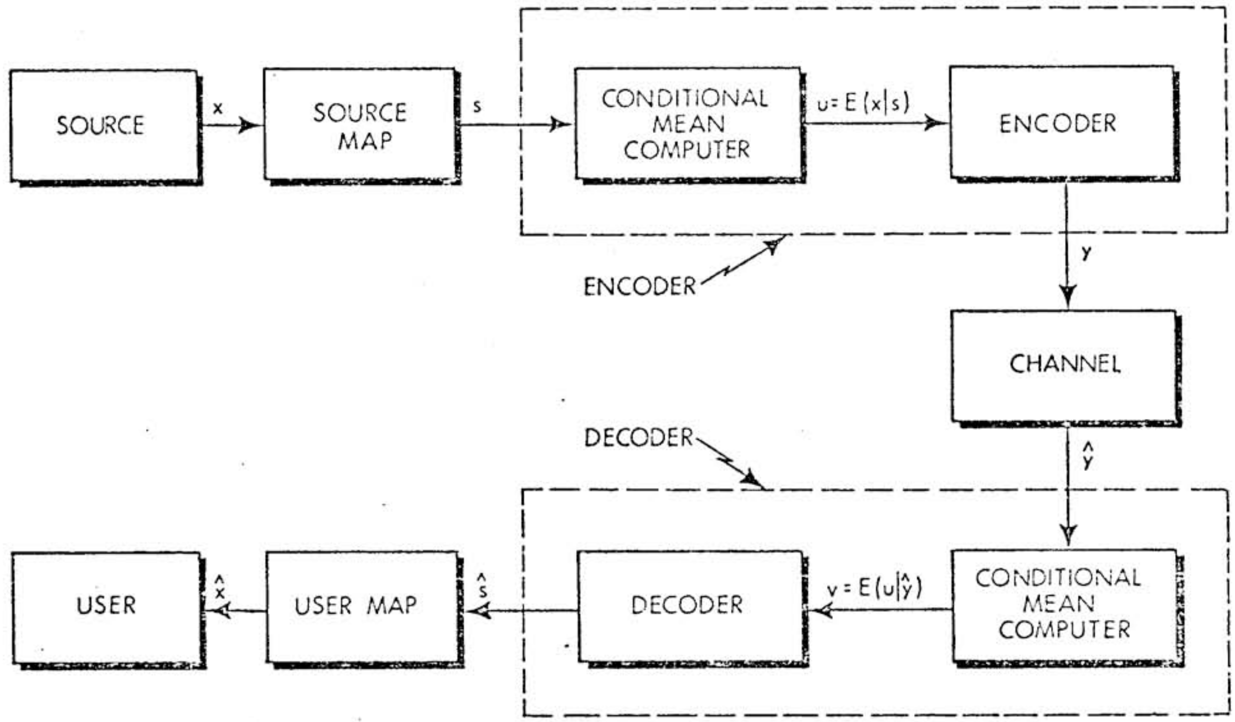

Figure 2.8 Structure of Optimal Communications System for Remote Source-User Problem

theorem needs to be proved. This has been done by Dobrushin and Tsybakov (41) for the more restrictive case where the source is Gaussian, the source and receiving maps are additive independent* Gaussian noise and naturally $T \to \infty$. Berger (20) proves coding theorems for two versions of the problem. In the more elementary case, the source puts out a sequence of independent identically distributed random variables and the source and user maps are memoryless random transformations. The prescribed distortion can be achieved for "long enough" sequences. In the more complex version, x is a stationary Gaussian random process

*Independence is not required for the source mapping noise.

(or sequence), the source and user maps are allowed to have memory, and the distortion measure is a "frequency weighted squared error criterion" (see(20)). The rate distortion function is defined as (referring to Fig. 2.7)

$$R(D) = \lim_{T\to\infty} \inf_{q(\hat{s}|s)\varepsilon Q_o} \frac{1}{T} I(s;\hat{s})$$

and the coding theorem assuring distortion level D for large enough T is proved in terms of this rate distortion function.

The basic purpose of this section has been to introduce the concept of the source and user maps as it has been used in the ITPWD problem. In particular we note that the four fundamental features of the ITP and ITPWD listed at the end of sections 2.2.3,4 are still found in the remote source and user problem: (1) Both the encoder and decoder are to be designed; (2) The performance index is an interval performance index; (3) While Eqs. (2.14), (2.15) hold for finite T, the solutions are still asymptotic statements; (4) The solutions are existence results.

Our next step is to begin to study the interaction between filtering, distortion, and information, based on the brief discription of the filtering and information theory problems (with and without distortion) given above. The approach to be followed is to first, in Section 2.3 summarize the work that has been done in this area and to point out the limitations associated with these contributions. Then in Chapter 3 a more natural information theoretic framework with which to treat the filtering problem will be given.

2.3 Information in Filtering: Survey of Previous Work

In this section we present a short survey of the work that has been done relating information, filtering and dynamical systems. Since our principal concern is filtering as defined in Section 2.1, contributions to (parameter) estimation are omitted. The order of presentation is almost chronological.

Goblick 1965. (27) The problem consider by Goblick is illustrated in Fig. 2.9. The signal $\{a(t),\ t\varepsilon[0,T]\}$ is a stationary process which is to be transmitted through a communication system that includes a channel of capacity C subject to a constraint of the form

$$\varepsilon = E\ \{\frac{1}{T}\int_0^T (a(t) - \hat{a}(t))^2\ dt\} \leq D \qquad (2.16)$$

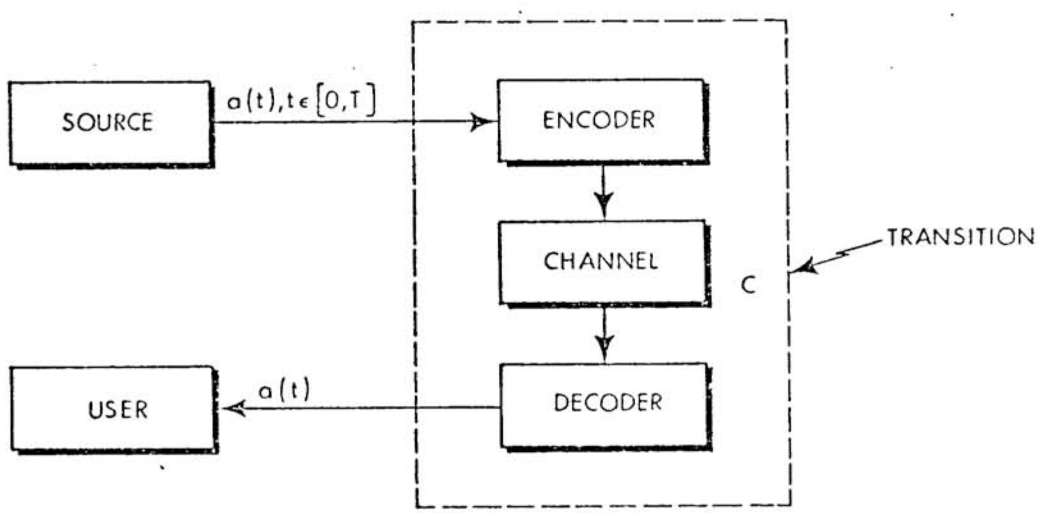

Figure 2.9 Model Considered By Goblick

The specific question considered is, given the source and the channel, what is the minimum error ε_{min} that can be obtained. Using results as presented in the previous section Goblick concludes that ε_{min} must be given the solution to the equation

$$R(\varepsilon_{min}) = C \tag{2.17}$$

where R(D) is the source's rate distortion function subject to the fidelity criterion defined by Eq. (2.16)

$$R(D) = \inf_{T} R_T(D)$$

$$R_T(D) = \min \frac{1}{T} I(a_o^t, \hat{a}_o^t)$$

which is of form analogous to Eq. (2.13). While Eq. (2.17) is very difficult to sovle in general since R(D) is very difficult to compute, Goblick considers some Gaussian examples where (2.17) can be used particularly in the above threshold (linear) region.

Weidemann and Stear 1970 (28,29). The situation considered by Weidemann-Stear(28) and by Weidemann(29) is illustrated in Fig. 2.8 where the say kth entry in the finite dimensional vectors $\underline{x}$, $\underline{y}$, $\underline{s}$, $\underline{w}$, etc. corresponds to the kth time sample. As in the Wiener filter problem the object is to produce an estimate of the ideal signal $\underline{y}$ obtained by a transformation of the signal $\underline{x}$ where no particular model for $\underline{x}$ is assumed. In this case, however, the transformation D, as well as all others, are not necessarily linear nor realizable (e.g., for $j<k$, y_j may depend on x_k). The distortion measure used is the entropy of the error signal $H(\tilde{y})$. Using information identities and the rate distortion function with respect to the error entropy fidelity criterion, Weidemann and Stear arrive at the following conclusions in regard to the model of Fig. 2.10:

(1) The entropy of the error can be lower bounded as follows:

$$H(y-\hat{y}) \geq H(y) - I(y;z)$$

(2) Minimizing $I(y-\hat{y};z)$ is equivalent to minimizing $H(y-\hat{y})$.

(3) Define

$$CT = I(x;z)$$

as the "channel transmittance" of the sensor. Then

$$H(y) - H(y-\hat{y}) \leq CT$$

so that if C is the channel capacity of the sensor channel (since by definition $I(x;z) \leq C$)

$$H(y) - H(y-\hat{y}) \leq CT \leq C \tag{2.18}$$

Eq. (2.18) in particular is a result similar to a coding theorem and a consequence of the "coding theorem nature" of Weidemann's formulation. In effect Eq. (2.18) says that the performance of the estimating system as measured by the left side of the inequality is bounded by the channel transmittance and channel capacity of the sensor.

Bucy 1968 (4). Bucy devotes a page of his book to the following very interesting result.

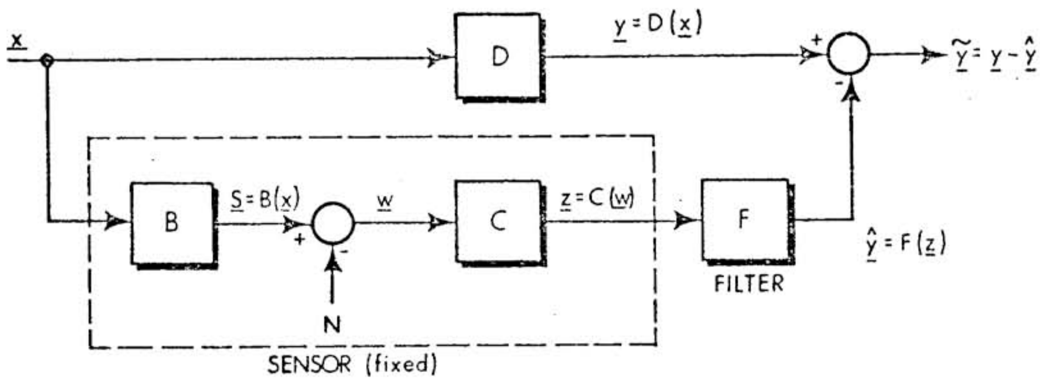

Figure 2.10 Problem Considered By Weidemann and Stear

Lemma 10 (Bucy). Consider the linear-Gaussian special case of the NLFP defined above where dynamcis and observation equations are given by:

$$dx_t = a(t)x_t \, dt + b(t) \, dB_t, \quad x_o\text{: Gaussian random variable}$$

$$dy_t = g(t)x_t \, dt + d\tilde{B}_t$$

Let $\mathcal{a}$ be the set of all maps $A:C(0,t)\to R$. Then a necessary condition for $A_o \varepsilon \mathcal{a}$ to be the optimum MSE estimator is that it maximizes $I(x_t; Ay_o^t)$ where $y_o^t = (y_s, 0 \leq s \leq t)$. Furthermore

$$I(x_t;y_o^t) = I(x_t;A_oy_o^t) = \sup_{A\varepsilon\mathcal{a}} I(x_t;A_{yo}^{\ t})$$

E.O.L.

The proof of this theorem relies completely on the Gaussianness of the random entitities involved. Specifically, Bucy computes $I(x_t;y_o^t)$ and $I(x_t;\hat{x}_t)$ where $\hat{x}_t$ is the Kalman-Bucy estimate, according to the formulas derived for Gaussian processes by

Gelfand-Yaglom(24) and shows that these two informations are equal. The results then follow from Lemma 7, above, since for all $A\varepsilon$, $I(x_t;Ay_o^t) \leq I(x_t;\ y_o^t)$.

In Chapter 3 we shall see that under the proper assumptions this Lemma 10 can be proved for the general NLFP.

Duncan 1967, 1969, 1970 (3,30,31,32,33).

Following Duncan (30), who considered the linear case, Kailath's likelihood ratio formula (31,32) can be used to prove the following result that we shall need later.

Lemma 11 (Kailath, Duncan). Consider the NLFP as defined above,

$$dx_t = a(x_t,t)dt + b(x_t,t)dB_t \ , \ t \geq 0$$

$$dy_t = g(x_t,t)dt + d\tilde{B}_t \quad , \quad (d\tilde{B}_t)^2 = R(t)dt$$

Then

$$I(x_o^t;y_o^t) = \tfrac{1}{2}E\int_0^t (g_s - \hat{g}_s)'\ R^{-1}(t)\ (g_s - \hat{g}_s)ds \qquad (2.19)$$

where

$$\hat{g}_s = E(g_s|y_o^s).$$

Eq.(2.19) is similar in nature to the formula derived by Gelfand-Yaglom(24) for the information about a Gaussian random variable x_t contained in a (jointly defined) Gaussian random process y_o^t,

$$I(x_t;y_o^t) = -\tfrac{1}{2}\log\left\{\left[E(x_t - E(x_t|y_o^t))^2\right]\left[E\ x_t^2\right]^{-1}\right\}$$

in that both depend on a filtering error. Unfortunately both of these formulas in themselves say little about the optimal estimate nor about suboptimal estimates.

Gray 1969, 1970 (34,35). Gray has computed the rate distortion function R(D) relative to (among other distortion measures) MSE of sources modeled as autoregressive processes:

$$x_n = - \sum_{k=1}^{n} a_k x_{n-k} + z_n$$

$$\varepsilon = \frac{1}{n} \sum_{k=1}^{n} E(x_k - \hat{x}_k)^2$$

and we recall from Eq. (2.13)

$$R_n(D) = \frac{1}{n} \min_{q \varepsilon Q_o} I(\underline{x};\hat{\underline{x}})$$

$$R(D) = \lim_{n \to \infty} R_n(D)$$

where as usual Q_o = {conditional densities $q(\hat{\underline{x}}|\underline{x}) : \varepsilon \leq D$}. He also proves the corresponding coding theorems in terms of R(D).

Toms and Berger 1971 (36,37). Toms and Berger compute the rate distortion function of sources modelled by discrete time linear dynamic systems and the channel capacity of channels also modelled as linear dynamic systems. A channel coding theorem is also proved for these models.

We observe that with the notable exception of Bucy's result all of the above investigations are strongly influenced by the ITPWD as typified by the four characteristics listed at the end of section 2.3. Thus note in Goblick's work the stationarity requirement, the interval nature of the performance index and especially the inherent interval and asymptotic nature (and apparent validity for large T only) of channel capacity and the rate distortion function. The ITPWD and coding theorem influences on the work of Werdemann-Stear is also evident as can be seen for example from the interval nature of the results (e.g. Eq. (2.18) as well as the non-causality of the situation (just as a block encoder is not causal within a block). Gray, Toms, and Berger are explicitly concerned with the ITPWD and their interest in using dynamic and autoregressive systems for models of channels and sources is directed towards obtaining more explicity results for the solution of the ITPWD.

The formula of Kailath-Duncan (Eq. (2.18), above),

$$I(x_o^t;y_o^t) = \tfrac{1}{2} E \int_0^t (g_s-\hat{g}_s)' R^{-1}(s) \quad (g_s-\hat{g}_s)ds \qquad (2.20)$$

is sometimes taken as establishing the relation (or even equivalency) of the ITP and NLFP. It should be noted however that while, it may be true that if for $0 \leq s \leq t$ $\hat{\hat{x}}_s$ is a "good" suboptimal estimate of x_s and $\hat{\hat{g}}_s = g(\hat{\hat{x}}_s, s)$ we may have that

$$I(x_o^t;y_o^t) = I(x_o^t;\hat{x}_o^t) \approx \tfrac{1}{2}E \int_0^t (g_s-\hat{\hat{g}}_s)^1 R^{-1}(s)(g_s-\hat{\hat{g}}_s)ds$$

it does not say that

$$I(x_o^t;\hat{\hat{x}}_o^t) \approx \tfrac{1}{2}E \int_o^t (g_s-\hat{\hat{g}}_s)^1 R^{-1}(s) \; (g_s-\hat{\hat{g}}_s)ds \qquad (2.21)$$

and in fact the right hand side of (2.21) is greater than the right hand side of (2.20) whereas one would expect that the left hand side of (2.21) be less than the left hand side of (2.20). Formula (2.20) also says little about the precise nature of the two problems. It thus seems more appropriate to consider the Kailath-Duncan formula as a relation -- and to be sure an important one -- between quantities arising in the ITP and NLFP.

We have not mentioned in this section what we consider to be the most important work we have encountered in this study, the paper "Lower and Upper Bounds ..." (38) by Zakai and Ziv, since we devote a full section to it in Chapter 5. We mention at this point, however, that while their derivation was restricted to stationary processes (apparently for pedagogical reasons) they were not hindered by the "coding theorem influences" typical in the above mentioned studies.

CHAPTER 3

A NEW INFORMATION THEORETIC FORMULATION OF THE FILTERING PROBLEM*

3.1 Preliminary Remarks

In the preceding chapter we presented a short description of the ITP and ITPWD emphasizing five key characteristics of these problems listed at the end of subsections 2.2.3 and 2.2.4 which we repeat here for easy reference. For both the ITP and ITPWD:

1. Both encoder and decoder are available for design.
2. The performance index is an interval PI.
3. The solution is asymptotic in nature (and so are the entities, such as channel capacity and rate distortion functions, in terms of which this solution is expressed).
4. The solution is an existence result.

And for the ITP:

5. The object is perfect reproduction.

We also summarized at the end of Chapter 2 the various contributions relating filtering to information and observed that (with the noted exceptions) these studies had been influenced

*We use "filtering" for "filtering problem" as was defined in Section 2.1.

by the above features of the ITPWD and implied that this influence was adverse in nature, thus hinting at the following important conclusion of our study:

> We claim that the above four features of the ITPWD are foreign to practical filtering and that the NLFP and ITPWD are basically different problems.

Thus, while in both problems the object is to reproduce at the destination a signal x(t) subject to some fidelity requirement, in the filtering problem the sensor (g in Eqs. (2.2), (2.4)) is not available for design while encoder design is an inherent part of the ITPWD.*

Further, the Performance Index (PI) in the NLFP is a point PI as opposed to an interval PI** in the ITPWD. Only in the case of stationary processes would interval and point PI's coincide and this is not, in general, the case of Eqs. (2.1), (2.3), nor is it even the case in the linear Kalman filter problem. It should also be noted that in "steady state filtering" the PI is also a point PI, namely the error at ∞ or large T, and behavior in $[0,\infty)$ or $[0,T)$ is not an issue.

The asymptotic nature of the solution to the IPTWD imposes severe restrictions on the processes that can be

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* This fact was observed by Weidemann-Stear (28, 29).

**See VanTrees (42) Chapter 3 for a discussion of point and interval PI's.

usefully considered since in order for the necessary quantities to merely exist, stationarity and some form of ergodicity must be required. These conditions are too demanding for the filtering problem since even the Kalman filtering problem is non-stationary in nature. Furthermore, the "long words" nature of the solution to the ITPWD eliminates causality -- an important feature of the filtering problem -- since block encoding is clearly not causal within the block (and the block must be long for the desired performance to be achieved).

Since, in view of the above comments the ITPWD and NLFP are different in nature, an existence result for the ITPWD would not contribute to the solution of the NLFP. In any case, existence results are not among the most pressing problems in filtering.

Finally, with regard to item (5) above, the object of the ITP is perfect reproduction which is clearly not a realistic goal in filtering.

The preceding remarks should not be taken as saying that information theory is useless in filtering, and in fact the remainder of this thesis is devoted to demonstrating that on the contrary information concepts and an information theoretic framework can be exceedingly useful in practical filtering. The point is that considerable advantage can be derived from bypassing the classical ITPWD as epitomized by the coding theorems -- that is by modifying the information theory problem to fit the filtering situation rather than by distorting filtering (or taking a very special case) in order to "shoe-horn-it" into the ITPWD.

The objectives of this chapter are: (1) to provide a formulation of the filtering problem -- including both the NLFP where filter structure is unconstrained as well as the case where the receiver is restricted to a given class -- in information theoretic terms that makes clear the interplay and usefulness of information concepts in filtering; and (2) as a byproduct, to derive a number of useful filter performance lowerbounds.

In Section 3.2 such a formulation is presented making use of the Bucy-Mortenson-Duncan representation and the concept of a random source map introduced in the subsection on Remote Sources and Users (subsection 2.2.5).

In order to understand how, based on the above formulation, filtering can be imbedded in information terms we introduce in Section 3.3 constrained rate distortion functions. This is the key concept in understanding the implications of information on filtering both when the filter structure is unconstrained as in the NLFP as well as when structure is constrained as is the case in reduced order filtering to be considered in the next chapter. Effectively constrained rate distortion functions imbed in an information framework not only optimal but also suboptimal filtering, and both at a conceptual level and in the form of design guidelines.

Also based in the formulation of Section 3.2, a number of useful performance lowerbounds for both optimal and suboptimal filters are derived in Section 3.4. The bound for optimum MSE will be further developed in Chapter 5 where explicit formulas are obtained and in chapter 6 where these formulas are applied to a concrete example, the phase lock loop. The bound on suboptimal MSE will be applied in Chapter 4 to the reduced order filter problem where, because of the gausianness of the situation, the computation of information loss is elementary.

Sections 3.2 through 3.4 are formulated in discrete time for clarity of presentation and in order to avoid unduly complicating the issues with measure theoretic concepts. It is very clear, however, that all results are valid in continuous time since no difficulty arises that could not be cured by a proper choice of a separable and measurable version of the processes involved (and these versions obviously exist for the kind of processes we consider). Furthermore, "translation" from discrete to continuous time can be done in a straightforward way upon providing the necessary technicalities since none of the arguments depend on the usual pitfalls (e.g., innovations reasoning, time limits, etc.). Consequently it is only necessary (and then only for reference in the sequel) to give in Section 3.5 continuous time versions of only a few of the results of the previous sections.

Finally we note that while special attention is given in this chapter to MSE the entire development is applicable to a wide variety of distortion measures.

The chapter closes with a short summary in Section 3.6.

3.2 A New Information Theoretic Formulation of the Filtering Problem

Our objective is to provide a framework for the filtering problem that, first, realistically models the problem; and second that allows treatment by information concepts while at the same time avoids the inadmissible features of the ITPWD discussed above. Such a formulation is illustrated for discrete time* in Fig. 3.1. It is described in this section and profitably used in Sections 3.3 and 3.4.

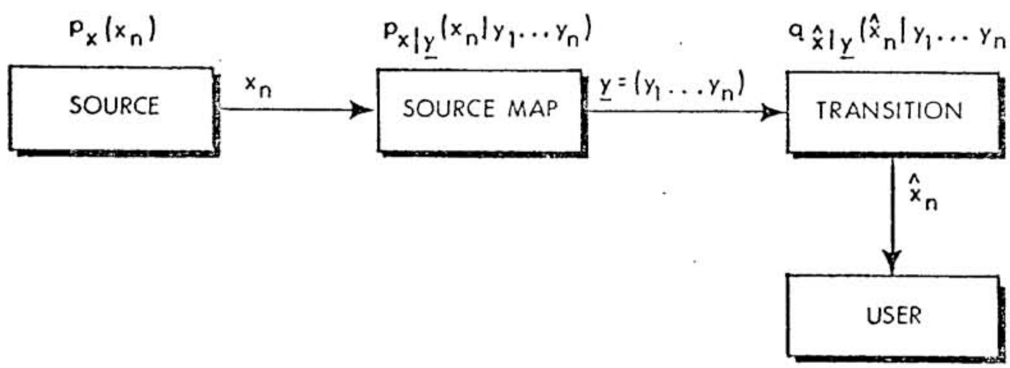

Figure 3.1 New Information Theoretic Formulation of the Filtering Problem in Discrete Time. (Variables are as Defined in Section 2.1)

*Continuous time counterparts can be obtained by replacing throughout x_n, y_1^n, $\hat{x}_n$ by x_t, y_o^t, $\hat{x}_t$.

First note that we are interested only in x_n and its reproduction $\hat{x}_n$, although n is arbitrary. (Alternatively we may consider the source to put out a sequence of independent identically distributed random variables $\ldots, x_n, x_n, \ldots$). We take as the associated distortion the point PI

$$d = (x_n - \hat{x}_n)^2 \tag{3.1}$$

The observation constraint is incorporated into the model by means of the random source map characterized by $p_{x|\underline{y}}$ whose existence in both discrete and continuous time is assured by the BMD representation* (Lemmas 1, 2 of Section 2.1). This source map is obviously a figment of the imagination and does not correspond to a physical device. Since in what follows we will operate on this problem using rate distortion theory we simply characterize the (causal) filter by a transition specified by its conditional probability density $q_{\hat{x}|y}$ as was noted in Section 2.2.4 (e.g., see Figs. 2.4, 2.6).

Since the three random entities $x_n, (y_1, \ldots, y_n), \hat{x}_n$ form a Markov chain we can write the following expressions noting that at this point our concern is that the quantities involved exist and are well defined rather than whether they can be easily computed.

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*As discussed below, $p_{x|y}$ specifies $p_{y|x}$ so that in effect $p_{x|y}$ determined the random source map.

$p_{x_n}(\cdot)$: Defined by Eq. (2.3)

$p_{\underline{y}}(\cdot)$: Defined by Eq. (2.4)

$p_{x_n|\underline{y}}(\cdot|\cdot)$: Defined by Lemma 2

$p_{x_n\underline{y}}(\cdot,\cdot)$: Defined by Bayes rule from the preceding PDF's

$p_{\underline{y}|x_n}(\cdot|\cdot)$: Defined by Bayes rule from the preceding PDF's

$q_{\hat{x}_n|\underline{y}}(\cdot|\cdot)$: Arbitrary

$p_{\hat{x}_n\underline{y}}(\cdot,\cdot)$: Defined by Bayes rule from the preceding PDF's

$p_{\hat{x}_n}(\cdot)$: Defined from the prededing PDF

$p_{\underline{y}|\hat{x}_n}(\cdot|\cdot)$: Defined by Bayes rule from the above PDF's

where $\underline{y} = (y_1, \ldots, y_n)$ and in what follows we suppress the n from x_n and $\hat{x}_n$ since no confusion is possible. Further:

$$p_{x\underline{y}\hat{x}} = p_{x|\underline{y}}\, q_{\hat{x}\underline{y}}\, p_{\underline{y}} \tag{3.2}$$

$$p_{x|\hat{x}} = \int_{\underline{y}\epsilon R^n} p_{x|\underline{y}}\, q_{\hat{x}\underline{y}}\, p_{\underline{y}}\, d\underline{y} \tag{3.3}$$

$$I(x;\underline{y}) = \int_R \int_{R^n} \left\{\log \frac{p_{x|\underline{y}}}{p_x}\right\} p_{x\underline{y}}\, dx\, d\underline{y} \tag{3.4}$$

$$I(x;\hat{x}) = \int_R \int_R \left\{\log \frac{p_{x\hat{x}}}{p_x\, p_{\hat{x}}}\right\} p_{xx}\, dx\, d\hat{x} \tag{3.5}$$

where the notation has been abused as is customary and x, $\hat{x}$, y_k have been taken in R for simplicity in notation but clearly without loss of generality. The above are all the information quantities we shall need. We note that the necessary probability density functions and Radon Nikodym derivatives exist both in discrete and continuous time and that, unlike in the ITPWD stationarity, ergodicity and the like are not issues here. (Alternatively we may consider the different rv's involved as being part of independent identically distributed sequences and the various maps as being memoryless.). Thus,. the above formulation contrasts sharply with that of classical rate distortion theory (the ITPWD) which is centered around a coding theorem as evident the development in Chapter 2. (For example compare with the formulations of Gray, Toms, Berger, Goeblik, Weidemann-Stear cited in Section 2.3.)

3.3 Constrained Rate Distortion Functions*

Based on the preceding formulation we introduce in this section several "constrained" rate distortion functions, CR1(D), CR2(D), CRN(D) corresponding to constraints in the variational problem defining the rate distortion function.

3.3.1 CR1(D) and CR2(D)

First consider the NLFP where filter structure is unconstrained.

---

*The word "rate" present in "constrained rate distortion function" is a leftover from rate distortion theory. In the present contact, however, information rate is irrelevant (unlike in the ITPWD) as evident from the formulation of the preceding section.

CR1(D) is defined, with reference to Fig. 3.1 and Eqs. 3.1 to 3.5, as follows:

$$d(q_{\hat{x}|\underline{y}}) = \int_R \int_{R^n} \int_R (x-\hat{x})^2 \, p_{x\underline{y}\hat{x}} \, dx d\underline{y} d\hat{x}$$

$$= \int_R \int_{R^n} \int_R (x-\hat{x})^2 \, p_{x|\underline{y}} \, q_{\hat{x}|\underline{y}} \, dx d\underline{y} d\hat{x} \tag{3.6}$$

ψ_1: set of all conditional PDF's $q_{\hat{x}|\underline{y}}$

$$CQ1_D = \left\{ q_{\hat{x}|\underline{y}} \, \varepsilon \psi_1 : d(q_{\hat{x}|\underline{y}}) \geq D \right\} \tag{3.7}$$

If D is such that $CQ1_D$ is not empty then

$$CR1(D) \triangleq \inf_{q_{\hat{x}|\underline{y}} \varepsilon \, CQ_D} I(x;\hat{x}) \tag{3.8}$$

where $I(x;\hat{x})$ is given by Eq. (3.5).

Observe that the minimization is not over all transitions $p_{\hat{x}|x}$ but rather over all transitions $q_{\hat{x}|\underline{y}}$ so that the constraint imposed by the sensor is incorporated into the definition since $p_{x|\underline{y}}$, as determined by the sensor, is fixed in Eq. (3.6). None of the five difficulties discussed above arise in the definition of CR1(D) as should be expected since the definition is based on a model that avoids these difficulties.

We now prove a number of properties of CR1(D). Before proceding to do so, however, we make two assumptions of a technical nature that will simplify the discussion and proofs throughout this chapter:

(1) It is assumed that whenever we speak of CR1(D), CR2(D), etc. these functions are defined, that is, the corresponding sets $CQ1_D$, $CQ2_D$, etc. are not empty. The intervals where they are not defined -- for example CR2(D) for D less than the optimum MSE in the NLFP -- are clearly not of interest.

(2) It is assumed that the infimums over the set of conditional probability density functions (CPDF's) used in defining CR1(D), CR2(D), etc. are achieved. This is a mere technicality which avoids clouding the relevant concepts by worrying constantly over whether or not infimums are achieved. We are dealing with minimizations over the set of CPDF's. Whether or not any infimum is achieved in a particular topology is not of practical significance (since performance within an arbitrarily small epsilon can be realized). Thus the problem is with the topology and not with the physical situation.

Lemma 12. CR1(D) is non-increasing in D.

Proof: Suppose $D_2 > D_1$. From (3.7) $CQ_{D_1} \subset CQ_{D_2}$ so that by (3.8) $CR1(D_1) > CR1(D_2)$. ∎

Theorem 1. CR1(D) is convex in D.

Proof: Assume that CR1(D') is achieved by $q'_{\hat{x}|\underline{y}}$ and CR1(D") is achieved by $q''_{\hat{x}|\underline{y}}$. Define $q^*_{\hat{x}|\underline{y}} = rq'_{\hat{x}|\underline{y}} + r'q''_{\hat{x}|\underline{y}}$, $r \in [0,1]$, $r' = 1-r$ which is a valid transition. Since $d(q_{\hat{x}\underline{y}})$ is linear in $q_{\hat{x}\underline{y}}$ we have

$$d(q^*_{\hat{x}\underline{y}}) = \int_R \int_{R^n} \int_R (x-\hat{x})^2 \, p_{x|\underline{y}} \, q^*_{\hat{x}|\underline{y}} \, p_{\underline{y}} \, dx \, d\underline{y} \, d\hat{x}$$

$$= r \, d(q'_{\hat{x}|\underline{y}}) + r' d(q''_{\hat{x}|\underline{y}})$$

$$= r \, D' + r' \, D''$$

so that $CQ1_{rD'+r'D'}$ is not empty, CR1(rD'+r'D") is defined, and $q^*_{\hat{x}|\underline{y}} \in CQ1_{rD'+r'D''}$.

By definition of CR1 (adding, for greater clarity, $q^*_{\hat{x}|\underline{y}}$)

$$CR1(rd'+r'D'') \leq I(x;\hat{x};q^*_{\hat{x}|\underline{y}})$$

$$= \int_R \int_R \left\{ \log \frac{p_{x\hat{x}}(q^*_{\hat{x}|\underline{y}})}{p_x \, p_{\hat{x}}(q^*_{\hat{x}|\underline{y}})} \right\} p_{x\hat{x}}(q^*_{\hat{x}|\underline{y}}) \, dx \, d\hat{x}$$

$$= \int_R \int_R \left\{ \log \frac{\int_{R^n} p_{x|\underline{y}} \, q^*_{\hat{x}\underline{y}} \, p_{\underline{y}} \, d\underline{y}}{p_x \cdot \int_R \int_{R^n} p_{x|\underline{y}} q^*_{\hat{x}\underline{y}} \, p_{\underline{y}} \, d\underline{y} dx} \right\} p_{x\hat{x}} \, (q^*_{\hat{x}|\underline{y}}) \, dx \, d\hat{x}$$

$$= \int_R \int_R \log \left\{ \frac{\left[r \int_{R^n} p_{x|\underline{y}} q'_{\hat{x}|\underline{y}} p_{\underline{y}} d\underline{y} \right] + \left[r' \int_{R^n} p_{x|\underline{y}} q''_{\hat{x}|\underline{y}} p_{\underline{y}} d\underline{y} \right]}{\left[r \, p_x \cdot \int_R \int_{R^n} p_{x|\underline{y}} q'_{\hat{x}|\underline{y}} p_{\underline{y}} d\underline{y} dx \right] + \left[r' p_x \cdot \int_R \int_{R^n} p_{x|\underline{y}} q''_{\hat{x}|\underline{y}} p_{y} dy dx \right]} \right\} \cdot$$

$$\cdot \; p_{x\hat{x}}(q^*_{\hat{x}|\underline{y}}) \, dx \, d\hat{x}$$

Using in the last expression the well known inequality

$$(v_1 + v_2) \log \frac{(v_1+v_2)}{(u_1+u_2)} \le \sum_{i=1}^{2} v_i \log \frac{v_i}{u_i}$$

for v_i, $u_i \ge 0$, we obtain

$$\mathrm{CR1}(rD' + r'D'') \le$$

$$\int_R\int_R \frac{\left[r \int_{R^n} p_{x|\underline{y}}\, q'_{\hat{x}|\underline{y}}\, p_{\underline{y}}\, d\underline{y}\right]}{\left[r \int_{R^n} p_{x|\underline{y}}\, q'_{\hat{x}|\underline{y}}\, p_{\underline{y}}\, d\underline{y}\right] + \left[r' \int_{R^n} p_{x|\underline{y}}\, q''_{\hat{x}|\underline{y}}\, p_{\underline{y}}\, d\underline{y}\right]} \cdot$$

$$\log \left\{ \frac{r \int_{R^n} p_{x|\underline{y}}\, q'_{\hat{x}\underline{y}}\, p_{\underline{y}}\, d\underline{y}}{r\, p_x \int_R\int_{R^n} p_{x|y}\, q'_{\hat{x}|y}\, p_y\, dy\, dx} \right\} p_{x\hat{x}}(q^*_{\hat{x}|y})\, dx d\hat{x}$$

$$+ \int_R\int_R \frac{\left[r' \int_{R^n} p_{x|y}\, q''_{\hat{x}|\underline{y}}\, p_{\underline{y}}\, d\underline{y}\right]}{\left[r \int_{R^n} p_{x|\underline{y}}\, q'_{\hat{x}\underline{y}}\, p_{\underline{y}}\, d\underline{y}\right] + \left[r' \int_{R^n} p_{x|\underline{y}}\, q''_{\hat{x}|\underline{y}}\, p_{\underline{y}}\, d\underline{y}\right]}$$

$$\log \left\{ \frac{r' \int_{R^n} p_{x|\underline{y}}\, q''_{\hat{x}\underline{y}}\, p_{\underline{y}}\, d\underline{y}}{r'\, p_x \int_R\int_{R^n} p_{x|\underline{y}}\, q''_{\hat{x}\underline{y}}\, p_{\underline{y}}\, d\underline{y}\, dx} \right\} p_{x\hat{x}}(q^*_{\hat{x}|\underline{y}})\, dx\, d\hat{x}$$

$$= \int_R\int_R \log \left\{ \frac{\int_{R^n} p_{x|\underline{y}}\, q'_{\hat{x}\underline{y}}\, p_{\underline{y}}\, d\underline{y}}{p_x \cdot \int_R\int_{R^n} p_{x|\underline{y}}\, q'_{\hat{x}|\underline{y}}\, p_{\underline{y}}\, d\underline{y}\, dx} \right\} \cdot$$

$$\cdot\; r\left[\int_{R^n} p_{x|\underline{y}}\, q'_{\hat{x}|\underline{y}}\, p_{\underline{y}}\, d\underline{y}\right] dx\, d\hat{x}$$

$$+ \int_R\int_R \log \left\{ \frac{\int_{R^n} p_{x|\underline{y}}\, q''_{\hat{x}|\underline{y}}\, p_{\underline{y}}\, d\underline{y}}{p_x \cdot \int_R\int_{R^n} p_{x|\underline{y}}\, q''_{\hat{x}|\underline{y}}\, p_{\underline{y}}\, d\underline{y}\, dx} \right\} \cdot$$

$$\cdot\; r'\left[\int_{R^n} p_{x|\underline{y}}\, q''_{\hat{x}|\underline{y}}\, p_{\underline{y}}\, d\underline{y}\right] dx d\hat{x}$$

$$= r\mathrm{CR1}(D') + r'\mathrm{CR1}(D'')$$

where the next to the last equality follows from the identity

$$p_{x\hat{x}}\ (q^{*}_{\hat{x}|\underline{y}} = \int_{R^n} p_{x|\underline{y}}\ q^{*}_{\hat{x}|\underline{y}}\ p_{\underline{y}}\ d\underline{y}$$

$$= r \int_{R^n} p_{x|\underline{y}}\ q'_{\hat{x}|\underline{y}}\ p_{\underline{y}}\ d\underline{y} + r' \int_{R^n} p_{x|\underline{y}}\ q''_{\hat{x}|\underline{y}}\ p_{\underline{y}}\ d\underline{y}$$ ■

Corollary 1. CR1(D) is continuous and strictly decreasing in D.

Proof: This follows immediately from the convexity proved in Theorem 1. ■

Lemma 13. $R(D) \leq CR1(D)$.

Proof: This follows from the definition of R(D) and CR1(D) since $CQ1_D \subset Q_D$. ■

From the statement just proved we conclude that CR1(D) has the shape and relation to R(D) illustrated in Fig. 3.2. For a given D, CR1(D) does not coincide with R(D) because of the effect of the random map that characterizes the sensor. It is reasonable to expect that the "better" the sensor the "closer" CR1(D) would be to R(D) and perhaps under some observability requirement they may coincide. Thus the effect of constraining the set of allowable transitions $p_{\hat{x}|x}$ is to "separate" the corresponding rate distortion curves. Since, as mentioned in the previous chapter, rate distortion curves can be viewed as distortion rate curves (defined as the minimum distortion D that can be achieved at a certain level of information I) we

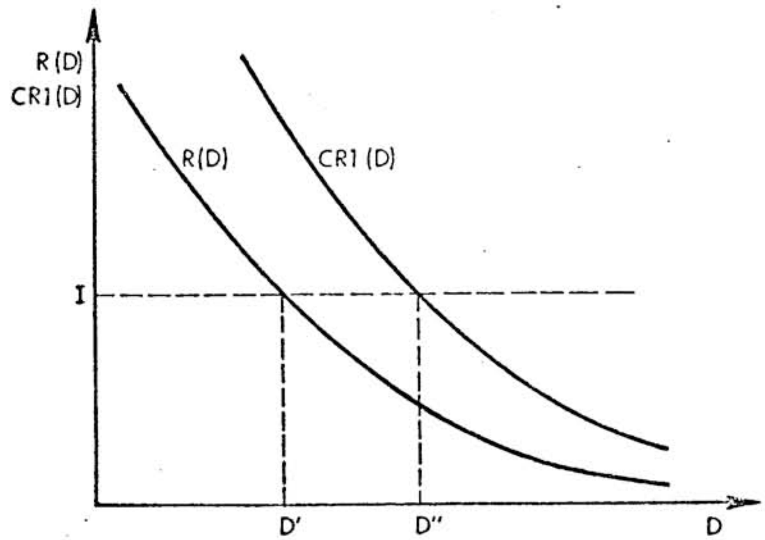

Figure 3.2 Comparison of R(D) and CR1(D)

see, as illustrated in Fig. 3.2, that "separation" corresponds to greater distortion for a given information level I.

In the definition of CR1(D) no constraint whatever was placed on the transition $q_{\hat{x}|\underline{y}}$ since ψ_1 includes all conditional PDF's. Thus the set $CQ1_D$ of allowable transitions for a given distortion contained not only measurable functions of the data, but also random transformations (see subsection 2.2.4) -- random filters -- of the data. While it is conceivable that random filters could perform better than deterministic ones, we must, if we are to realistically model the filtering problem, restrict transitions to measurable functions* of the data. We recognize this as a further difference between

*As discussed by Duncan in (3), differential (and difference) equation filters are measurable functions of the data. The solution to the NLFP is of course a measurable function of the data.

ITPWD and filtering problem concepts and are therefore led to the following definition (compare with Eqs. (3.6), (3.7)).

$$d(q_{\hat{x}|\underline{y}}) = \int_R \int_{R^n} \int_R (x-\hat{x})^2 \, p_{x|\underline{y}} \, q_{\hat{x}|\underline{y}} \, p_{\underline{y}} \, dx \, d\underline{y} \, d\hat{x} \qquad (3.9)$$

$$CQ2_D = \left\{ q_{\hat{x}|y} \in \psi_2 : d(q_{\hat{x}|\underline{y}}) \le D \right\}$$

ψ_2 = set of all CPDF's of the form $q_{\hat{x}|\underline{y}}(\xi|\underline{\eta}) = \delta(\xi - f(\underline{\eta}))$ for f a measurable function of $\underline{y}$.

$$CR2(D) = \inf_{q_{\hat{x}|\underline{y}} \in CQ2_D} I(x;\hat{x}) \qquad (3.10)$$

It is important to emphasize that while in what follows we will speak loosely of "CR2", CR2(D) is the constrained rate distortion function <u>of</u> the source x_n <u>relative</u> to a given distortion measure (MSE here), <u>relative</u> to the sensor, and <u>relative</u> to the set ψ_2 of allowed filters.

CR2(D) has the following properties.

<u>Lemma 14.</u> CR2(D) is defined for all $D \ge \varepsilon^* = E(x_n - E(x_n|\underline{y}))^2$ and is non-increasing in D.

<u>Proof:</u> Clearly CR2(D) is defined as $D = \varepsilon^*$. Consider now the estimate of x_n given by (b is a constant)

$$\hat{x}_n = \hat{x}_n^* + b$$

We have:

$$\begin{aligned} d(\hat{x}_n) &= E(x_n - \hat{x}_n^* - b)^2 \\ &= E(x_n - \hat{x}_n^*)^2 - 2bE(x_n - \hat{x}_n^*) + b^2 \\ &= \varepsilon^* + b^2 \end{aligned}$$

Thus $CQ2_D$ is not empty for all $D \geq \varepsilon^*$. The second assertion follows from the fact that for $D_2 \geq D_1$, $CQ2_{D_1} \subseteq CQ2_{D_2}$. ∎

Lemma 15. $R(D) \leq CR1(D) \leq CR2(D)$.

Proof: This follows from the definitions of R, CR1, CR2. ∎

The shape of CR2(D) is therefore as illustrated in Fig. 3.3.

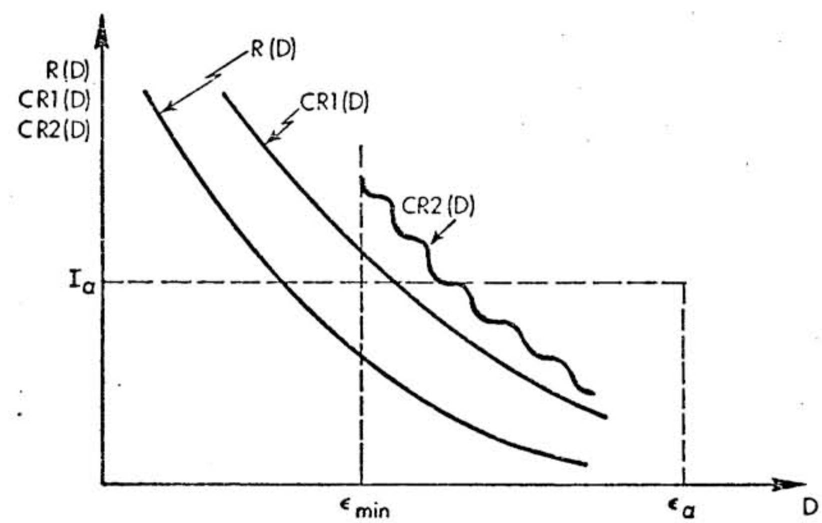

Figure 3.3 Illustration of CR2

Associated with CR2(D) we can define a constrained distortion-rate function CD2(R) as was done in Section 2.2 for the rate distortion function.

$$CD2(R) = \inf_{q_{\hat{x}|\underline{y}} \, \varepsilon \, CO2_R} d(q_{\hat{x}|\underline{y}}) \tag{3.11}$$

$$CO2_R = \left\{ q_{\hat{x}|y} \, \varepsilon \, \psi_2 \,|\, I(x_n; \hat{x}_n) = R \right\}$$

ψ_2: as defined for Eq. 3.18.

Thus CD2(R) is the minimum error that can be achieved at a given level of information $I(x_n; \hat{x}_n)$ subject to the constraints of the NLFP.

In order for CR2(D) to be useful in the NLFP, it must have certain properties as will become evident in the development of this section. We are therefore led to make the following definition which will be used in the form of an assumption in the hypothesis of all propositions in this section.

Definition. CR2(D) is said to be Acceptable if it satisfies the following conditions:

A. CR2(D) is continuous in D.

B. CR2(D) is strictly decreasing* in D in the sense that for any point (D,I) on CR2 (i.e., any point such that CR2(D) = I), if there is an element of ψ_2 producing information $I(x; \hat{x}) = I + \Delta I$, $\Delta I > 0$, then for some $\Delta D > 0$, $CR2(D - \Delta D)$ is defined and $CR2(D - \Delta D) = I + \Delta I$.

The assumption that CR2(D) be acceptable allows us to prove the following lemma which in turn is essential to the other theorems in this section.

Lemma 16. If CR2(D) is acceptable, then:

*A function f(x) will be called strictly decreasing if $x_1 > x_2$ implies $f(x_1) < f(x_2)$.

A. CR2(D) and CD2(R) form identical curves*.

B. The infimums on Eqs. (3.10) and (3.11) are achieved by an element of ψ_2 which determines a point on the CR2(D) and CD2(R) curves.

Proof: A. This follows from the definitions of Eqs. (3.10) and (3.11) just as in the case of the (unconstrained) rate distortion function. B. Consider CR2(D_1) and suppose that the infimum for D_1 is achieved at $q^2_{\hat{x}|\underline{y}}$ and that $d(q^2_{\hat{x}|\underline{y}}) = D_2 < D_1$. Then CR2(D) is constant for $D \varepsilon [D_1, D_2]$ violating the strictly decreasing hypothesis.□

Unfortunately the convexity proof of CR1 in Theorem 1 does not extend easily to CR2 since ψ_2, unlike ψ_1, is not convex. Neither is it easy to show that CR2 so that CR2 and CD2 need not coincide and the infimums need not be achieved on the curves. Nevertheless the concepts of CR2 and CD2 effectively imbed filtering in an information framework. In fact with every filter we can associate a transition $q^{\alpha}_{\hat{x}|\underline{y}}$ which specifies a point $(\varepsilon_\alpha, I_\alpha)$ in information-distortion space as

*The question may be raised why not deal with CD2 and forget about CR2. Three reasons not to do this are: (1) CR2 is useful only when it coincides with CD2, and naturally when they coincide and can deal with either. (2) CR2 seems much easier to deal with than CD2 when proving properties of the curve. See for example the trivial proof of Lemma 14 and consider the difficulty of showing that CD2(R) is defined for every R. (3) Traditionally information theory has used the rate distortion formulation is more convenient when accessing traditional results such as the Shammon lowerbound on the rate distortion function used extensively in this study. On the other hand, the CD2 may be more intuitively pleasing.

illustrated in Fig. 3.3. We say that all filters that lie on I_α belong to the same "iso-information surface" (IIS) and we thus can parametrize the IIS by associating with a particular filter "iso-information parameters" (IIP) -- that is parameters that without changing the information, may change the error. We see that according to Lemma 8 and the discussion of subsection 2.2.2 filters that produce mutually subordinate random variables lie on the same IIS and transformations relating these mutually subordinate random variables constitute IIPs. Thus the concepts of subordination and everywhere dense presented in Chapter 2 allow the closest* characterization of the IIS, IIP concepts.

To give an example of the meaning of IIS and IIP consider the filter

$$\hat{x}_n = f_1(\underline{y})$$

We can in principle compute $I(x_n; \hat{x}_n)$. Consider now the filter whose output is $\hat{\hat{x}}_n$ as follows (remember $\underline{y}$, $\hat{x}_n$, $\hat{\hat{x}}_n$ are random variables)

$$\hat{x}_n = f_1(\underline{y})$$

$$\hat{\hat{x}}_n = f_2(\hat{x}_n)$$

where f_2 is a one-one function. Clearly (see Lemma 7)

*Closer than one-one. See for example proof of Theorem 3.

$$I(x_n;\ \hat{x}_n) = I(x_n;\ \hat{\hat{x}}_n)$$

but, in general

$$\hat{\varepsilon}_n \neq \hat{\hat{\varepsilon}}_n$$

where $\hat{\varepsilon}_n$ and $\hat{\hat{\varepsilon}}_n$ are the respective MSE's. Thus filters $\hat{x}$ and $\hat{\hat{x}}$ are on the same IIS (say, they both lie on I_α in Fig. 3.3), and the parameters that determine f_2 are IIP's of the filter. While the computations involved are certainly non-trivial -- as is simply inherent in general nonlinear problesm -- we can put them immediately to use in the case of the linear-Gaussian reduced order filter considered in the next chapter.

If CR2(D) is acceptable then by Lemma 16 CR2(D) and CD2(D) coincide and every point in these curves can be achieved by an acceptable filter. The objectives of filter design are then, as illustrated on Fig. 3.4, (1) to optimize information, and (2) whether or not a particular filter optimizes information, to manipulate the IIP's to bring the filter to operate on CR2 (hence as close as possible to CR1 and R). The following two theorems concern each of these two steps.

Theorem 2. Assume that CR2(D) is acceptable. If a given filter α achieves information $I(x;\ \hat{x}_\alpha) = I_\alpha$ and MSE ε_α, then another filter α' in the same I_α IIS achieves MSE $\varepsilon_{\alpha'} \leq \varepsilon_\alpha$ given by the solution to the equation

$$CR2(\varepsilon_{\alpha'}) = I_\alpha$$

Proof: Theorem follows from Lemma 16. □

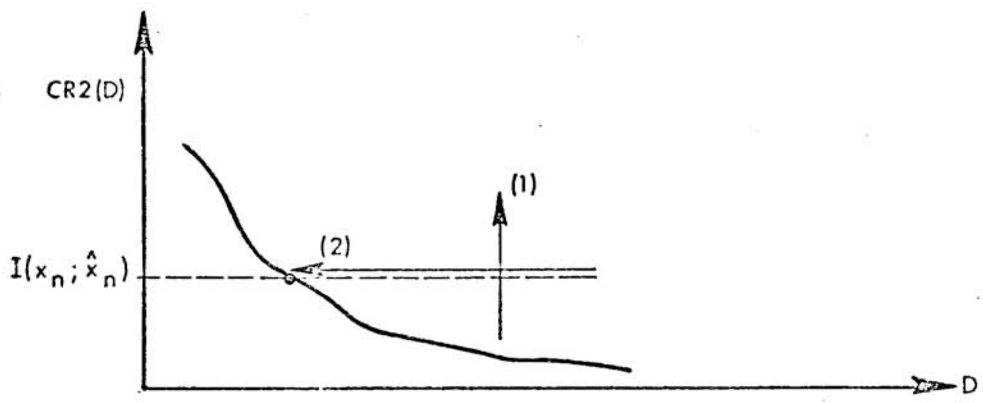

Figure 3.4 Illustration of Objectives of Filter Design for the MLFP (CR2(D) Assumed Strictly Decreasing)

Theorem 3. Consider the NLFP of Section 2.1 and assume that CR2(D) is acceptable. Let $\mathcal{a}$ be the set of all measurable maps A: $R^n \to R$, $\underline{y} \to \hat{x}_n$. Then a necessary condition for $A_o \in \mathcal{a}$ to be the (Kushner) optimum MSE filter so that $\hat{x}_n^* = E(x_n | \underline{y}) = A_o \underline{y}$ is that it maximizes $I(x_n; A\underline{y})$ over all $A \in \mathcal{a}$. Furthermore the Kushner filter looses no information:

$$IL(\hat{x}_n^*) \triangleq I(x_n; \underline{y}) - I(x_n; A_o \underline{y})$$
$$= I(x_n; \underline{y}) - \sup_{A \in \mathcal{a}} I(x_n; A\underline{y}) = 0 \qquad (3.12)$$

(An equivalent result holds for the continuous time NLFP).

Proof: The first assertion is clear from the hypothesis that CR1 be acceptable.

From Lemma 7

$$I(x_n;\ \underline{y}) \geq I(x_n;\ \hat{x}_n^*) \tag{3.13}$$

Let $Y = (R^n,\ S_{\underline{y}},\ P_Y)$ be the probability space associated with the measurements $\underline{y}$. Clearly the measure algebra $(Y(P_Y),P_Y)$ (Halmos (78)) associated with Y is separable, non-atomic, and normalized. By Halmos' Isomorphism Theorem (78) there is an isomorphism A_o' between $(Y(P_Y),P_Y)$ and the measure algebra of the unit interval. Thus $\underline{y}$ and $A_o'\ \underline{y}$ are mutually subordinate* and there is at least one map that looses no information. By the previous theorem (Theorem 2) there is a filter in the same IIS as A_o' with error $\varepsilon_{\alpha'}$ such that $CR2(\varepsilon_{\alpha'}) = I(x_n;\ \underline{y})$. Since CR2 is acceptable, if $I(x_n;\ \underline{y}) > I(x_n;\ \hat{x}_n^*)$ then there would be a filter even better than the Kushner filter. Hence only the equality can hold in Eq. (3.13).□

The prededing two theorems emphasized the role of CR2 in filter design. The following theorem is concerned with filter comparison (see Fig. 3.5)

Theorem 4. Assume that CR2 is acceptable. Suppose that filters α, $\beta\varepsilon\psi_2$ achieve respectively informations $I(x;\ \hat{x}_\alpha) = I_\alpha$ and $I(x;\hat{x}_\beta) = I_\beta$, MSE's ε_α, ε_β, and that $I_\alpha > I_\beta$. Then we can obtain a filter α' in the I_α IIS with MSE $\varepsilon_{\alpha'}$ such that for any filter β' with MSE $\varepsilon_{\beta'}$ in the I_β IIS

*Basically A_o' is "almost surely one-one."

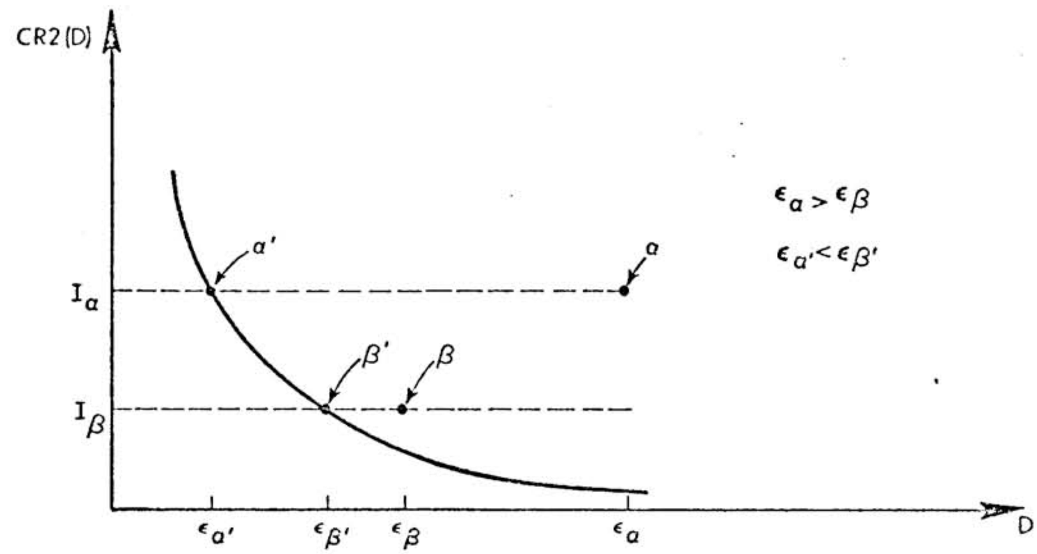

Figure 3.5 Illustration of Filter Comparison According to Theorem 4

$$\varepsilon_{\alpha'} < \varepsilon_{\beta'}$$

irrespective of whether $\varepsilon_\alpha > \varepsilon_\beta$, $\varepsilon_\alpha = \varepsilon_\beta$, $\varepsilon_\alpha < \varepsilon_\beta$.

<u>Proof:</u> Follows from Lemma 16. ∎

3.3.2 CRN(D)

Consider now the case where, unlike the NLFP, constraints are placed upon the allowed filters (e.g., constraining the estimator to be recursive, to be of a certain dimension, etc.). We then replace ψ_2 in the definition of CR2 (Eqs. (3.9), (3.10)) with one of its subsets ψ_N containing only filters that obey the prescribed constraint and proceed to define CRN(D).

$$d(q_{\hat{x}|\underline{y}}) = \int_R \int_{R^n} \int_R (x-\hat{x})^2 \; p_{x|\underline{y}} \; q_{\hat{x}|\underline{y}} \; p_{\underline{y}} \; dx \; d\underline{y} \; d\hat{x}$$

$$CQN_D = \left\{ q_{\hat{x}|\underline{y}} \; \varepsilon \; \psi_N : d(q_{\hat{x}|\underline{y}}) \leq D \right\} \qquad (3.14)$$

ψ_N = subset of ψ_2 satisfying prescribed constraints associated with N

$$CRN(D) = \inf_{q_{\hat{x}|\underline{y}} \; \varepsilon \; CQN_D} I(x;\hat{x}) \qquad (3.15)$$

As with CR2 we emphasize that while in what follows we will speak loosely of "CRN", CRN(D) is the constrained rate distortion function <u>of</u> the source x_n <u>relative</u> to a given distortion measure (MSE here), <u>relative</u> to the sensor, and <u>relative</u> to the set ψ_N of allowed filters. We can also define the corresponding constrained distortion rate function CDN(R) in fashion analogous to Eq. (3.11):

$$CDN(R) = \inf_{q_{\hat{x}|\underline{y}} \; \varepsilon \; CON_R} d(q_{\hat{x}|\underline{y}}) \qquad (3.16)$$

$$CON_R = (q_{\hat{x}|\underline{y}} \; \varepsilon \; \psi_N : I(x_n; \hat{x}_n) = R)$$

ψ_N = as in Eq. (3.15)

The entire development of CR2 in the previous subsection can be duplicated for CRN with exceptions as noted below. We begin with the corresponding definition of an "acceptable CRN" and then summarize all results for CRN in Lemma 17 and Theorem 5.

<u>Definition.</u> CRN(D) is said to be <u>Acceptable</u> if it satisfies the following conditions:

A. CRN(D) is continuous in D.

B. CRN(D) is strictly decreasing* in D in the sense that for any point (D,I) on CRN (i.e., any point such that CRN(D) = I), if there is an element of ψ_N producing information $I(x;\hat{x}) = I + \Delta I$, $\Delta I > 0$, then for some $\Delta D > 0$, CRN(D-ΔD) is defined and CRN(D-ΔD) = I + ΔI.

Lemma 17.

A. If CRN(D) is defined at $D = \varepsilon$ then it is defined for $D \geq \varepsilon$. (Compare with Lemma 14; observe that while the proof of Lemma 14 may not hold for CRN, the definition of CRN alone is sufficient proof.)

B. R(D) ≤ CR1(D) ≤ CR2(D) ≤ CRN(D). (Compare with Lemma 15.)

C. If CRN(D) is acceptable then CRN(D) and CDN(R) form identical curves and the infimums on Eqs. (3.15), (3.16) are achieved by an element of ψ_N which determines a point on the CR2(D), CD2(R) curves. (Compare with Lemma 16.)

Theorem 5.

A. Assume CRN(D) is acceptable. If a given filter $\alpha\varepsilon\psi_N$ achieves information $I(x;\hat{x}_\alpha) = I_\alpha$ and MSE ε_α, then another filter $\alpha'\varepsilon\psi_N$ in the same I_α IIS achieves MSE $\varepsilon_{\alpha'} \leq \varepsilon_\alpha$ given by the solution to the equation

---

*See subsection 3.3.1.

$CRN(\varepsilon_{\alpha'}) = I_{\alpha}$

(compare with Theorem 2.)

B. Assume CRN(D) is acceptable. Then a necessary condition for an element $\alpha \varepsilon \psi_N$ to achieve minimum MSE is that it maximizes $I(x; \hat{x}_{\alpha})$ over ψ_N. (Compare with Theorem 3; observe that the proof of the zero loss of information does not carry through in this case since A_o', along with many other elements of ψ_2, may not belong to ψ_N.)

C. Assume that CRN(D) is acceptable. Suppose that filters α, $\beta \varepsilon \psi_N$ achieve respectively informations $I(x; \hat{x}_{\alpha}) = I_{\alpha}$ and $I(x; \hat{x}_{\beta}) = I_{\beta}$, MSE's ε_{α}, ε_{β}, and that $I_{\alpha} > I_{\beta}$. Then we can obtain a filter $\alpha' \varepsilon \psi_N$ in the I_{α} IIS with MSE $\varepsilon_{\alpha'}$ such that for any filter $\beta' \varepsilon \psi_N$ with MSE $\varepsilon_{\beta'}$ in the I_{β} IIS

$$\varepsilon_{\alpha'} < \varepsilon_{\beta'}$$

irrespective of whether $\varepsilon_{\alpha} > \varepsilon_{\beta}$, $\varepsilon_{\alpha} = \varepsilon_{\beta}$, $\varepsilon_{\alpha} < \varepsilon_{\beta}$. (Compare with Theorem 4.)

3.3.3 CRN(D), $N \geq 2$, For Non-MSE Distortion Measures

Up to this point we have considered exclusively MSE as a measure of disto-tion. While MSE is a useful PI many physical situations arise where other measures of distortion are more relevant. Shannon's rate distortion theory is applicable to a wide variety of distortion measures and therefore it is only natural that information concepts can be equally applied to filter design under non-MSE distortion and in fact the preceding development for CR2 and general CRN can be identically reproduced (with the obvious exceptions noted below) for non-MSE measures is distortion.*

First we define distortion following Shannon and in reference to the formulation of Section 3.2 (so that we only need to consider a per letter or point PI). A distortion measure ρ is defined as a function**

$$\rho : (x,\hat{x}) \to r, \; r \in R$$

such that

$$\rho(\cdot,\cdot) \geq 0 \text{ for all } x, \hat{x} \in R$$

$$\rho(x,\hat{x}) = 0 \text{ for at least one } (x,\hat{x}) \text{ pair}$$

Average distortion d is defined as

$$d = E\, \rho(x(\omega),\, \hat{x}(\omega))$$

provided the expectation exists and is finite. In what follows the terms "distortion," "distortion measure" are used for either distortion or average distortion, the meaning being clear from the context.

---

* See Fig. 3.6 of Section 3.4 for an example of a non-MSE distortion measure.

**As in the prèvious development the state x and estimate $\hat{x}$ are taken as scalar for simplicity in notation, but all the results apply in analogous way to the vector case.

Consider any $N \geq 2$*. Let be an arbitrary distortion measure as just defined. We can then define CRN by replacing Eqs. (3.9), (3.14) with Eq. (3.17):

$$d(q_{\hat{x}|\underline{y}}) = \int_R \int_{R^n} \int_R \rho(x,\hat{x})\; p_{x|\underline{y}}\; q_{\hat{x}|\underline{y}}\; p_{\underline{y}}\; dx\; d\underline{y}\; d\hat{x} \qquad (3.17)$$

$$CQN_D = \left\{ q_{\hat{x}|\underline{y}} \; \varepsilon \psi_N \; : \; d(q_{\hat{x}|\underline{y}}) \leq D \right\}$$

ψ_N = subset of ψ_2 satisfying prescribed constraints (if any) associated with N.

$$CRN(D) = \inf_{q_{\hat{x}|\underline{y}} \; \varepsilon \; CQN_D} I(x;\hat{x}) \qquad (3.18)$$

As with CR2, CRN of the previous subsections we empahsize that while in what follows we will speak loosely of "CRN," CRN(D) is the constrained rate distortion function <u>of</u> the source x_n <u>relative</u> to the given distortion measure ρ, <u>relative</u> to the sensor, and <u>relative</u> to the set ψ_N of allowed filters. We can also define the corresponding constrained distortion rate function CDN(R) in fashion analogous to (3.16)

$$CDN(R) = \inf_{q_{\hat{x}|\underline{y}} \; \varepsilon \; CON_R} d(q_{\hat{x}|\underline{y}}) \qquad (3.19)$$

---

*We recall that for N=2 we deal (as in subsection 3.3.1) with ψ_2, CR2 and the NLFP where receiver structure is unconstrained. For filter constraints characterized by N, we deal (as in subsection 3.3.2) with ψ_N, CRN.

$$CON_R = \left\{ q_{\hat{x}|\underline{y}} \, \varepsilon \psi_N : I(x_n;\hat{x}_n) = R \right\}$$

ψ_N = as in Eq. (3.17)

As in the previous subsections we now give the definition of the acceptable CRN and then summarize in Lemma 18 and Theorem 6 the relevant version of the previous results.

Definition. CRN is said to be Acceptable for the distortion measure ρ if it satisfies the following conditions:

A. CRN(D) is continuous in D.

B. CRN(D) is strictly decreasing* in D in the sense that for any point (D,I) on CRN (i.e., any point such that CRN(D) = I), if there is an element of ψ_N producing information $I(x;\hat{x}) = I + \Delta I$, $\Delta I > 0$, then for some $\Delta D > 0$, $CRN(D-\Delta D)$ is defined and $CRN(D-\Delta D) = I + \Delta I$.

Lemma 18.

A. If CRN(D) is defined at D = d then it is defined for $D \geq d$. (Compare with Lemmas 14, 17A; comments in Lemma 17A apply.)

B. $R(D) \leq CR1(D) \leq CR2(D) \leq CRN(D)$. (Compare with Lemmas 15, 17B.)

C. If CRN(D) is acceptable then CRN(D) and CDN(R) form identical curves and the infimums on Eqs. (3.18), (3.19) are achieved by an element of ψ_N which determines a point on the CRN(D), CDN(R) curves. (Compare with Lammas 16, 17C).

*See subsection 3.3.1.

Theorem 6.

A. Assume CRN(D) is acceptable. If a given filter $\alpha\varepsilon\psi_N$ achieves information $I(x;\hat{x}_\alpha) = I_\alpha$ and distortion d_α, then another filter $\alpha'\varepsilon\psi_N$ in the same I_α IIS achieves distortion $d_{\alpha'} \leq d_\alpha$ given by the solution to the equation

$$CRN(d_{\alpha'}) = I_\alpha$$

(Compare with Theorems 2, 5A.)

B. Assume CRN is acceptable. Then a necessary condition for an element $\alpha\varepsilon\psi_N$ to achieve minimum distortion is that it maximizes $I(x;\hat{x}_\alpha)$ over ψ_N. (Compare with Theorems 3, 5B; comments in Theorem 5B apply; also observe that the Kushner estimate is a minimum MSE estimate.)

C. Assume that CRN(D) is acceptable. Suppose that filters α, $\beta\varepsilon\psi_N$ achieve respectively informations $I(x;\hat{x}_\alpha) = I_\alpha$ and $I(x;\hat{x}_\beta) = I_\beta$, distortions d_α, d_β, and that $I_\alpha > I_\beta$. Then we can obtain a filter α' in the I_α IIS with distortion $d_{\alpha'}$ such that for any filter β' with distortion $d_{\beta'}$ in the I_β IIS

$$d_{\alpha'} < d_{\beta'}$$

irrespective of whether $d_\alpha > d_\beta$, $d_\alpha = d_\beta$, $d_\alpha < d_\beta$. (Compare with Theorems 4, 5C).

3.3.4 Summary

Summarizing, the concept of the constrained rate distortion function introduced in this section, used in connection with the filtering problem furmulation of Section 3.2,

(1) Establishes the relation between information and distortion, including MSE, in filtering.

And provided the relevant constrained rate distortion functions are acceptable,

(2) Establishes the following two step filter design procedure:

A. Optimize the information that the estimate has about the state (alternatively minimize the information lost in the filter).

B. Adjust the IIP's to achieve good performance relative to any desired distortion measure.

Observe that the first step is independent of any distortion measure. In the second step, by adjusting the IIP's of a filter we can tailor filter performance to minimize a particular distortion measure. Naturally the particular IIP's that are best for one distortion measure may not be desirable for a different one. Thus information design is more general than desing for a particular distortion measure. Furthermore, the ensuing decoupling in processor space search evident in this design procedure may be computationally advantageous. (See Fig. 3.4 and Theorems 2, 5A, 6A, 3, 5B, 6B.)

(3) Establishes a basis for suboptimal filter comparison in terms of the information about the state contained in the suboptimal estimate (alternatively in terms of the information lost in a suboptimal filter). Thus, regardless of the relative performance of any two filters (e.g., regardless of their MSE's), the filter with more information (alternatively the filter that looses less information) has the potential for better performance. (See Fig. 3.5 and Theorems 4, 5C, 6C.)

The questions that remained unanswered are:

(1) Whether the constrained rate distortion functions are acceptable.

(2) Given a filter what are the IIP's that can be used to obtain optimum performance for a distortion measure.

That the constrained rate distortion functions are acceptable is a reasonable thing to expect in view of -- among other reasons -- both the intuitive and formal definitions of Shannon information.

In regards to the second question we note that the theorems in this section have been worded in terms of "... then there is another filter in the IIS..." rather than "... then by adjusting IIP's another filter can be achieved..." Clearly both statements are equivalent since by definition IIP's parametrize the IIS so that indeed any element of the IIS can be achieve by "adjusting IIP's." Thus the relevant question is not whether IIP's exist but rather what are the IIP's of a problem.

Finally we remark that all the concepts of this section will be clearly illustrated in terms of a simple example when the design of reduced order filters is consdiered in the next chapter.

3.4 Distortion Lowerbounds for Optimum and Suboptimum Filters

Computation of the constrained rate distortion functions of the previous section is in general very difficult. In this section, using the Shannon lowerbound on the rate distortion function applied to the formulation of Section 3.2, we derive formulas for distortion lowerbounds for optimum and suboptimum filters that, in addition to being useful in their own right, ressemble and illustrate Theorems 2, 4 and their counterparts on subsections 3.3.2, 3.3.3.

3.4.1 MSE Distortion Measure

The Shannon lowerbound on the rate distortion function is as follows.*

Lemma 19. (Shannon). Let R(D) be the (per letter) rate distortion function with respect to the MSE fidelity criterion defined as in Eq. (2.11). Then

$$R(\varepsilon) \geq H(x) - \tfrac{1}{2}\log(2\pi e\varepsilon) \tag{3.20}$$

As was noted in subsection 2.2.4, $R(\varepsilon)$ is the effective entropy (average self-information) of the source. Thus, intuitively, Eq. (3.20) says that the entropy of the source H(x) is decreased by a quantity which increases with the MSE ε.

We now obtain a lowerbound on optimum MSE for the discrete time NLFP of Section 2.1 (the continuous time results is given in Section 3.5). This bound is implemented in Chapter 5 and applied in Chapter 6 to the Phase Locked Loop.

Theorem 7. Consider the discrete NLFP of Section 2.1 as modeled by the formulation of Section 3.2. Let (recall $\underline{y} = (y_1, \ldots, y_n)$)

$$\hat{x}_n^* = E(x_n|\underline{y}), \quad \varepsilon_n^* = E(x_n - \hat{x}_n)^2$$

---

*Zakai and Ziv (38) were the first to apply Shannon's lower bound to compute a lower bound on optimal filtering MSE for messages modeled by dynamical systems. See Section 5.2 for a summary of their paper.

Then

$$\varepsilon_n^* \geq \frac{1}{2\pi e} \exp 2 \left(H(x_n) - I(x_n; \underline{y})\right) \qquad (3.21)$$

Proof: By definition of CR2 and Lemma 7,

$$CR2(\varepsilon_n^*) \leq I(x_n; x^*) \leq I(x_n; \underline{y}) \qquad (3.22)$$

By Lemmas 19 and 15

$$H(x_n) - \tfrac{1}{2}\log(2\pi e\ \varepsilon_n^*) \leq R(\varepsilon_n^*) \leq CR2(\varepsilon_n^*) \qquad (3.23)$$

Combining Eqs. (3.22) and (3.23)

$$H(x_n) - \tfrac{1}{2}\log(2\pi e\ \varepsilon_n^*) \leq I(x_n; \underline{y}) \qquad (3.24)$$

Upon solving for ε_n^* in Eq. (3.24) we get Eq. (3.21).□

The above proof could be made simpler by suppressing CR2 from Eqs. (3.22), (3.23) since CR2 plays no essential part in this proof (nor is it relevant whether CR2 is acceptable). The extra detail just illustrates where "bounding" takes place.

Perhaps more important than a lower bound on optimum MSE is a lower bound for suboptimum MSE which would tell us, given the information loss of a particular suboptimum filter (or alternatively the information about the state contained in the estimate of the filter), how well can this filter perform Such a bound is derived in Theorem 8 and will be applied in Chapter 4 to reduced order filter design where computation of information is elementary. Before proving Theorem 8, we derive a formula for information loss in the following lemma.

Lemma 20. Consider the discrete NLFP of Section 2.1. Let $\hat{x}_n$ be an optimum or suboptimum estimate of x_n based on the data $\underline{y}$, i.e., $\hat{x}_n$ is a measurable function of $\underline{y}$. Define the information loss produced by the filter (i.e., produced by the transition in Fig. 3.1) as

$$IL(\hat{x}_n) \triangleq I(x_n; \underline{y}) - I(x_n; \hat{x}_n) \qquad (3.25)$$

Then the information loss is given by

$$IL(\hat{x}_n) = I(x_n; \underline{y}|\hat{x}_n) \qquad (3.26)$$

Proof: By Lemma 6 (Kolmogorov's formula) we have

$$\begin{aligned} I(x_n; \underline{y},\hat{x}_n) &= I(x_n; \underline{y}|\hat{x}_n) + I(x_n; \hat{x}_n) \\ &= I(x_n; \hat{x}_n|\underline{y}) + I(x_n; \underline{y}) \qquad (3.27) \end{aligned}$$

Now $\hat{x}_n$ is a measurable function of $\underline{y}$ and afortiori subordinate to $\underline{y}$. Hence by Lemma 9,

$$I(x_n; \hat{x}_n|\underline{y}) = 0$$

Eq. 3.13 then gives

$$I(x_n; \underline{y}|\hat{x}_n) + I(x_n; \hat{x}_n) = I(x_n; \underline{y})$$

so that indeed

$$IL(\hat{x}_n) = I(x_n; \underline{y}|\hat{x}_n) \quad \blacksquare$$

We note that Eq. (3.26) "makes sense" from an intuitive point of view since the information about x_n lost is that remaining in the data $\underline{y}$ if the estimate $\hat{x}_n$ is given.

Theorem 8. Consider the formulation of the filtering problem of Section 3.2 and suppose that $\hat{x}_n$ is the estimate of x_n produced by a certain filter ($\hat{x}_n$ is a measurable function of the data). Suppose further that, using the terminology of Section 3.3, this filter is constrained to belong to a class ψ_N where as before we let N=2 for the NLFP). Let

$$\varepsilon_n = E(x_n - \hat{x}_n)^2$$

Then,

$$\varepsilon_n \geq \frac{1}{2\pi e} \cdot \exp 2 \left(H(x_n) - I(x_n; \hat{x}_n)\right) \tag{3.28}$$

so that

$$\varepsilon_n \geq \frac{1}{2\pi e} \cdot \exp 2 \left(IL(\hat{x}_n)\right) \cdot \exp 2 \left(H(x_n) - I(x_n; \underline{y})\right) \tag{3.29}$$

and

$$\varepsilon_n \geq \frac{1}{2\pi e} \cdot \exp 2 \left(I(x_n; \underline{y}|\hat{x}_n)\right) \cdot \exp 2 \left(H(x_n) - I(x_n; \underline{y})\right) \tag{3.30}$$

Proof: Clearly if (3.28) is true, (3.29) and (3.30) follow immediately so we need only show (3.28). By the definition of CR2,

$$CR2(\varepsilon_n) \le I(x_n;\ \hat{x}_n)$$

By Lemmas 19 and 15,

$$H(x_n) - \tfrac{1}{2}\log(2\pi e\ \varepsilon_n) \le R(\varepsilon_n) \le CR2(\varepsilon_n)$$

so that

$$H(x_n) - \tfrac{1}{2}\log(2\pi e\ \varepsilon_n) \le I(x_n;\ \hat{x}_n)$$

Solving for ε_n gives (3.28).□

The formulas derived in Theorems 7 and 8 are intuitively pleasing. Eq. (3.21) says that the greater $I(x_n;\ \underline{y})$ is -- hence the greater the information about x_n contained in the data $\underline{y}$, i.e., the better the sensor is -- the smaller will the best possible error ε_n^* be.

The bound for suboptimum error given in Eq. (3.29) has two terms. The first term is identical to the right hand side of Eq. (3.21) and accounts for the error due to sensor performance. The second term, $\exp 2(IL(\hat{x}))$ accounts for the error produced by the information lost in the filter, the greater this loss the greater the expected error.

Finally we note that the suboptimum error bound given by Theorem 8 clearly applies to all filters in the same IIS as the filter under consideration.

3.4.2 Non-MSE Distortion Measures

Consider now the general class of distortion meausres defined on subsection 3.3.3. The Shannon lower bound (Lemma 19) is readily applicable to difference distortion measures (See Berger (20), Shannon (22)) as follows.

<u>Lemma 19A</u> (Shannon). Let R(D) be the rate distortion function of a source x relative to a distortion measure as defined in subsection 3.3.3 where in addition* $\rho(x,y) = \rho(x-y)$. Let d be the associated average distortion as in Eq. (3.17). Then R(D) is lower bounded by

$$H(x) - \phi(D) \leq R(D) \tag{3.31}$$

where

$$\phi(D) = \sup_{g \varepsilon G_D} \int_{-\infty}^{\infty} [-\log g(\xi)]\ g(\xi)\ d\xi \tag{3.32}$$

$$G_D = \left\{ g: \text{PDF}: \left[\int_{-\infty}^{\infty} \rho(z)\ g(z)\ dz \right] \leq D \right\}$$

Using the calculus of variation on Eq. (3.21) gives

$$\phi(D) = \int_{-\infty}^{\infty} \left[-\log g_s(\xi)\right]\ g_s(\xi)\ d\xi$$

where

$$g_s(u) = \frac{\varepsilon^{s\rho(u)}}{\int_{-\infty}^{\infty} \varepsilon^{s\rho(z)}\ dz}$$

---

*I.e., ρ is a difference distortion measure.

for s such that

$$\int_{-\infty}^{\infty} \rho(z)\; g_s(z)\; dz = D$$

The function $\phi(D)$ is positive, non-decreasing and convex $\cap$ in D. It is evaluated for different distortion measures in the literature. In particular we mention Pinkston (43) who considered the interesting distortion measure of Fig. 3.6.

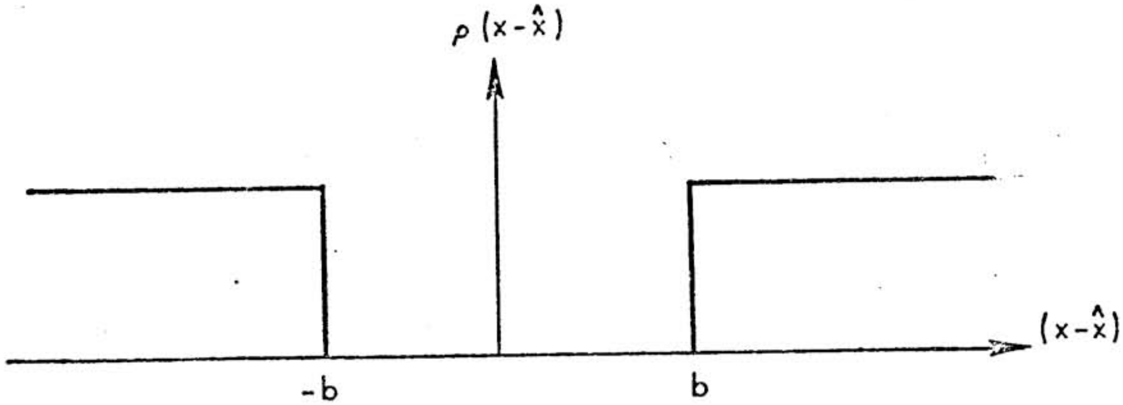

Figure 3.6 A Distortion Measure Considered By Pinkston

Clearly as b becomes small we obtain a lower bound for MAP filtering.

The counterparts of Theorems 7 and 8 can now be obtained.

Theorem 7A. Consider the discrete MLFP of Section 2.1 as modeled by the formulation of Section 3.2 but with arbitrary difference distortion measure d_n. Then the optimal error d_n^* that can be achieved is lower bounded by

$$d_n^* \geq \phi^{-1} \; (H(x_n) - I(x_n;\; \underline{y})) \tag{3.33}$$

where ϕ^{-1} is one of the solutions of

$$\phi(D) = H(x_n) - I(x_n;\; \underline{y})$$

<u>Theorem 8A.</u> Consider the formulation of the filtering problem of Section 3.2 and suppose that $\hat{x}_n$ is the estimate of x_n produced by a certain filter ($\hat{x}_n$ is a measurable function of the data). Suppose further that, using the terminology of Section 3.3, this filter is constrained to belong to a class ψ_N (where as before we let N=2 for the NLFP). Let ρ be a distortion measure as defined on subsection 3.3.3 where in addition $\rho(x,y) = \rho(x-y)$. Let d_n be the associated average distortion as in Eq. (3.105) so that

$$d_n = d_n(x_n - \hat{x}_n)$$

Then

$$d_n \geq \phi^{-1} \; (H(x_n) - I(x_n;\; \hat{x}_n)) \tag{3.34}$$

so that

$$d_n \geq \phi^{-1} \; (IL(\hat{x}_n) + H(x_n) - I(x_x;\; \underline{y})) \tag{3.35}$$

and

$$d_n \geq \phi^{-1} \; (I(x_n;\; \underline{y}|\hat{x}_n) + H(x_n) - I(x_n;\; \underline{y})) \tag{3.36}$$

where again $\phi^{-1}(\alpha)$ is one of the solutions of $\phi(D) = \alpha$.

Proofs: The proofs of the above theorems follow from the properties of ϕ as those of Theorems 4 and 5. ☐

Observe that since ϕ is strictly increasing in D the same comparison of Eqs. (3.33) and (3.35) is possible as in the case of Eqs. (3.27) and (3.29) for MSE. Thus, the bound in the distortion increases with information loss above the bound for optimal error.

3.5 Continuous Time Results

All of the preceding definitions and result extend without difficulty to the continuous time case. In particular we note that the existence of the source map of Fig. 3.1 is assured by the Bucy-Mortenson-Duncan representation theorem. We give here continuous time counterparts only of the bound theorems since we shall need then in the sequel and to give more would be repetitious.

Theorem 7C. Consider the continuous NLFP of Section 2.1. Let

$$\hat{x}_t^* = E(x_t | y_o^t), \quad \varepsilon^*(t) = E(x_t - \hat{x}_t^*)^2$$

Then

$$\varepsilon^*(t) \geq \frac{1}{2\pi e} \exp 2 \, (H(x_t) - I(x_t; y_o^t))$$

where $H(x_t)$ is the entropy of x_t.

Theorem 8C. Consider the continuous NLFP of Section 2.1 and suppose that $\hat{x}_t$ is the estimate of x_t produced by a certain filter based on the data y_o^t ($\hat{x}_t$ is a measurable function of the data). Let

$$\varepsilon(t) = E(x_t - \hat{x}_t)^2$$

Then,

$$\varepsilon(t) \geq \frac{1}{2\pi e} \exp 2\,(H(x_t) - I(x_t;\, x_t))$$

so that

$$\varepsilon(t) \geq \frac{1}{2\pi e} \exp 2\,(IL(\hat{x}_t)) \cdot \exp 2\,(H(x_t) - I(x_t;\, y_o^t))$$

and

$$\varepsilon(t) \geq \frac{1}{2\pi e} \exp 2\,(I(x_t;\, y_o^t | \hat{x}_t)) \cdot \exp 2\,(H(x_t) - I(x_t;\, y_o^t))$$

where $IL(\hat{x}_t)$ is the information loss associated with the filter.

In Chapter 5 a concrete formula will be provided for the lowerbound of Theorem 7C.

3.6 Summary and Conclusions

In this chapter we first gave a comfortable formulation of the filtering problem, a formulation that avoids all the unnatural* features of the ITPWD arrived at in Chapter 2.

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*Insofar as the filtering problem is concerned.

Based on this formulation the concepts of CR1, CR2, CRN were introduced. The constrained rate distortion functions effectively imbed the filtering problem in an information theoretic framework and establish design and comparison basis for optimum and suboptimum filters, for MSE and non-MSE distortion measures. Distortion lowerbounds also applicable to optimum and suboptimum filters, MSE and non-MSE distortion measures, were then developed based on the above mentioned formulation and the Shannon lowerbound on the rate distortion function.

The net result has been to establish the relation* between "Information and Distortion in Filtering Theory."** In particular we note that information is not distortion in filtering but rather is related to distortion. That, if the relevant constrained rate distortion functions are acceptable, Shannon information is information in the common sense of the word which can be used to minimize any distortion measure. And that, when properly formulated, Shannon information is relevant in the filtering problem outisde of a coding context.

In the next chapter it will be possible to put the concepts of Chapter 3 to good use in the design of reduced order filters.

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* Perhaps "a relation" would be more appropriate terminology.
**The quotes emphasize that this is the title of this study.

CHAPTER 4

REDUCED ORDER FILTERING

4.1 The ROFP

The objective of this chapter is to illustrate the usefulness of the concepts developed in the previous chapter by applying them to a very relevant problem, the design of reduced order filters, which on account of being set in a linear gaussian environment exhibits special computational tractability.

In the reduced order filter problem (hereafter abbreviated as ROFP) we start with the usual linear gaussian filtering problem with MSE performance index defined by *,**

$$x_{k+1} = \Phi(k+1,\ k)\ x_k + G_k\ w_k \tag{4.1}$$

$$y_k = H_k\ x_k + v_k \tag{4.2}$$

where

$$\begin{aligned} x_o &= N(O,\ P_o) \\ w_k &= N(O,\ Q_k) \\ v_k &= N(O,\ R_k) \end{aligned} \tag{4.3}$$

are mutually uncorrelated as usual. While the Kalman filter

*The development in this chapter is in discrete time matching the presentation of Chapter 3 and simplifying the numerical computations. An analogus development can be carried out in continuous time.

**The dimensions of all the vectors and matrices are of course assumed to compatible.

provides the best performance, it presents very often an unacceptable computational burden for online implementation (see Gelb et.al.(79), Chapters 7,8). The search for a "smaller" suboptimum filter of "reduced order" characterizes the ROFP.

There are many variations of the ROFP but the basic problem is to estimate some (or all) of the n state variables in x_k (or linear combinations of them) by a filter constrained to belong to a given class. The optimal solution is inherently a two point boundary value problem (i.e., a terminal cost is present) as has been shown by Johansen(80) and as is evident from the matrix minimum principle (81), (82). This solution, however is usually more computationally taxing than the Kalman filter and optimization over everything need not be the best design approach. Different techniques of reduced order filtering have appeared in the literature among them Aoki-Huddle (88), Center (84), Damiani (90), Joseph (85), Hutchinson-D'Appolito (83), Meditch (86), Pentecost (87), Uttam-O'Halloran (94), (95).

The approach taken in this chapter is as follows. With one exception the development in this chapter is given in terms of a simple example, that illustrates concretely (by specific graphs and equations) the abstract notions of the previous chapter. This simple example (defined in Section 4.2) will be refered to as the ROFE (reduced order filter example) to distinguish it from the general ROFP. The one exception is the derivation of a performance lowerbound (Section 4.3) which is applicable to the general ROFP. Section 4.4 contains a number of necessary computational lemmas which are then used in Section 4.5 to implement the information design and comparison for the ROFE.

4.2 The ROFE and Its Information Theoretic Formulation

The example that we will consider is an oscillator with a random bias that can be modeled as a three dimensional continuous time system by the equations (Damiani (90))

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega^2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w_t \qquad (4.4)$$

with random initial condition

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = N(0, \begin{bmatrix} x_{01} & 0 & 0 \\ 0 & x_{02} & 0 \\ 0 & 0 & x_{03} \end{bmatrix}) \qquad (4.5)$$

and process noise

$$w_t = N(0,q) \qquad (4.6)$$

The elliptical unforced (w_t=0) phase plane trajectory is illustrated in Fig. 4.1. We take noisy discrete measurements of x_3 according to

$$y_{t_k} = [0 \quad 0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + v_k \qquad (4.7)$$

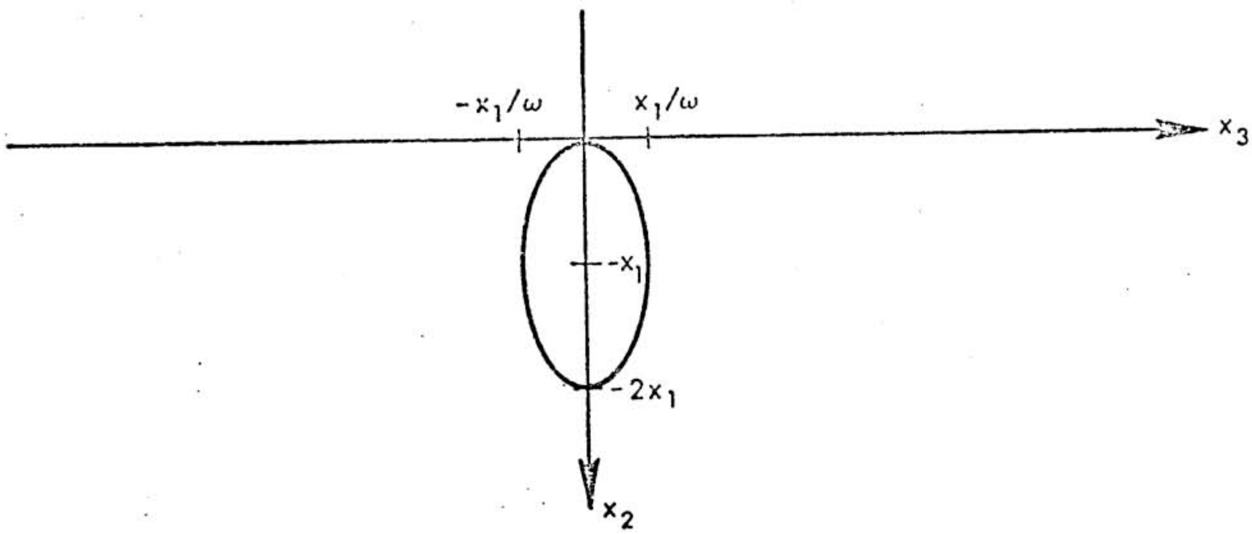

Figure 4.1 Unforced Continuous Time Trajectory of the ROFE

where

$$v_k = N(0, R) \tag{4.8}$$

We are interested in estimating the bias x_1 with a <u>one dimensional</u> filter, that is, without recourse to the full three dimensional* Kalman filter.

The numerical values of the parameters used are as follows:

$x_{o1} = 2^2$, $x_{02} = (.02)^2$, $x_{03} = (.25)^2$

ω = 21 radians/hour , $q = (.05)^2$, $R = (.02)^2$

Sampling rate: One sample every 3 minutes

Units of time variable: hours

*This loose terminology of course ignores the Ricati variables as filter state.

We observe that the system is detectable (91,92) (but not observable).

Equations (4.4) to (4.8) can be cast into the discrete time in the usual manner so that we obtain the equations of the ROFE:

$$x_{k+1} = \Phi(t_{k+1}, t_k)\, x_k + W_k \tag{4.9}$$

$$y_{k+1} = H\, x_k + V_k \tag{4.10}$$

where

$$\left[x_k = x_{1,k}\ x_{2,k}\ x_{3,k}\right]'$$

$$\Phi(t_{k+1}, t_k) = \begin{bmatrix} 1 & 0 & 0 \\ -1+\cos\omega(t_{k+1}-t_k) & \cos\omega(t_{k+1}-t_k) & -\omega\sin(t_{k+1}-t_k) \\ \frac{1}{\omega}\sin\omega(t_{k+1}-t_k) & \frac{1}{\omega}\sin\omega(t_{k+1}-t_k) & \cos\omega(t_{k+1}-t_k) \end{bmatrix} \tag{4.11}$$

$$W_k = N(0, Q_k)$$

$$Q_k = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, s) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q \end{bmatrix} \Phi'(t_{k+1}, s)\, ds$$

$$H = [0\ \ 0\ \ 1]$$

and v_k and the initial conditions are as in Eqs. (4.5), (4.8). Letting

ψ_3 = {class of all one dimensional filters of the form* $\hat{x}_{k+1} = A_k\, \hat{x}_k + B_k\, y_k$ where A_k, B_k are chosen sequentially at each k}

*The bias x_1 has zero mean so that it is not necessary to include a bias in the class of all one dimensional filters.

the problem in the ROFE is to select an element of ψ_3 that gives good performance in terms of the MSE*

$$\varepsilon_k = E\ (x_1 - \hat{x}_k)^2 \tag{4.13}$$

Naturally the best performance possible is obtained by $\hat{x}_k* = E(x_k | y_o^k)$ as given by the Kalman filter and illustrated in Fig. 4.2. By the definition of the problem, however, we cannot use the Kalman filter and are forced use the Kalman filter and are forced to choose a suboptimal scheme within ψ_3. We choose for comparison** a very interesting approach to the ROFP advanced by Hutchinson and D'Appolito (83) who obtain filters of lower order by using for filter dynamics the projection of the dynamics of the original system to the subspace of the part of the state that is of interest.*** For the ROFE the subspace of interest is that of x_1 and the Projection of Dynamics filter (PDF) takes the form

$$\hat{x}_{k+1} = x_k + K_{PD}\ y_{k+1} \tag{4.14}$$

K_{PD}: Chosen according to the Projection of Dynamics algorithm (Eqs. (4.54) of Section 4.4)

The performance of the PDF for the ROFE is also shown in Fig.4.2.

---

*Since the bias is constant we often omit the "k" in $x_{1,k}$.

**We emphasize that the object here is not to assess the relative performance of this suboptimal scheme in general nor even on this example but rather to illustrate how information concept enter into reduced order filter design.

***The equations for the projection of dynamic filter of (83) are listed in subsection 4.4.2.

A question that naturally arises is what performance results if we choose filter gain in Eq. (4.14) by optimizing the information $I(x_1 ; \hat{x}_k)$:

$$x_{k+1} = x_k + K_{INFO}\, y_{k+1} \tag{4.15}$$

K_{INFO}: Selected at every k to optimize $I(x_1; \hat{x}_k)$(Eq. (4.48) of Section 4.4)

As will be shown later such a filter obtains the maximum information $I(x_1 ; \hat{x}_k)$ (minimum information loss) over all (one dimensional) filter in class ψ_3 of Eq. (4.12). The performance of this filter is as shown in Fig. (4.2).

Without the theory of Chapter 3 one may be tempted to reach the following conclusions:

(1) Select (possibly as a starting point in the design process) the PDF, or in any case do not select the information optimizing filter.

(2) Information says little if any about MSE design.

Or else,

(2′) Information is just a different performance index which perhaps is useful in some problem.

The theory of Chapter 3, however, says otherwise and in the remainder of this chapter we will demonstrate how all three of these conclusions are incorrect.

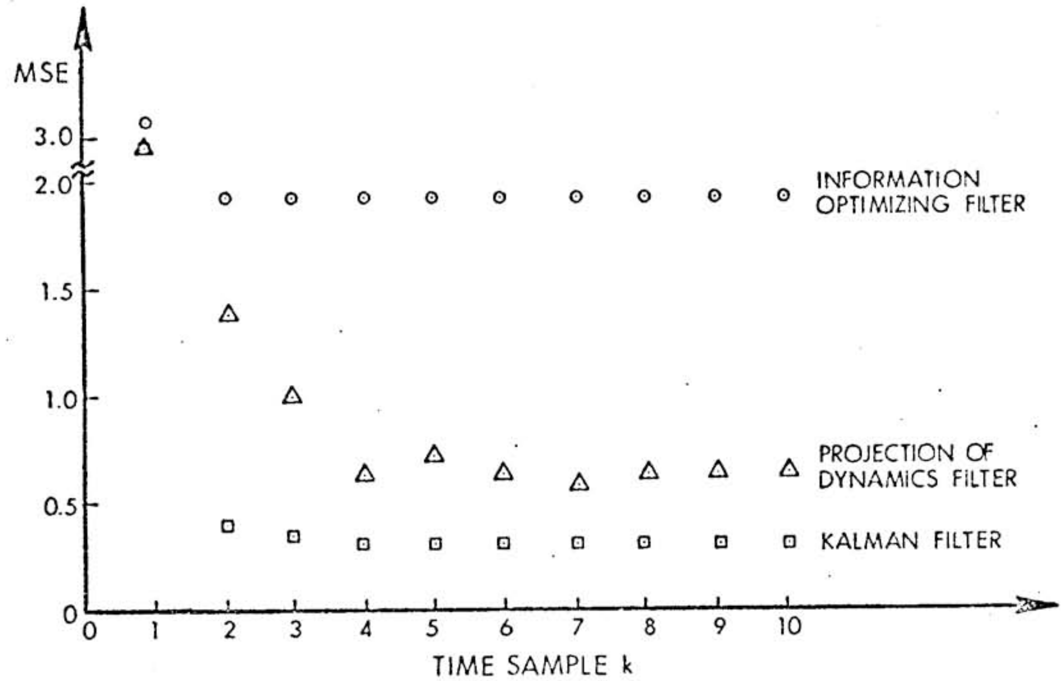

Figure 4.2 Comparison for the ROFE of the Performance of the Kalman, Projection of Dynamics, and Information Optimizing Filters

We now start to introduce the theory of Chapter 3 into the ROFE. The information theoretic formulation of the filtering problem of Section 3.2 is illustrated in Fig. 4.3. The source puts out only $x_{1,k}$ (although k is arbitrary)* which gets mapped into y_o^k by the random map characterized by $p(x_{1,k}|y_o^k)$. The filter is restricted to the class ψ_3 of Eq.(4.12) and it will be useful to associate with any filter** $\hat{x}_k$ in ψ_3 its information loss at sample k

*Or else the source puts out a sequence $x_{1,k}$, $x_{1,k}$, $x_{1,k}$, ... of independent identically distributed rvs.

**As is common practice we confuse the concepts of "estimate" and "estimator" by saying "the filter $\hat{x}_k$".

$$IL\ (\hat{x}_k) \triangleq I(x_{1,k}\ ;\ y_o^k) - I(x_{1,k}\ ;\ \hat{x}_k) \quad (4.16)$$

$$= I(x_1\ ;\ \hat{x}^*_{1,k}) - I(x_{1,k}\ ;\ \hat{x}_k) \quad (4.17)$$

where Eq. (4.17) follows from the fact that according to Lemma 10 (Bucy) of Chapter 2 the Kalman filter looses no information.

As in Eq. (3.15) of subsection 3.3.3 we define the constrained rate distortion function CR3(D) for the ROFE:

$d(q_{\hat{x}|\underline{y}})$ = as in Eq. (3.14)

ψ_3 = as in Eq. (4.12)

$CQ3_D = \{q_{x|\underline{y}}\ \varepsilon\psi_3\ :\ d(q_{\hat{x}|y}) \leq D\}$

$$CR3(D) = \inf_{q_{\hat{x}|\underline{y}}} I(x_1\ ;\ x_k) \quad (4.18)$$

Eq. (4.18) defines the constrained rate distortion function of the source $x_{1,k}$ relative to the MSE distortion measure ε_k of Eq. (4.13), relative to the given sensor, and relative to the class ψ_3 of allowed filters. We can also defined the corresponding constrained distortion rate function as in Eq. (3.16)

$$CDE(R) = \inf_{q_{\hat{x}|\underline{y}}\ \varepsilon\ CO_R} d(q_{\hat{x}|\underline{y}}) \quad (4.19)$$

$CO3_R = \{q_{\hat{x}|\underline{y}}\ \varepsilon\ \psi_3\ :\ I(x_1\ ;\ \hat{x}_k) = R\}$

The role and interaction of CR3 and CD3 is as discussed in Chapter 3.

Finally we introduce IIP's into the picture as illustrated in Fig. 4.4. We pass the estimate $\hat{x}_k$ through a non-singular (i.e., $C_k \neq 0$) output map to obtain an estimate $\hat{\xi}_k$ according to

$$\hat{\xi}_k = C_k \hat{x}_k \quad , \quad \hat{x}_k = C_k^{-1} \hat{\xi}_k \tag{4.20}$$

Not only is the resulting filter $\hat{\xi}_k$ in class ψ_3 but also on the same IIS as $\hat{x}_k$ since

$$I(x_1 \; ; \hat{x}_k) = I(x_1 \; ; \hat{\xi}_k) \tag{4.21}$$

so that C_k is an IIP of the filter. Getting more specific we introduce IIP's into the filters of Eqs. (4.14), (4.15) to obtain from Eqs. (4.20),

$$\hat{\xi}_{k+1} = C_{k+1} C_k^{-1} \hat{\xi}_k + C_{k+1} K_{PD,k+1} y_{k+1} \tag{4.22}$$

$$K_{PD,k+1} = \text{As in Eq. (4.14)}$$

$$\hat{\xi}_{k+1} = C_{k+1} C_k^{-1} \hat{\xi}_k + C_{k+1} K_{INFO,\, k+1,} y_{k+1} \tag{4.23}$$

where Eq. (4.21) clearly holds for both filters.

We now pause to derive a lowerbound theorem in the next section and some computational results in Section 4.4 both of which we shall need before we proceed with the application of information concepts to the ROFE.

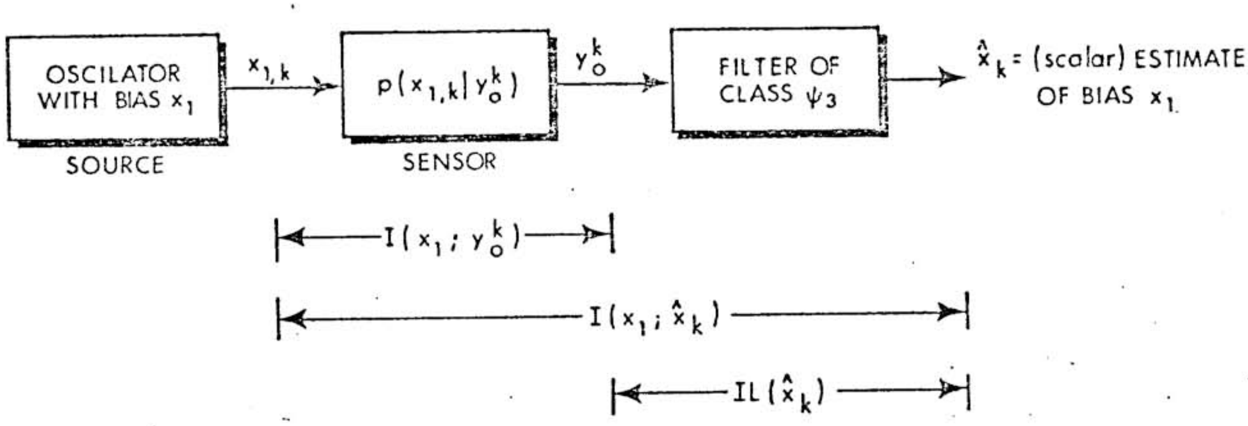

Figure 4.3 Information Theoretic Formulation of the ROFE.

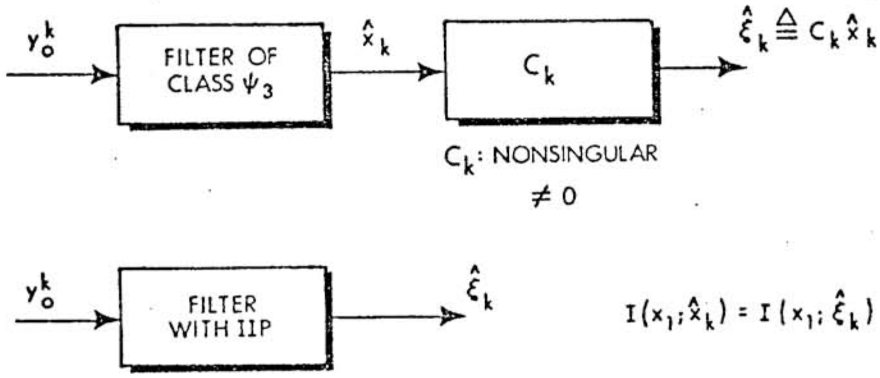

Figure 4.4 Introduction of IIP's

4.3 The Lowerbound Theorem

The following theorem is simply an application of Theorem 8 of Chapter 3, Section 3.4. As evident from the hypothesis, this theorem is applicable to the general ROFP.

Theorem 9. Consider Eqs. (4.1) to (4.3) in the context of the formulation of Section 3.2 so that we have a source putting out an n dimensional vector x_k and passed through the sensor random map. Suppose that a (suboptimal) filter produces an m dimensional estimate $\hat{x}_k$, $m \leq n$ of the first m components of x_k. Using the notation

$$x_k = (x_{1,k}, \ldots, x_{j,k}, \ldots, x_{n,k})$$

$$\hat{x}_k = (\hat{x}_{1,k}, \ldots, \hat{x}_{j,k}, \ldots, \hat{x}_{m,k})$$

let as usual for $j = 1, \ldots, m$

$$IL(\hat{x}_{j,k}) = I(x_{j,k} \, ; \, y_o^k) - I(x_{j,k} ; \hat{x}_{j,k})$$

$$\varepsilon_{j,k} = E(x_{j,k} - \hat{x}_{j,k})^2$$

be the information lost by the filter and MSE for each component. Let the optimal Kalman errors be

$$\varepsilon^*_{j,k} = E(x_{j,k} - \hat{x}^*_{j,k})^2$$

$$\hat{x}^*_{j,k} = E(x_{j,k} | y_o^k)$$

Then we have the following equivalent* relations (ordered from conceptually enlightening to computationally expedient):

*The right hand sides of Eqs. (4.24) to (4.26) are identical.

$$\varepsilon_{j,k} \geq \varepsilon^*_{j,k} \quad \exp\{2 \quad IL(\hat{x}_{j,k})\} \tag{4.24}$$

$$\varepsilon_{j,k} \geq E(x^2_{j,k}) \exp\{-2 \quad I(x_{j,k} \ ; \ \hat{x}_{j,k})\} \tag{4.25}$$

$$\varepsilon_{j,k} \geq E(x^2_{j,k}) \quad (1 - r_{j,k}) \tag{4.26}$$

where

$$r_{j,k} = \frac{[E(x_{j,k} \ \hat{x}_{j,k})]^2}{E(x^2_{j,k}) \ E(\hat{x}^2_{j,k})} \tag{4.27}$$

for $j = 1, \ldots, m$.

<u>Proof</u>: From Theorem 8 Chapter 3,

$$\varepsilon_{j,k} \geq \frac{1}{2\pi e} \exp 2\{H(x_{j,k}) - I(x_{j,k} \ ; \ \hat{x}_{j,k})\} \tag{4.28}$$

Substituting

$$H(x_{j,k}) = \frac{1}{2} \log \{2\pi e \ E(x^2_{j,k})\} \tag{4.29}$$

in (4.28) we get

$$\varepsilon_{j,k} \geq E(x^2_{j,k}) \exp\{-2 \ I(x_{j,k}; \ \hat{x}_{j,k})\} \tag{4.30}$$

proving (4.25). Further substituting

$$I(x_{j,k} \ ; \ \hat{x}_{j,k}) = \frac{1}{2} \log \frac{E(x^2_{j,k}) \quad E(\hat{x}^2_{j,k})}{E(x^2_{j,k})E(\hat{x}^2_{j,k}) - [E(x_{j,k} \ \hat{x}_{j,k})]^2} \tag{4.31}$$

in (4.30) proves (4.26).

We now prove (4.24). From (4.28) we have

$$\varepsilon_{j,k} \geq \frac{1}{2\pi e} \exp\{2\ H(x_{j,k}) - 2I(x_{j,k};\underline{y})\} \exp\{2\ IL(x_{j,k})\} \tag{4.32}$$

Now sense $I(x_{j,k}\ ;\ \underline{y}) = I(x_{j,k}\ ;\ \hat{x}^*_{j,k})$, using an expression similar to (4.31) together with (4.29) we get

$$\frac{1}{2\pi e} \exp\{2\ H(x_{m,k}) - 2I(x_{j,k}\ ;\ \underline{y})\}$$

$$= \frac{E(x^2_{j,k})\ E(\hat{x}^{*2}_{j,k}) - [E(x_{j,k}\ \hat{x}^*_{j,k})]^2}{E(\hat{x}^{*2}_{j,k})} \tag{4.33}$$

But $E(x_{j,k} - \hat{x}^*_{j,k})\ \hat{x}^*_{j,k} = 0$ so that $E(x_{j,k}\ \hat{x}^*_{j,k}) - E(\hat{x}^{*2}_{j,k})$

so that

$$(4.33) = E(x^2_{j,k}) - E(\hat{x}^{*2}_{j,k})$$

$$= \varepsilon_{j,k} \blacksquare$$

Corollary 2. Hypothesis of Theorem 9. Then letting $[x_k]_m$ be the vector formed by the first m components of x_k,

$$\text{trace } E\{([x_k]_m - \hat{x}_k)\ ([x_k]_m - \hat{x}_k)'\} \geq$$

$$\sum_{j=1}^{m} \varepsilon^*_{j,k} \exp\ \{2\ IL\ (\hat{x}_{j,k})\} =$$

$$= \sum_{j=1}^{m} E(x^2_{j,k}) \exp\ \{-2\ I(x_{j,k}\ ;\ \hat{x}_{j,k})\}$$

$$= \sum_{j=1}^{m} E(x^2_{j,k})\ \ (1 - r_{j,k}) \tag{4.34}$$

where $r_{j,k}$ is as in Eq. (4.27).

Proof: Follows immediately from Theorem 9. ∎

Corollary 3. Continuous time version of Theorem 9. Hypothesis of Theorem 9 with k replaced by t. Then:

$$\varepsilon_{j,t} \geq \varepsilon^*_{j,t} \exp \{2\ IL(\hat{x}_{j,t})\} \tag{4.35}$$

$$\varepsilon_{j,t} \geq E(x^2_{j,t}) \exp \{-2\ I(x_{j,t}\ ;\ \hat{x}_{j,t})\} \tag{4.36}$$

$$\varepsilon_{j,t} \geq E(x^2_{j,t})\ (1-r_{j,t}) \tag{4.37}$$

where

$$r_{j,t} = \frac{[E(x_{j,t}\ \hat{x}_{j,t})]^2}{E(x^2_{j,t})\ \ E(\hat{x}^2_{j,t})}$$

for $j=1,...,m$.

Proof: Clearly the proof of Theorem 9 holds in continuous time. ∎

Eq. (4.24) is the most pleasing of the three bounds of Theorem 9. Basically the bound is obtained by multiplying $IL \geq 0$. Thus Eq. (4.26) illustrates how the best possible performance of a suboptimal filter deteriorates as the information loss by the filter increases starting from the performance of the Kalman filter.

In Eq. (4.25) when $I(x_{j,k};\ \hat{x}_{j,k})=0$ then as would be expected the best performance that can be achieved is given by the apriori error $E(x^2_{j,k})$. The estimation process effectively multiplies the apriori error by a factor less than one, allowing better performance as the estimation process becomes more effective (i.e., as $I(x_{j,k}\ ;\ \hat{x}_{j,k})$ increases).

Eq. (4.26) provides an easy to compute MSE lowerbound in terms of the correlation coefficient and has interpretation similar to that of Eq. (4.25).

These bounds provide an indication of how good a performance of a filter and its lower bound gives an indication of whether we are putting to use all the available information.

Using the equations derived in the computational lemmas of the next section we will see in Section 4.5 that these bounds are actually achieved for the ROFE (i.e., equality holds in Eqs. (4.24) to (4.26)).

4.4 Computational Lemmas

4.4.1 The General Element of ψ_3

This subsection simply considers the general element of ψ_3 as defined in Eq. (4.12) and lists formulas for the relevant quantities (such as MSE, covariances, information, etc.) involved in the ROFE. Since all these equations can be "cranked out" straightforwardly no proofs are given. The formulas derived will be put to use in Section 4.5.

The general element of ψ_3 is of the form

$$\hat{x}_{k+1} = A_{k+1}\ \hat{x}_k + B_{k+1}\ \hat{y}_{k+1} \tag{4.38}$$

Upon introducing IIP's as in Eqs. (4.20) to (4.23) of Section 4.2 we get

$$\hat{\xi}_{k+1} = C_{k+1} A C_k^{-1} \hat{\xi}_k + C_{k+1} B y_{k+1} \tag{4.39}$$

where for brevity we abbreviate A_{k+1}, B_{k+1} by A, B. We also use the notation

$$T \triangleq [\ 1 \quad 0 \quad 0\ ] \tag{4.40}$$

so that

$$x_{1,k} = T x_k \tag{4.41}$$

<u>Lemma 21</u>. Consider the general element of ψ_3 as defined in Eq. (4.3b). Then the MSE $\varepsilon_{k+1} = E(Tx_{k+1} - \hat{\xi}_{k+1})$ and partinent correlations are given by:

A. $\varepsilon(C_{k+1}\ ,\ A\ C_k^{-1},\ \left[\frac{B}{AC_k^{-1}}\right]) =$

$$= \left[AC_k^{-1}\right] C_{k+1}^2 \{E(\hat{\xi}_k\hat{\xi}_k) + 2\left[\frac{B}{AC_k^{-1}}\right] H \quad E(x_k\hat{\xi}_k)$$

$$+ \left[\frac{B}{AC_k^{-1}}\right]^2 [H\Phi E(x_kx_k')\Phi'H+HE(W_kW_k')H'+E(v_kv_k)]\}$$

$$-2\left[AC_k^{-1}\right] C_{k+1}\{T\Phi E(x_k\hat{\xi}_k) + \left[\frac{B}{AC_k^{-1}}\right]\left[H\Phi E(x_kx'_k)\Phi'T'+ HE(W_kW_k')T'\right]\} \tag{4.42}$$

$$+ T\Phi E(x_kx_k')\Phi'T' + T\ E(W_kW_k')T'$$

B. $E(x_{k+1}\ x'_{k+1}) = \Phi E(x_k\ x'_k)\Phi' + E(W_k\ W'_k)$

C. $E(\hat{\xi}_{k+1}\hat{\xi}_{k+1}) = (C_{k+1}AC_k^{-1})^2\ E(\hat{\xi}_k\hat{\xi}_k) +$

$$+ 2(C_{k+1}AC_k^{-1})(C_{k+1})(C_{k+1}B)\ H\Phi E(x_k\hat{\xi}_k)$$

$$+ (C_{k+1}B)^2\left[H\Phi E(x_k x'_k)\Phi' H' + HE(W_k W'_k)H' + E(v_{k+1}v_{k+1})\right]$$

D. $E(x_{k+1}\hat{\xi}_{k+1}) = \left[C_{k+1}A\ C_k^{-1}\right]\Phi E(x_k\hat{\xi}_k) + \left[C_{k+1}B\right]\Phi E(x_k x'_k)H'$

$$+ \left[C_{k+1}B\right]E(W_k W'_k)H'$$

Lemma 22. Hypothesis of Lemma 21.

A. Minimizing Eq. (4.42) over C_{k+1} gives, calling C^*_{k+1} the minimum,

$$\varepsilon_{k+1}\left(C^*_{k+1},\ AC_k^{-1},\ \frac{B}{AC_k^{-1}}\right) = \varepsilon_{k+1}\left(\frac{B}{AC_k^{-1}}\right)$$

$$\varepsilon_{k+1}\left(\frac{B}{AC_k^{-1}}\right) = T\Phi E(x_k\ x'_k)\Phi' T' + TE(W_k W'_k)T'$$

$$\frac{\left\{T\Phi E(x_k\hat{\xi}_k) + \left[\frac{B}{AC_k^{-1}}\right]\left[H\Phi E(x_k x'_k)\Phi' H' + HE(W_k W'_k)T\right]\right\}^2}{E(\hat{\xi}_k\hat{\xi}_k) + 2\left[\frac{B}{AC_k^{-1}}\right]H\Phi E(x_k\hat{\xi}_k) + \left[\frac{B}{AC_k^{-1}}\right]^2\left[H\Phi E(x_k x'_k)\Phi' H' + HE(W_k W'_k)H' + E(v_{k+1}v_{k+1})\right]}$$

(4.43)

4.4.2 Selected Elements of ψ_3

In this subsection we given expressions for the parameters of selected elements of ψ_3, namely:

(1) Gain K_{INFO} in Eq. (4.15) for the filter that maximizes information (Lemma 23).

(2) Parameters A_{k+1}, B_{k+1} in Eq. (4.38) for the optimal filter in ψ_3 (Lemma 24).

(3) Gain K_{PD} for the projection of dynamics filter of Eq. (4.14) (Lemma 25, Hutchinson-D'Appolito (83)).

<u>Lemma 23.</u> For the ROFE the gain K_{INFO} in

$$\hat{x}_{k+1} = \hat{x}_k + K_{INFO}\, y_{k+1} \tag{4.47}$$

K^*_{INFO}, that maximizes $I(Tx_{k+1};\ \hat{x}_{k+1})$ at each (k+1) is given by

$$K^*_{INFO} = \frac{\left[H\Phi E(x_k\hat{\xi}_k)\right]\left[T\Phi E(x_k\hat{\xi}_k)\right] - E(\hat{\xi}_k\hat{\xi}_k)\left[T\Phi E(x_k x_k')\Phi' H' + TE(W_k W_k')H'\right]}{\left[H\Phi E(x_k\hat{\xi}_k)\right]\left[T\Phi E(x_k x_k')\Phi' H' + TE(W_k W_k')H\right] - \left[T\Phi E(x_k\hat{\xi}_k)\right]\left[H\Phi E(x_k x_k')\Phi' H' + H\, E(W_k W_k')H' + E(v_{k+1}\, v_{k+1})\right]} \tag{4.48}$$

The information achieved by this filter is the maximum that can be achieved among elements of ψ_3. (This last statement is evident from the parameterization in terms of B/AC_k^{-1} in Eqs. (4.44), (4.45).)

E.O.L.

Lemma 24. For the ROFE the element

$$\hat{x}_{k+1} = A^*_{k+1}\ \hat{x}_k + B_{k+1}\ y_{k+1} \tag{4.49}$$

of ψ_3 that minimizes $E(TX_{k+1} - \hat{x}_{k+1})^2$ is defined by

$$A^*_{k+1} = \frac{T\Phi E(x_k\hat{\xi}_k)\left[H\Phi E(x_k x_k)\Phi' H' + HE(W_k W_k)H' + E(v_{k+1}\ v_{k+1})\right. - H\Phi E(x_k\hat{\xi}_k)\left[H\Phi E(x_k x_k')\Phi' T' + H\Phi E(W_k W_k)T'\right]}{DEN} \tag{4.50}$$

$$K^*_{k+1} = \frac{\left\{E(\hat{\xi}_k\hat{\xi}_k)\left[H\Phi E(x_k x_k')\Phi' T' + H\Phi E(W_k W_k)T\right] - T\Phi E(x_k\hat{\xi}_k)\ H\Phi E(x_k\hat{\xi}_k)\right\}}{DEN} \tag{4.51}$$

where

$$DEN = E(\hat{\xi}_k\hat{\xi}_k)\left[H\Phi E(x_k x_k')\Phi' H' + HE(W_k W_k')H' + E(v_{k+1}v_{k+1})\right] - \left[H\Phi E(x_k\hat{\xi}_k)\right]^2 \tag{4.52}$$

E.O.L.

Lemma 25 (Hutchinson-D'Appolito (83)). For the ROFE the gain of the Projection of Dynamics filter as given in (83) are*:

$$\hat{x}_{k+1} = \hat{x}_k + K_{PD}\ \hat{y}_{k+1} \tag{4.53}$$

where K_{PD} is obtained according to the following equations. For "extrapolation:"

*As listed here these equations do not illustrate the interesting way in which the "projection of dynamics effect" is achieved. See (83).

T^+ = pseudo inverse of T

I = identify matrix

$$G_{k+1|k} = \tilde{\Phi}_k \; G_{k|k} \; \tilde{\Phi} + \tilde{Q}_k$$

$$\tilde{\Phi}_k = \left[\begin{array}{c|c} \Phi & (T^+T - I)\;\Phi\; T^+ \\ \hline 0 & T\Phi T^+ \end{array}\right]$$

$$\tilde{Q}_k = \left[\begin{array}{c|c} E(W_k W_k) & 0 \\ \hline 0 & 0 \end{array}\right]$$

$$G_{k+1|k} = \left[\begin{array}{c|c} P_{k|k-1} & M_{k|k-1} \\ \hline M'_{k|k-1} & Y_{k|k-1} \end{array}\right]$$

For update:

$$K_{PD} = TP_{k+1|k} \; H'(H \; P_{k+1|k}H' + E(v_{k+1} \; v_{k+1}))^{-1} \qquad (4.54)$$

$$G_{k+1|k+1} = B_{k+1} \; G_{k+1|k} \; B'_{k+1} + \tilde{K}_{k+1} \; E(v_{k+1} \; v_{k+1}) \; \tilde{K}'_{k+1}$$

$$G_{k+1|k+1} = \left[\begin{array}{c|c} P_{k+1} & M_{k+1} \\ \hline M'_{k+1} & Y_{k+1} \end{array}\right]$$

$$B_{k+1} = \left[\begin{array}{c|c} (I - T^+K_{PD}H) & 0 \\ \hline -K_{PD} \; H & I \end{array}\right]$$

E.O.L.

4.5 Information Design and Comparison for the ROFE

In Section 4.2 we started to apply the results of Chapter 3 to the ROFE. In this section we continue this development armed with the formulas derived in Sections 4.3, 4.4.

First we compute the lowerbound of Theorem 9,

$$\varepsilon_k \geq \varepsilon_k^* \exp \{2 \text{ IL } (\hat{x}_k)\} \tag{4.55}$$

to the filters whose performance appeared in Fig. 4.2. The results are as illustrated in Fig. 4.5. For the Kalman filter* $IL(\hat{x}_k^*) = 0$ and its lowerbound coincides with its performance. The information optimizing filter of Eq. (4.15) is the filter in ψ_3 that looses least information and consequently it is natural that its performance lowerbound be below the projection of dynamics filter.

The difference observed in the lowerbounds is caused by the difference in information loss or alternatively by the difference in the information $I(Tx_k; \hat{x}_k)$ about the bias Tx_k contained in the different estimates. $I(Tx_k; \hat{x}_k)$ is plotted in Fig. 4.6 from the formulas of Section 4.4. We know from Theorem 5A,B,C that provided CR3 is acceptable we may have the situation of Fig. 3.5 as also illustrated on Fig. 4.7. Not only is CR3 acceptable but by varying the C_k IIP we can achieve the performance of the lowerbounds of Fig. 4.5 as proved in the following lemmas.

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*See Lemma 10 (Bucy) in Chapter 2

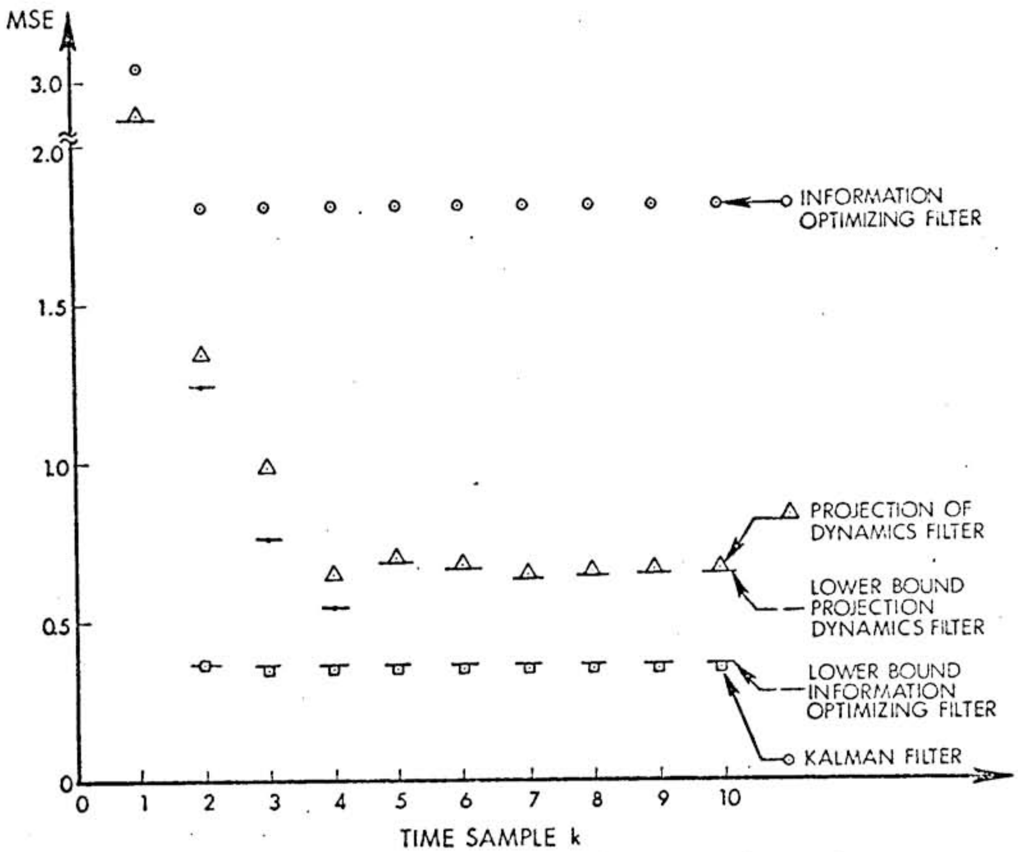


Figure 4.5 Theorem 21 lowerbounds for the filters whose performance appears in Fig. 4.2. Observe that for almost all samples the lowerbound for the Information Optimizing Filter is slightly above the Kalman filter; and that the lowerbound for the Projection of Dynamics Filter is almost achieved by the filter as it approaches steady state.

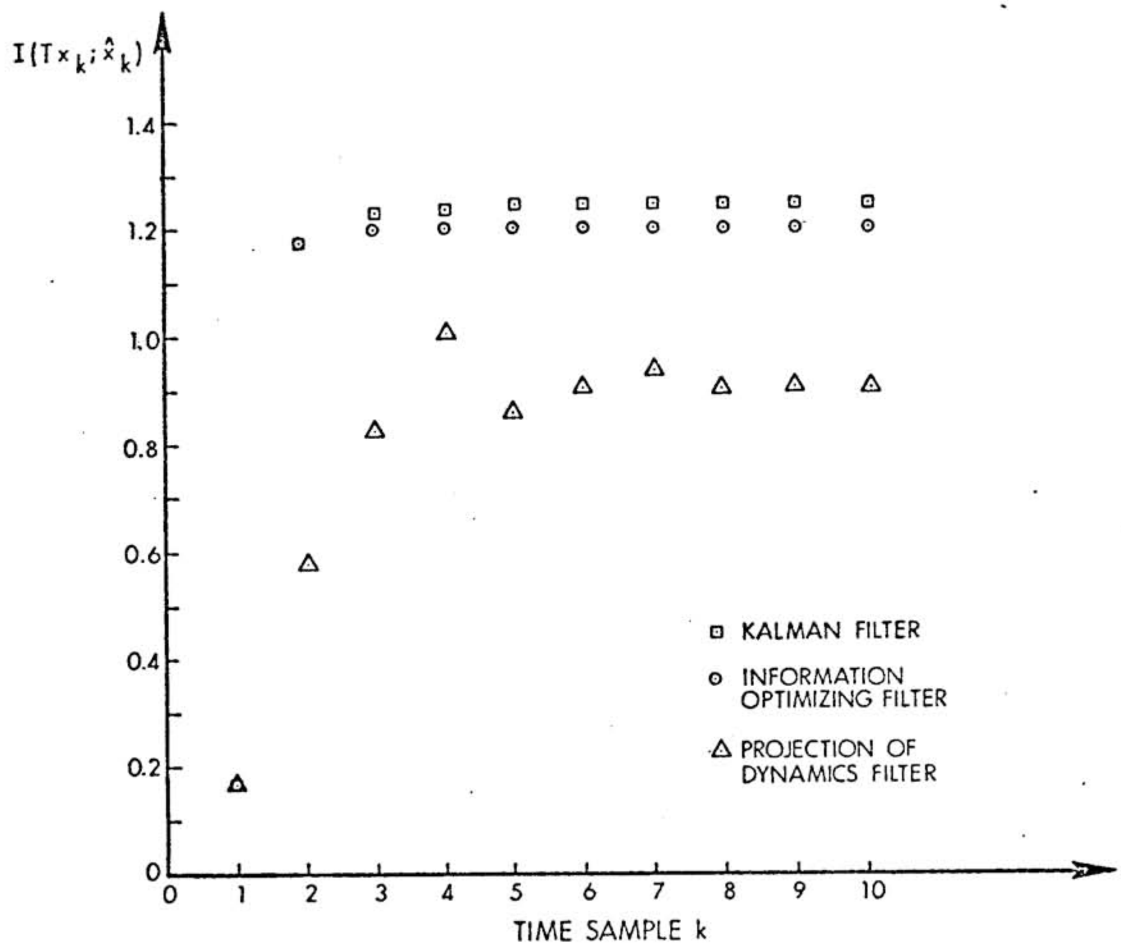
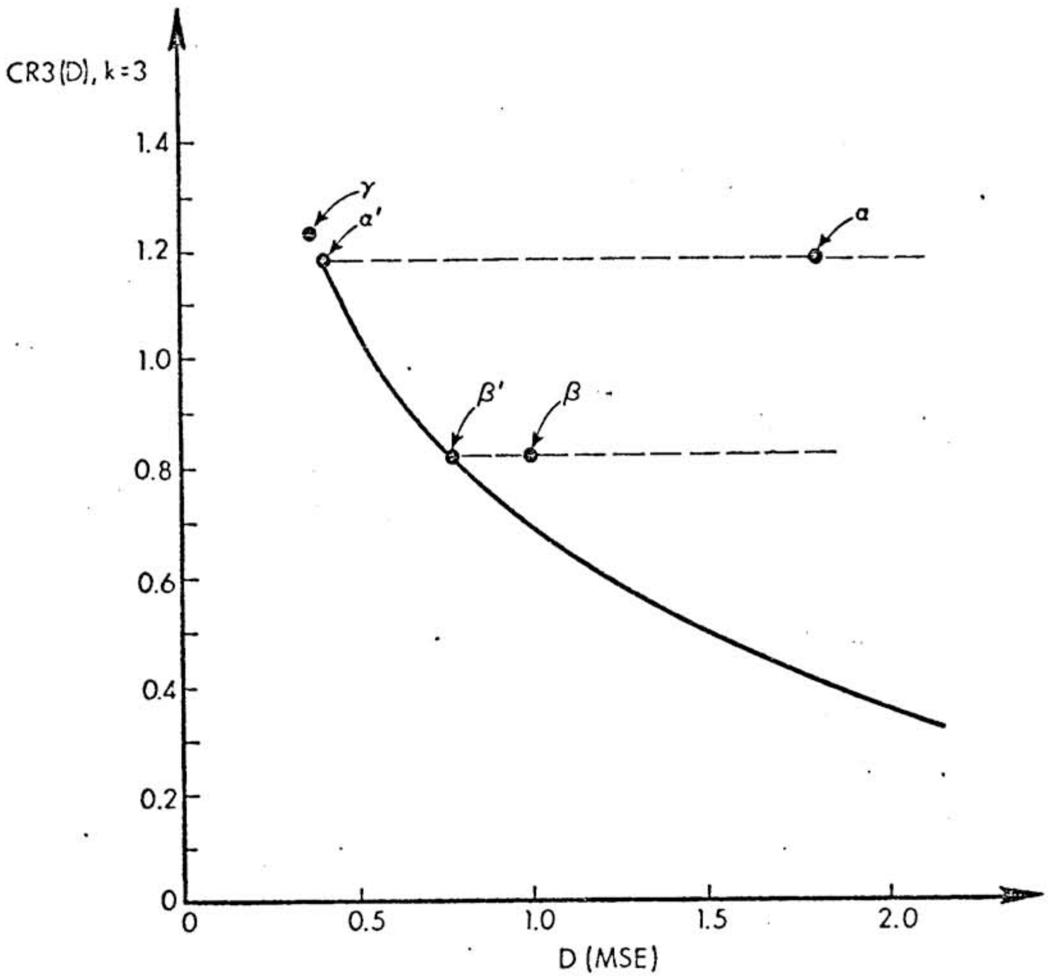


Figure 4.6 Comparison of Information About The Bias Contained in the Different Estimates.



α, α' : IMFORMATION OPTIMIZING FILTER BEFORE, AFTER IIP OPTIMIZATION
β, β' : PROJECTION OF DYNAMICS FILTER BEFORE, AFTER IIP OPTIMIZATION
γ : KALMAN FILTER

Figure 4.7 CR3(D) for the ROFE at Sample k=3 (compare with Fig. 3.6).

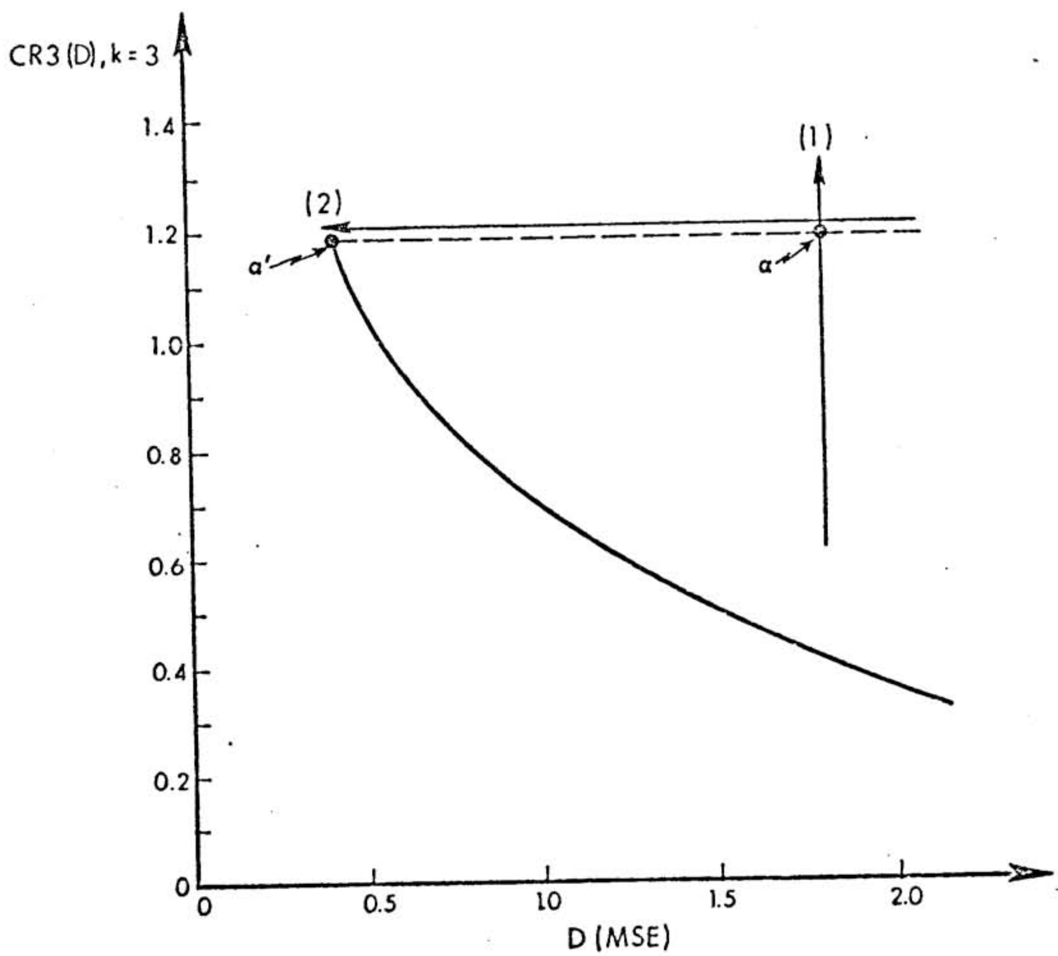


Figure 4.8 Illustration of the Two Step Design Procedure for the ROFE

Lemma 26. Consider an arbitrary filter $\alpha\varepsilon\psi_3$ that achieves at sample k+1 information $I_{\alpha,k+1}$. Then by choosing appropriately the IIP C_{k+1} in Eq. (4.39) we can achieve the lowerbound of Theorem 9 so that as evident from Eq. (4.55) a performance of

$$\varepsilon_{k+1} = \varepsilon^*_{k+1} \exp \{2\ I(Tx_{k+1}\ ;\ \mathcal{Y}) - 2\ I_{\alpha,k+1}\}$$

is obtained.

Proof: Follows by noticing that Eqs. (4.43) and (4.46) of Lemma 22 are identical. ◘

Lemma 27. CR3 is acceptable for the ROFE (in fact $CR3\ (\varepsilon_k) = (I(Tx_k\ ;\ \underline{y}) - \frac{1}{2} \log \frac{\varepsilon_k}{\varepsilon^*_k}$ where $\underline{y} = [y_1\ \ldots\ y_k]'$).

Proof: Consider an arbitrary element of ψ_3 with parameters A_o, B_o in Eq. (4.38) which achieves a point (ε_o, I_o) in the information distortion plane. By Lemma 26 by proper choice of IIP C_{k+1} we can achieve a point (ε_o', I_o) on the lowerbound to CD3 and hence on CD3. Furthermore as evident from Eqs. (4.44), (4.45) information $((Tx_{k+1}\ ;\ \hat{\xi}_{k+1})$ is a continuous function of the filter parameters B/A that achieves a bounded maximum (Lemma 23). Thus by varying B/A we see that below this maximum CD3 coincides with its lowerbound, the rate distortion function and CR3 thus making clearly acceptable. (The given expression for CR3 follows from the equality that holds in Eq. (4.55). ◘

By virtue of lemma 26 and 27 we can now give the following versions of parts A, B, C of Theorem 5.

Theorem 5A for the ROFE (filter design): If at sample k+1 a given filter $\alpha\varepsilon\psi_3$ achieves information I_α and MSE ε_α then by proper choice of IIP C_{k+1} in Eq. (4.55) one can obtain a filter $\alpha'\varepsilon\psi_3$ in the I_α IIS with MSE $\varepsilon_{\alpha'} \leq \varepsilon_\alpha$ given by the solution to

$$CR3(\varepsilon_{\alpha'}) = I_\alpha$$

As illustrated in Fig. 4.7 this is the case for both the information optimizing filter (filters α, α') and the projection of dynamics filter (filters β, β')*. Thus in the design process one can proceed along the alternatives that maximize information confident that at the end one can select the IIP that would translate the information available into good MSE performance.

Theorem 5B for the ROFE (filter design): A necessary condition for a filter $\alpha\varepsilon\psi_3$ to achieve minimum MSE at sample k+1 is that it maximizes the information $I(Tx_{k+1} ; x_{\alpha,k+1})$. In fact as can be shown from Lemmas 23, 24 of subsection 4.4.2 the information of the MSE-optimal element of ψ_3 (as defined by Eqs. (4.49) to (4.52)) and that of the information optimizing filter (as defined by Eqs. (4.47), (4.48)) are identical (and naturally their MSE performance coincides after IIP optimization.

Together the two preceding paragraphs suggest the two step design procedure of Fig. 3.4 reproduced again in Fig. 4.8.

*It is interesting to note that in steady state the projection of dynamics filter practically achieves its lowerbound.

Theorem 5C for the ROFE (<u>filter comparison</u>): suppose that filters $\alpha, \beta \varepsilon \psi_3$ achieve respectively at sample (k+1) informations $I(Tx_{k+1} ; \hat{x}_{\alpha,k+1}) = I_\alpha$, $I(Tx_{k+1} ; \hat{x}_{\beta,k+1}) = I_\beta$, MSE's ε_α, ε_β, and that $I_\alpha > I_\beta$. Then by adjusting the C_{k+1} IIP in Eq. (4.55) we can obtain a filter $\alpha' \varepsilon \psi_3$ in the I_α IIS with MSE $\varepsilon_{\alpha'}$ such that for any filter $\beta' \varepsilon \psi_3$ with MSE $\varepsilon_{\beta'}$ in the I_β IIS

$$\varepsilon_{\alpha'} < \varepsilon_{\beta'}$$

irrespective of whether $\varepsilon_\alpha < \varepsilon_\beta$, $\varepsilon_\alpha = \varepsilon_\beta$, $\varepsilon_\alpha > \varepsilon_\beta$. Thus when comparing two (or more) filters it is rather irrelevant to look at their MSE. Instead one should ask the question, how much information do these filters lose (or else how much information is contained in these estimates).

We note that for the ROFE we have considered MSE as a measure of distortion and in the design process we have adjusted the IIP's to translate the information available in the estimate into good MSE performance. By a different choice of IIP we could translate this information into good performance for another non-MSE distortion measure provided the relevant constrained rate distortion functions were acceptable. The starting point in both of these situations is the same: Maximize information, minimize information loss*.

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*An application that comes to mind is in the navigation system of, say, an aircraft where it may be desirable to use different performance measures for different missions. An information optimizing filter would only require a change in the IIP program.

Summarizing, two objectives have been achieved in this chapter. First in a general context an easy to compute performance lowerbound was developed that gives (as dramatically illustrated in the ROFE) an indication of whether all the information contained in an estimate is being put to good use. And second a simple example -- the ROFE -- has been used to illustrate the concepts developed in Chapter 3 and the features of this approach to filter design and comparison listed in subsection 3.3.4. In particular, because of the ease with which information quantities could be computed for the ROFE, the imposition of an "information-distortion grid" on the design procedure was easy to implement as was the decoupling in the search over processor space. I.e. rather than over R^2, the search was over $K_{INFO} \in R$ and then over IIP $C_{k+1} \in R$.

CHAPTER 5

A LOWER BOUND ON OPTIMUM MSE

The main objective of this chapter is to implement the formulas derived in Theorems 7 and 7C of the previous chapter for a lower bound on optimal filtering MSE for both the discrete and continuous time MLFP. The resulting expressions for the bound will be in terms of the moments of the $g(x_s,s)$ process of the NLFP and will be applied in the next chapter to the phase locked loop.

For comparison and historical perspective we begin the chapter (Section 5.1) with a summary of the important paper by M. Zakai and J. Ziv (38) where the Shannon lower bound (Lemma 16) was first used in the dynamical systems context. In Section 5.2 we derive the main formula in continuous time for the implementation of Theorem 7C and compare our approach with that of Zakai and Ziv. The next two sections, Sections 5.3 and 5.4, are devoted to evaluating the two "components" (denominator and numerator) of the formula derived in Section 5.2. The results developed in Sections 5.2 to 5.4 are applied in Chapter 6 to the phase locked loop. Finally in Section 5.5 we consider the important discrete time case and derive the main formula for the lower bound on optimum MSE based on Theorem 7.

5.1 Zakai and Ziv's Paper (38)*

For our purposes we divide the results obtained in (38) into two areas. The development dealing directly with the lower bound is presented in some detail in subsection 5.1.1. Other results obtained in the paper are summarized in subsections 5.1.2.

5.1.1 The Zakai-Ziv Lower Bound

Zakai and Ziv (38) derive a lower bound on optimum MSE for a subclass of the NLFP based on the Shannon lower bound. The specific problem they consider is as follows. The message model is given by (see Fig. 5.1)

$$dx_1(t) = x_2(t)dt$$

$$dx_2(t) = x_3(t)dt$$

$$\vdots$$

$$dx_n(t) = a(\underline{x}(t))dt + b(\underline{x}(t))dw(t), \quad t\varepsilon\ (-\infty,\infty) \qquad (5.1)$$

where w(t) is standard Brownian motion. The observation model is given by

$$dy(t) = g(x_k(t),t)dt + \sqrt{N_o}\ dv(t) \quad t\varepsilon(-\infty,\infty) \qquad (5.2)$$

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*The kind of patient assistance in the understanding of this paper extended by Prof. Moshe Zakai is gratefully acknowledged.

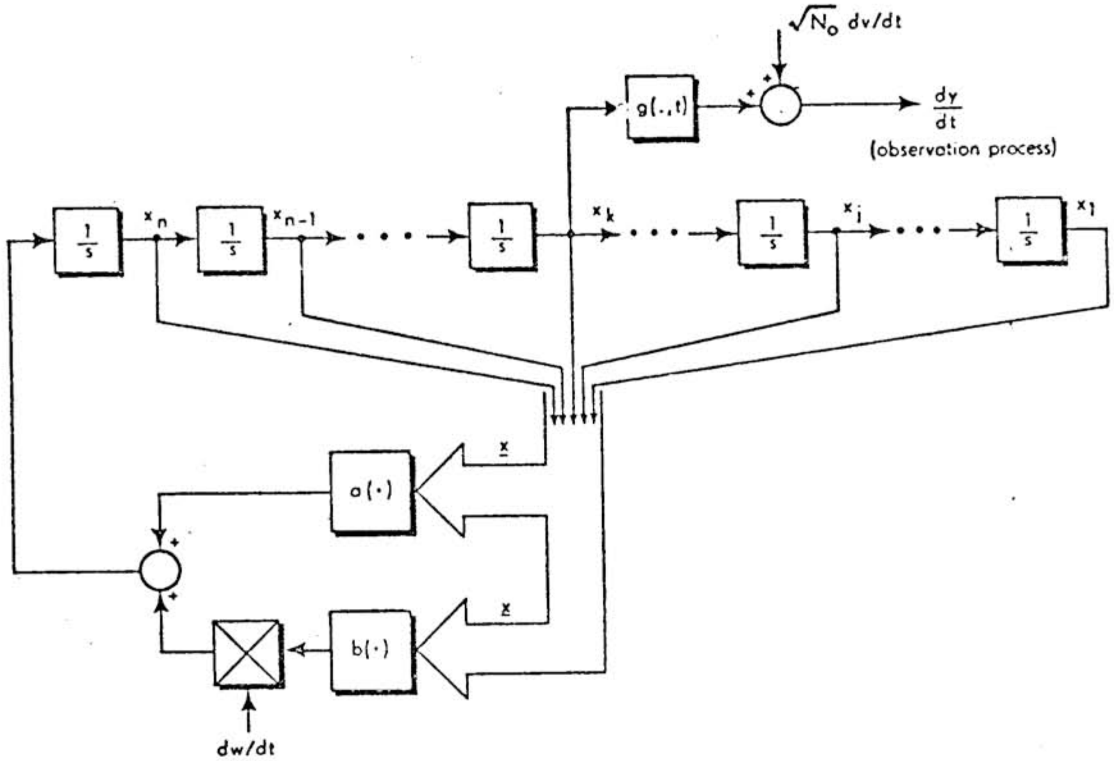

Figure 5.1 Model Considered by Zakai and Ziv (k and j need not be in the order shown)

where again v(t) is standard Brownian motion independent from w(t). The filtering problem considered is

$$\text{estimate } x_j(t),\ j\varepsilon[1,n] \quad \text{based on } (y(s),\ s\varepsilon[0,t])$$

Thus the bound desired is a lower bound on

$$\varepsilon_A = E(x_j(t) - E(x_j(t)|y(s),\ s\varepsilon[-\infty,t]))^2 \tag{5.3}$$

The reason for choosing the above model is that stationarity of $\underline{x}(t)$ will be used in what follows and $\underline{x}(t)$ as given by Eq. (5.1) is indeed stationary since a and b do not depend on t, w(t) is standard Brownian motion (so that the associated white noise has stationary covariance), and the process starts at $t = -\infty$.

Zakai and Ziv begin by considering two other errors in addition to that of ε_A in Eq. (5.3) as follows:

$$\varepsilon_A = E(x_j(t) - E(x_j(t)|y_{-\infty}^t))^2 \tag{5.3}$$

$$\varepsilon_B = E\left(x_j(t) - E(x_j(t)|y_{-\infty}^t , x(0))\right)^2 \tag{5.4}$$

$$\varepsilon_C(t,x(0)) = E\left\{\left(x_j(t) - E_j(x(t)|y_o^t, x(0))\right)^2 x(0)\right\} \tag{5.5}$$

Observe that ε_A is a constant because of the stationarity assumption and that ε_B, ε_C are functions of the indicated arguments. Further note that on an inutitive basis it is clear that the following relation holds among those three errors (making use of stationarity of $\underline{x}(t)$),

$$\sup_t E\{\varepsilon_C(t,x(0))\} = \sup_t \varepsilon_B(t) \le \varepsilon_A \tag{5.6}$$

since $\varepsilon_B(t) = E\{\varepsilon_C(t,x(0))\}$ on account of (x(t),y(t)) being a Markov process.

The objective is therefore to derive a lower bound on

$$\varepsilon_C(t,x(0)) = E\{(x_j(t) - E(x_j(t)|y_o^t, x(0))^2|x(0)\}$$

since such a bound will yield a lower bound on ε_A upon averaging over all x(0) as indicated in Eq. (5.6). Thus, conditioning throughout by x(0), consider the following two inequalities for an estimate $\hat{x}_j(t)$ based on y_o^t and x(0) and its corresponding error ε_C:

$$H_j(t|x(0)) - \tfrac{1}{2}\log 2\pi e\ \varepsilon_C \le R(\varepsilon_C, x_j(t)|x(0)) \qquad (5.7)$$

$$R(\varepsilon_C, x_j(t)|x(0)) \le I(x_j(t);\ \hat{x}_j(t)\big|x(0)) \le \qquad (5.8)$$

$$\le I(x_j(t);\ y_o^t|x(0)) = \qquad (5.9)$$

$$= I(x_k(t);\ y_o^t|x(0)) \le \qquad (5.10)$$

$$\le I(x_k({}_o^t);\ y_o^t|x(0)) = \qquad (5.11)$$

$$= \frac{1}{2N_o}\int_0^t E_{x(0)}\,[g(x_k(s),s) - \hat{g}(x_k(s),s)]^2 ds \qquad (5.12)$$

$$\le \frac{1}{2N_o}\int_0^t E_{x(0)}\,[g(x_k(s),s) - Eg(x_k(s),s)]^2 ds \qquad (5.13)$$

where $\hat{g}(x_k(s),s) = E\{g(x_k(s),s)|y_0^s\}$

$H_j(t|x(0))$: conditional entropy of $x_j(t)$ conditioned by x(0)

$R(\varepsilon_C, x_j(t)|x(0))$: MSE rate distortion function of $x_j(t)$ conditioned on x(0)

and Eq. (5.7) follows from Shannon's lower bound (Lemma 16) and Eq. (5.12) from Kailath-Duncan formula (Lemma 11). Solving for ε_C in the leftmost and rightmost terms of (5.7) through (5.13), substituting in (5.6), and using Jensen's convex inequality (Lemma 28A, below) gives the lower bound

$$\sup_{t>0} \left\{\frac{1}{2\pi e} \exp\left[2\overline{H}_j(t) - \frac{1}{N_o} \int_0^t \sigma_g^2(s)\right]\right\} ds \leq \varepsilon_A \qquad (5.14)$$

where $\overline{H}_j(t) = E\{H_j(t|x(0))\}$ and σ_g^2 is the apriori variance of g.

5.1.2 Other Results

Zakai and Ziv also develop an upper bound based on linear filtering arguments. Specifically, they produce a (suboptimal) linear filter whose error is easy to compute. Naturally this error constitutes an upper bound to the optimum MSE. They further show that their lower and upper bounds are tight up to a factor of 1.6 for small observation noise -- that is in the "above threshold" or approximately linear region of operation of the receiver.

Appendix I of (38) is devoted to deriving bounds on moments of the process $\underline{x}(t)$ of Eq. (5.1) making use of its stationarity. These bounds are used in the section on systems with small observation noise as well as in some of the examples. In Appendix B a very interesting method of lower bounding the entropy $\overline{H}_j$ is given which makes use of the equivalence between the measure induced by Eqs. (5.1) and the Gaussian measure induced by an appropriately chosen system of linear equations.

5.2 Main Formula -- Continuous Time

5.2.1 Motivation and Comparison

In this section we present the basic formula for the implementation of the continuous time lower bound on optimum filtering MSE of Theorem 7. The derivation is carried out in subsection 5.2.2. In subsection 5.2.3 properties of the two terms composing the bound will be studied. Machinery to evaluate these terms will be presented in the next two sections (Sections 5.3 and 5.4).

The formula derived here differs from that of Zakai and Ziv presented in the previous section in the following three respects.

(1) Our formula is applicable to the important discrete time observation case (e.g., radar tracking). The Zakai-Ziv formula is based on the Kailath-Duncan formula (Lemma 11) which does not have an immediate counterpart in discrete time since the "innovations process" in discrete time is not Gaussian with observation noise covariance (see (31), (32), (33)).

(2) In reference to the inequality in Eqs. (5.8) to (5.13), our formula is based on Eq. (5.10) which is a tighter bound on the rate distortion

function that Eq. (5.11) especially in situations where $I(x_t;\ y_0^t) << I(x_0^t;\ y_0^t)$. The exact factor of improvement is difficult to evaluate in general since just as $I(x_0^t;\ y_0^t)$ cannot be evaluated exactly but has to be bounded by Eq. (5.13), $I(x_t;\ y_0^t)$ cannot be evaluated exactly either and will have to be bounded as done in Section 5.3 and 5.4. Nevertheless, we can conjecture that the relation << will hold in the highly noisy and nonlinear region (below threshold) since it is possible to argue from an intuitive point of view that when the observation noise is considerable we may have

$$I(x_0^t;\ y_0^t) \approx \sum_{i=0}^{N} I(x_{t_i-\delta}^{t_i};\ y_{t_i-\delta}^{t_i})$$

$$>> I(x_{t-\delta}^t;\ y_{t-\delta}^t) \qquad (5.15)$$

and

$$I(x_t;\ y_t) \approx I(x_{t-\delta}^t;\ y_{t-\delta}^t) \approx I(x_{t-\delta}^t;\ y_0^t)$$

$$\approx I(x_y;\ y_0^t) \qquad (5.16)$$

where $t_i = i\delta$, δ is "small", and $t_N = t$. While whether or not this conjecture is true in general is difficult if not impossible to ascertain, the results of Chapter 6 seem to support its veracity.

(3) The formula derived here is directly applicable to the general NLFP and not limited to a specific canonical form or stationary processess over infinite intervals. While the Zakai-Ziv procedure is upon slightly modifying some of the development equally applicable, the importance in filtering for dynamical systems of non-stationary processes and systems of arbitrary form cannot be overemphasized.

5.2.2 Derivation of Main Formulas

We will need in what follows the following well known results.

Lemma 28. (Jensen-Holder). Let X be a random variable and f, g, h be scalar valued functions of a real variable. Then, provided the obvious integrability requirements hold:

A. $[g \text{ convex} \cup] \rightarrow [g(EX) \leq E\, g(X)]$

B. $[f \text{ convex} \cap] \rightarrow [E\, f(X) \leq f(EX)]$

C. $\int_a^b h(t)dt\ ^r \leq (b-a)^{r-1} \left[\int_a^b h^r(t)dt\right]$

Theorem 10. Consider the continuous time NLFP of Section 2.1. Let $\varepsilon^*(t)$ be the associated optimum MSE. Then a lower bound on $\varepsilon^*(t)$ is given by

$$\varepsilon^*(t) \geq \frac{1}{2\pi e} \exp 2H(t) \cdot \left\{ \frac{\exp E_{P_{\underline{x}\cdot y}} \log E_{P_x}(\exp \zeta_t)}{E_{P_{\underline{x}\cdot y}}(\exp \zeta_t)} \right\}^2 \tag{5.17}$$

where

$$\zeta_t = \zeta_t(x_0^t, y_0^t, t) = \int_0^t g(x_s,s)R^{-1}(s)dy_s' - $$
$$- \tfrac{1}{2} \int_0^t g(x_s,s)R^{-1}(s)g(x_s,s)'ds$$
$$= \tfrac{1}{2} \int_0^t g(x_s,s)R^{-1}(s)g(x_s,s)'ds +$$
$$+ \int_0^t g(x_s,s)R^{-1}(s)d\tilde{B}_s' \tag{5.18}$$

$H(t)$: Entropy of x_t

provided the necessary expectations exist. (If the processes defined by the NLFP are stationary then $\varepsilon^*(t)$ is constant and bounded by the supreme over all t of the right-hand side of Eq. (5.17)).

<u>Proof:</u> By Lemma 4,

$$I(x_t;\, y_0^t) = E_{P_{x\cdot y}} \log \frac{dP_{\underline{x}\cdot y}}{dP_{xxy}} = E_{P_{x\cdot y}} \log \frac{dP_x^{\mathcal{Y}_t}}{dP_x}$$

where $\mathcal{Y}_t = \sigma\{y_s, s\varepsilon[0,t]\}$. Using Eq. (2.5) of Lemma 1 (continuous Bucy-Mortenson-Duncan representation),

$$I(x_t;\, y_0^t) = E_{P_{x\cdot y}} \log \left\{ \frac{E_{P_x}^{\sigma\{x_t\}}(\exp \zeta_t)}{E_{P_x}(\exp \zeta_t)} \right\}$$

$$= E_{P_{x\cdot y}} \log E_{P_x}^{\sigma\{x_t\}}(\exp \zeta_t) - E_{P_{x\cdot y}} \log E_{P_x}(\exp \zeta_t) \tag{5.19}$$

where ζ_t is as given by Eq. (5.18). By Jensen's inequality (Lemma 28B) we have

$$E_{P_{x \cdot y}} \log E_{P_x}^{\sigma\{x_t\}} (\exp \zeta_t) \leq \log E_{P_{x \cdot y}} E_{P_x}^{\sigma\{x_t\}} (\exp \zeta_t)$$

$$= \log E_{P_{x \cdot y}} (\exp \zeta_t) \qquad (5.20)$$

where use was made of the smoothing property of the conditioned expectation. Substituting (5.20) into (5.19) gives

$$I(x_t;\, y_0^t) \leq \log E_{P_{x \cdot y}} (\exp \zeta_t) - E_{P_{x \cdot y}} \log E_{P_x} (\exp \zeta_t)$$

Substituting this last expression into that given in Theorem 7C (Section 3.4) gives (5.17). The stationarity part follows from reasoning analogous to those of Zakai-Ziv presented in Section 5.1. ∎

A more computable expression can be obtained for the numerator in Eq. (5.17) as follows.

Lemma 29. Hypothesis of Theorem 10. Let

$$N(t) = E_{P_{x \cdot y}} \log E_{P_x} (\exp \zeta_t(\underline{\omega})) \qquad (5.21)$$

where $\zeta_t(\underline{\omega})$ is as defined by Eqs. (5.19). Then

$$N(t) = E_{P_{x \cdot y}} \int_0^t \frac{E_{P_x}(g_s' R^{-1}(s) g_s \psi_s)}{E_{P_x}(\psi_s)} ds$$

$$- \tfrac{1}{2} E_{P_{x \cdot y}} \int_0^t \frac{E_{P_x}(\psi_s g_s') R^{-1}(s)\, E_{P_x}(\psi_s g_s)}{(E_{P_x} \psi_s)^2} ds \qquad (5.22)$$

where $\psi_s(\underline{\omega}) = \exp \zeta_s(\underline{\omega})$ and $g_s = g(x_s, s)$.

<u>Proof:</u> Define

$$\gamma_t(\tilde{\omega}) = E_{P_x} \exp \zeta_t$$

$$\Gamma_t(\tilde{\omega}) = \log E_{P_x} \exp \zeta_t$$

Observe that $\exp \zeta_t > 0$ a.s. $P_{x \cdot y}$, $E_{P_x} \exp \zeta_t > 0$ a.s. P_x so that $\Gamma_t(\tilde{\omega})$ is defined taking values on the reals. Consider the Ito process defined by Eq. (5.18),

$$d\zeta_t = \tfrac{1}{2}\, g(x_t, t)'\, R^{-1}(t)\, g(x_t, t)dt + g(x_t, t)'\, R^{-1}(t)d\tilde{B}_t$$

By the Ito differential rule (change of variable formula) $\psi_t(\underline{\omega}) = \exp \zeta_t$ satisifes the equation

$$d\psi_t = (\exp \zeta_t)d\zeta_t + \tfrac{1}{2}\, g(x_t, t)'R^{-1}(t)\, g(x_t, t) \exp \zeta_t\, dt$$

or equivalently

$$d\psi_t = \{g(x_t, t)'R^{-1}(t)\, g(x_t, t)\psi_t\}dt + \{\psi_t\, g(x_t, t)'R^{-1}(t)\}d\tilde{B}_t$$

Invoking the Fubini theorem we have $E_{P_x} d\psi_t = d(E_{P_x} \psi_t)$ so that

$$d\gamma_t \equiv d(E_{P_x} \psi_t) = E_{P_x}\{g(x_t, t)'R^{-1}(t)\, g(x_t, t)\psi_t\}dt + $$

$$+ E_{P_x}\{\psi_t\, g(x_t, t)'R^{-1}(t)\}d\tilde{B}_t$$

$$\gamma_o = E_{P_x} \exp \zeta_o = 1$$

Similarly, $\Gamma_t(\tilde{\omega}) = \log \gamma_t$

$$d\Gamma_t = \frac{1}{\gamma_t} d\gamma_t - \frac{1}{2\gamma_t^2} E_{P_x}\{\psi_t g'(x_t,t)\} R^{-1}(t) R(t) R^{-1}(t) E_{P_x}\{g(x_t,t)\psi_t\} dt$$

so that

$$d\Gamma_t = \frac{E_{P_x}\{\psi_t \; g_t' \; R^{-1}(t) \; g_t\}}{\gamma_t} dt - \frac{1}{2} \frac{E_{P_x}\{\psi_t \; g_t'\} R^{-1}(t) E_{P_x}\{g_t \psi_t\}}{\gamma^2} dt$$

$$+ \frac{E_{P_x}\{\psi_t \; g_t' \; R^{-1}(t)\}}{\gamma_t} d\tilde{B}_t$$

$$\Gamma_o = \log \gamma_o = 0$$

Making use of the usual properties of the Ito integral the previous equation becomes

$$N(t) = E_{P_{x \cdot y}} \Gamma_t = E_{P_{x \cdot y}} \int_0^t \frac{E_{P_x}\{g_s' \; R^{-1}(s) \; g_s \; \psi_s\}}{E_{P_x}\{\psi_s\}} ds$$

$$- \frac{1}{2} E_{P_{x \cdot y}} \int_0^t \frac{E_{P_x}\{\psi_s \; g_s'\} R^{-1}(s) E_{P_x}\{g_s \; \psi_s\}}{(E_{P_x}\{\psi_s\})^2} ds$$

as desired. ∎

Theorem 11. Hypothesis of Theorem 10. A lower bound on $\varepsilon^*(t)$ is given by

$$\varepsilon^*(t) \geq \left\{\frac{1}{2\pi e} \exp[2H(t)]\right\} \cdot \left\{\frac{\exp\{N(t)\}}{D(t)}\right\}^2 \qquad (5.23)$$

where

$$N(t) = E_{P_{x \cdot y}} \int_0^t \frac{E_{P_x}(g_s' R^{-1}(s) g_s \psi_s)}{E_{P_x}(\psi_s)} ds$$

$$- \frac{1}{2} E_{P_{x \cdot y}} \int_0^t \frac{E_{P_x}(\psi_s g_s') R^{-1}(s) E_{P_x}(\psi_s g_s)}{(E_{P_x} \psi_s)^2} ds$$

$$D(t) = E_{P_{x \cdot y}}(\psi_t)$$

where again $\psi_s = \exp \zeta_s$, ζ_s is as in Eq. (5.18) and $g_s = g(x_s,s)$. (As with Theorem 10, if the processes defined by the NLFP are stationary then $\varepsilon^*(t)$ is constant and bounded by the supremum over all t of the right-hand side of Eq. (5.23)).

Proof: Eq. (t.23) follows upon substituting Eq. (5.22) into Eq. (5.20).■

The formula given in Theorem 11 is the main formula of this section. We note that by the use of the Jensen inequality in Eq. (5.20) the difficult to compute conditional expectation in function space present in the BMD representations has been eliminated -- this step being the only bound (in addition to the Shannon lower bound) used in the derivation of Eq. (5.23). While Eq. (5.23) may seem formidable, we will be able to apply it in the next chapter to the phase locked loop.

For problems of form similar to that considered by Zakai and Ziv an equivalent result can be given. Specifically we have the following corollary to Theorem 11 for a case a little more general than that of Eqs. (5.1) to (5.3).

Corollary 4. Consider the subclass of the NLFP resulting when Eqs. (2.1) and (2.2) are replaced by

$$dx_1(t) = x_2(t)dt$$

$$dx_2(t) = x_3(t)dt$$

$$\vdots$$

$$dx_n(t) = a(\underline{x}(t),t)dt + b(\underline{x}(t),t)dB_t, \quad t\varepsilon[0,t]$$

$$dy(t) = g(x_k(t),t)dt + d\tilde{B}_t \tag{5.24}$$

all other conditions and notation remaining unchanged. For $j\varepsilon[1,n]$ let

$$\varepsilon_j^*(t) = E(x_j(t) - E(x_j(t)|y_s,\ s\varepsilon[0,t]))^2$$

Then a lower bound on $\varepsilon_j^*(t)$ is given by

$$\varepsilon_j^*(t) \geq \left\{\frac{1}{2\pi e}\exp[2H_j(t)]\right\}$$

$$\cdot\left\{\frac{\exp\left\{E_{P_{X\cdot Y}}\int_0^t \frac{E_{P_X}(g_s^2/R(s))}{E_{P_X}(\psi_s)}\,ds - \frac{1}{2}E_{P_{X\cdot Y}}\int_0^t\left[\frac{E_{P_X}(g_s\psi_s)}{E_{P_X}(\psi_s)}\right]^2\frac{ds}{R(s)}\right\}}{E_{P_{X\cdot Y}}(\psi_t)}\right\} \tag{5.25}$$

where as before $\psi_s = \exp \zeta_s$, ζ_s is given by Eq. (5.18) and $g_s = g(x_k(s),s)$. (If the processes defined by Eq. (5.24) are stationary then $\varepsilon_g^*(t)$ is constant and bounded by the supremum over all t of the right-hand side of Eq. (5.25).)

Proof: Eq. (5.25) follows upon substituting Eq. (5.24) in (5.23).∎

The above corollary gives a lower bound on optimum MSE for the j^{th} component of x(t) for systems of the form of Eq. (5.24). For the general NLFP we can obtain a formula for a bound on the j^{th} component (and consequently on the entire error) by the following theorem which we mention in passing (we will not make use of this result in this study).

Theorem 12. Consider the continuous time NLFP of Section 2.1. Let $\varepsilon_j^*(t) = E(x_g(t) - E(x_g(t)|y_0^t))^2$. A lower bound for $\varepsilon_j^*(t)$ is given by

$$\varepsilon_j^*(t) \geq \left\{\frac{1}{2\pi e} \exp 2H_j(t)\right\} / E_{P_{X \cdot Y}} \left\{ \frac{E_{P_X}^{\sigma\{x_t\}}\{\exp \zeta_t\}}{E_{P_X}\{\exp \zeta_t\}} \right\} \tag{5.26}$$

for all j so that

$$\varepsilon^*(t) \geq \left\{\sum_{j=1}^{n} \frac{1}{2\pi e} \exp 2H_j(t)\right\} / E_{P_{X \cdot Y}} \left\{ \frac{E_{P_X}^{\sigma\{x_t\}}\{\exp \zeta_t\}}{E_{P_X}\{\exp \zeta_t\}} \right\} \tag{5.27}$$

where $H_j(t)$ is the entropy of $x_j(t)$ and ζ_t is as in Eq. (5.18) above provided the necessary expectations exist.

Proof: If (5.26) is true so is (5.27). By the definition of mutual information,

$$I(x_j(t);\ y_0^t) = E_{P_{X\cdot Y}} \log \frac{dP_{X|Y,X_j}}{dP_{X,X_j}}$$

where $X_j = \sigma\{x_j(t)\}$. Hence

$$I(x_j(t);\ y_0^t) = E_{P_{X\cdot Y}} \log E_{P_X}^{X_j} \frac{dP_{X|Y}}{dP_X}$$

$$\leq \log E_{P_{X\cdot Y}} \frac{dP_{X|Y}}{dP_X}$$

$$= \log E_{P_{X\cdot Y}} \left\{ \frac{E_{P_X}^{\sigma\{x(t)\}}\{\exp \zeta_t\}}{E_{P_X}\{\exp \zeta_t\}} \right\}$$

where we have made use of the properties of the conditional expectation, the Jensen unequality (Lemma 28) and the Bucy-Mortensen-Duncan representation (Lemma 1). Eq. (5.26) now follows from Lemma 16. ∎

5.2.3 Properties of Terms

The computation of the main formula developed in the previous subsection, Eq. (5.23) of Theorem 11, depends on the evaluation of two terms* which for obvious reasons we call the

*We do not address hereto the computation of the entropy H(t) since for the partiuclar example to which this material is applied in Chapter 6 this computation presents no problem. As mentioned in Section 5.1 Zakai and Ziv (38) present a useful bound on the entropy. In the most general case the entropy can be obtained by solving Kolmogorovs equation which while certainly a difficult task is of an order of magnitude easier than solving the Kirshner equation.

numerator N(t) and denominator* D(t) given by

$$N(t) = E_{P_{X\cdot Y}} \log E_{P_X} \exp \zeta_t$$

$$= E_{P_{X\cdot Y}} \int_0^t \frac{E_{P_X}(g_s' R^{-1}(s)\, g_s\, \psi_s)}{E_{P_X}(\psi_s)}\, ds$$

$$- \tfrac{1}{2} E_{P_{X\cdot Y}} \int_0^t \frac{E_{P_X}\{\psi_s\, g_s'\} R^{-1}(s)\, E_{P_X}\{g_s\, \psi_s\}}{(E_{P_X}\psi_s)^2}\, ds \quad (5.28)$$

$$D(t) = E_{P_{X\cdot Y}} \exp \zeta_t \quad (5.29)$$

In this subsection we study some of the properties of the terms composing N(t) and D(t) leaving for Sections 5.3 and 5.4 their evaluation.

First we address the question of the hypothesis of Theorem 10 and the other propositions of the previous section specifically in what regards to integrability requirements. What is at issue is whether the expectations in Eqs. (5.25) and (5.29) exist and are finite.

Lemma 30. Let as before $\psi_t(\omega,\tilde{\omega}) = \exp \zeta_t$ and $\Gamma_t(\tilde{\omega}) = \log E_{P_X} \exp \zeta_t$ where ζ_t is as in Eq. (5.18). If $g(x_t,t)$ is used in the definition of the NLFP is P_X – a.s. bounded then for $s \varepsilon [0,t]$ (1) $E_{P_{X\cdot Y}} \psi_s$ exists and is finite (ψ_s is integrable); (2) $E_{P_{X\cdot Y}} \Gamma_s$ exists and is finite (Γ_s is integrable).

*D(t) is not to be confused with distortion D.

<u>Proof:</u> (1) Since ζ_t satisfies

$$d\zeta_t = \tfrac{1}{2}\, g(x_t,t)'R^{-1}(t)\, g(x_t,t) + g(x_t,t)'R^{-1}(t)d\tilde{B}_t, \quad \zeta_o = 0$$

$\psi_t = \exp \zeta_t$ satisfies

$$\psi_t = 1 + \int_0^t g(x_s,s)'R^{-1}(s)\, g(x_s,s)\psi_s\, ds + $$
$$+ \int_0^t \psi_s\, g(x_s,s)'R^{-1}(s)d\tilde{B}_s$$

Since $\psi_t \geq 0$, $E_{P_{X \cdot Y}}\, \psi_t$ exists and we may apply the Fubini theorem to obtain

$$E_{P_{X\cdot Y}}\, \psi_t = 1 + \int_0^t E_{P_{X\cdot Y}}\{g(x_s,s)'R^{-1}(s)\, g(x_s,s)\psi_s\} ds$$

since the expectation of the stochastic integral vanishes. By hypothesis there exists a constant $K < \infty$ such that

$$0 \leq g(x_s,s)'R^{-1}(s)\, g(x_s,s) \leq K \qquad P_{X\cdot Y} - \text{a.s.}$$

Hence since $\psi_s \geq 0$,

$$E_{P_{X\cdot Y}}\, \psi_t = 1 + K \int_0^t E_{P_{X\cdot Y}}\, \psi_s\, ds$$

which by the Gronwal-Belman inequality gives the desired result

$$E_{P_{X\cdot Y}} \psi_t \leq \varepsilon^{Kt}$$

(2) First note that

$$E_{P_X} \exp \zeta_t < \infty \qquad P_{X \cdot Y} - \text{a.s.}$$

since, as just proved, $\exp \zeta_t$ is integrable. Further

$$0 < E_{P_X} \exp \zeta_t \qquad P_{X \cdot Y} - \text{a.s.}$$

since, for K as in part (1), $E_{P_{X \cdot Y}} \zeta_t \leq Kt$ so that in fact ζ_t is a.s. finite. Hence

$$-\infty < \log E_{P_X} \exp \zeta_t < \infty \qquad P_{X \cdot Y} - \text{a.s.}$$

which implies that $E_{P_{X \cdot Y}} \Gamma_t$ exists and is finite.□

It is possible to relax the hypothesis that g_s be a.s. bounded and still carry out all the development of this chapter by dealing with stopped processes (40, 50, 47) in the same way that Duncan (3) was able to extend Mortensen's (6) version of the Bucy-Mortensen-Duncan representation (Lemma 1) by using such arguments. In this study we will not go into the details involved in relaxing the a.s. boundedness of g_s for the following reasons:

(1) This is a technicality which, although very important, has been effectively resolved in the literature. See for example Duncan's (3) excellent treatment of the subject.

(2) The hypothesis that g_s be a.s. bounded is sufficient for the particular example considered here, the phase locked loop of Chapter 6.

(3) Bounded modulator or sensor output is not an unreasonable requirement especially for nonlinear systems.

The second item considered in this subsection is the martingale (49, 50, 47, 44, 45) nature of the expressions defining N(t) and D(t). We have the following two lemmas.

Lemma 31. Hypothesis of Lemma 30* $\{\exp \zeta_t, \underline{B}_t, t \geq 0\}$ is a submartingale (where as before ζ_t is given by Eq. (5.18) and $\underline{B}_t$ is defined in the definition of the NLFP in Chapter 2).

Proof: ζ_t is certainly $\underline{B}_t$ measurable since

$$dx_t = a(x_t,t)dt + b(x_t,t)dB_t$$

$$d\zeta_t = \tfrac{1}{2}\, g(x_t,t)'R^{-1}(t)\, g(x_t,t)dt + g(x_t,t)R^{-1}(t)\, d\tilde{B}_t$$

where the usual conditions are satisfied. Making use of Jensen's inequality (Lemma 18) for $0 \leq s < t$,

*Again if the hypothesis of g_s a.s. bounded is not satisifed an analogous treatment can be followed using local martingale treatment (3, 44, 47).

$$E^{\underline{B}_s}(\exp \zeta_t) \geq \exp E^{\underline{B}_s} \zeta_t =$$

$$= \exp E^{\underline{B}_s} \left\{ \int_0^t \tfrac{1}{2} g_r' R^{-1}(r) g_r \, dr + \int_0^t g_r' R^{-1}(r) d\tilde{B}_r \right\}$$

$$= \exp E^{\underline{B}_s} \left\{ \int_0^s \tfrac{1}{2} g_r' R^{-1}(r) g_r \, dr + \int_0^s g_r' R^{-1}(r) d\tilde{B}_r \right\}$$

$$\cdot \exp E^{\underline{B}_s} \left\{ \int_s^t \tfrac{1}{2} g_r' R^{-1}(r) g_r \, dr + \int_s^t g_r' R^{-1}(r) d\tilde{B}_r \right\}$$

$$= \exp \zeta_s \cdot \exp E^{\underline{B}_s} \left\{ \int_s^t \tfrac{1}{2} g_r' R^{-1}(r) g_r \, dr + \right.$$

$$\left. + \int_s^t g_r' R^{-1}(r) d\tilde{B}_r \right\}$$

where $E^{\underline{B}_s} = E^{\underline{B}}_{\overline{P}_{X\cdot Y}}$, $g_r = (x_r, r)$, and the last equality follows from ζ_s being $\underline{B}_s$ measurable. Since the conditional expectation of the stochastic integral vanishes,

$$E^{\underline{B}_s}(\exp \zeta_t) \geq \exp \zeta_s \exp E^{\underline{B}_s} \left\{ \int_s^t \tfrac{1}{2} g_r' R^{-1}(r) g_r \, dr \right\}.$$

Further

$$\exp E^{\underline{B}_s} \int_s^t \tfrac{1}{2} g_r R^{-1}(r) g_r \, dr \geq 1$$

so that as desired

$$E^{\underline{B}_s}_{\overline{P}_{X\,Y}}(\exp \zeta_t) \geq \exp \zeta_s \qquad P_{X\cdot Y} - \text{a.s.}^* \blacksquare$$

---

*This lemma can also be proved by showing that $\{\zeta_t, \underline{B}_t, t \geq 0\}$ is a submartingale and then using the standard submartingale formation theorem of Doob (49).

Lemma 32. Let $\gamma_t(\tilde{\omega}) = E_{P_X} \exp \zeta_t$, $\tilde{\omega} \in \tilde{\Omega}$, and as before ζ_t is as given by Eq. (5.18). $\{\gamma_t, \tilde{B}_t, t \geq 0\}$ is a submartingale.

Proof: $\psi_t(\omega,\tilde{\omega}) = \exp \zeta_t$ satisfies the equation

$$d\psi_t = g_t' R^{-1}(t)g_t \psi_t \, dt + \psi_t g_t' R^{-1}(t)d\tilde{B}_t$$

$$\psi_o = 1$$

where $g_t = g(x_t,t)$ and $\psi_t \geq 0$. Hence by the Fubini theorem $E_{P_X} d\psi_t = d E_{P_X} \psi_t$ so that

$$d\gamma_t = d E_{P_X} \psi_t = E_{P_X} \left\{g_t' R^{-1}(t)g_t \psi_t\right\} + E_{P_X}\left\{\psi_t g_t' R^{-1}(t)\right\} d\tilde{B}_t$$

$$\gamma_o = 1$$

or

$$\gamma_t = 1 + \int_0^t E_{P_X} \left\{g_r' R^{-1}(r)g_r \psi_r\right\} dr +$$

$$+ \int_0^t E_{P_X}\left\{\psi_r g_r' R^{-1}(r)\right\} d\tilde{B}_r$$

For $0 \leq s < t$ and $E^{\tilde{B}_s} = E^{\tilde{B}_s}_{P_{X\cdot Y}} = E^{\tilde{B}_s}_{P_Y}$,

$$E^{\tilde{B}_s}\gamma_t = 1 + E^{\tilde{B}_s} \int_0^s E_{P_X}\left\{g_r' R^{-1}(r)g_r \psi_r\right\}dr +$$

$$+ E^{\tilde{B}_s} \int_0^s E_{P_X}\left\{\psi_r g_r' R^{-1}(r)\right\}d\tilde{B}_r +$$

$$+ E^{\tilde{B}_s} \int_s^t E_{P_X} \left\{g_r' R^{-1}(r)g_r \psi_r\right\}dr +$$

$$+ E^{\tilde{B}_s} \int_s^t E_{P_X}\left\{\psi_r g_r' R^{-1}(r)\right\} d\tilde{B}_r$$

Noting that the first two integrals are $\tilde{B}_s$ measurable and that the last term vanishes we have

$$E^{\tilde{B}_s}\gamma_t = \gamma_s + E^{\tilde{B}_s} \int_s^t E_{P_X}\left\{g_r' R^{-1}(r) g_r \psi_r\right\} dr$$

so that as desired

$$E^{B_s}\gamma_t \geq \gamma_s \qquad P_{X \cdot Y} - \text{a.s.}$$

since

$$E^{\tilde{B}_s} \int_s^t E_{P_X}\{g_r' R^{-1}(r) g_r \psi_r\} dr \geq 0 \quad \blacksquare$$

These lemmas describe the "general behavior" of the terms in the numerator and denominator in terms of the classical properties of submartingales (Doob, 49). For example we know that D(t) is monotone increasing in t and that ψ_t, γ_t obey the standard submartingale inequalities and convergence theorems (Doob, 49). In addition Lemma 32 makes possible an alternate derivation of Lemma 29 (hence of the main formula of Theorem 11) using the following result of Ito-Watenabe (55) and Doleans-Dade (53).

Lemma 33 (Doleans-Bade, 53, 54). Let (γ_t, ζ_t) be a local submartingale (44) with Meyer decomposition (47, 51, 52) $\gamma_t = M_t + A_t$. Then

$$\gamma_t = \varepsilon(U)\ \varepsilon(V) \tag{5.30}$$

where

$$U_t = \int_0^t \frac{dM_s}{d\gamma_{s^-}} \text{ : a local martingale}$$

$$V_t = \int_0^t \frac{dA_s}{d\gamma_{s^-}} \text{ : a natural increasing process}$$

and for z_t a local semimartingale with z_t^c as the continuous part of the associated martingale,

$$\varepsilon(z_t) = \exp\left\{z_t - \tfrac{1}{2}\langle z_t^c, z_t^c\rangle\right\}$$

For the particular case of γ_t as in Lemma 32, Eq. (5.30) reduces to

$$\gamma_t = \exp\left\{V_t - \tfrac{1}{2}\langle U_t^c, U_t^c\rangle\right\}$$

thus providing an explicit expression for N(t) in Eq. (5.28) without the logarithm and exponential as follows:

$$\begin{aligned} N(t) &= E_{P_{X\cdot Y}} \log E_{P_X} \exp \zeta_t \\ &= E_{P_{X\cdot Y}} \log \gamma_t \\ &= E_{P_{X\cdot Y}} \log \exp\left\{V_t - \tfrac{1}{2}\langle U_t^c, U_t^c\rangle\right\} \\ &= E_{P_{X\cdot Y}} V_t - \tfrac{1}{2} E_{P_{X\cdot Y}} \langle U_t^c, U_t^c\rangle \qquad (5.31) \end{aligned}$$

The EV_t and $-\frac{1}{2}E\langle U_t^c, U_t^c\rangle$ terms are obtained by the following prescription:

(1) Define $W_t = \int_0^t \frac{d\gamma_s}{\gamma_s} ds$;

(2) Meyer decompose W_t into $W_t = W_t^c + W_t^r$;

(3) Meyer decompose $W_t^c W_t^c$ into $W_t^c W_t^c = \left[W_t^c W_t^c\right]^c + \left[W_t^c W_t^c\right]^r$;

(4) Then $EV_t = EW_t$ and $E\langle U_t^c, U_t^c\rangle = E\left[W_t^c W_t^c\right]^r$ where the expectations are $E_{P_{X\cdot Y}}$.

Evaluated in this manner the EV and $E\langle U,U\rangle$ terms of Eq. (5.31) correspond to the two terms in Eq. (5.22) of Lemma 29 thus providing an alternate proof of this lemma and hence to the main formula of this chapter given in Eq. (5.23) of Theorem 11. Evaluation of EV and $E\langle U,U\rangle$ will be considered in Section 5.4.

5.3 Evaluation of Main Formula: Denominator

The objective of this section is to derive formulas for the numerical evaluation of the denominator* of the main formula (Eq. (5.23) of Theorem 11), that is, the evaluation of

$$E_{P_{X\cdot Y}} = E_{P_{X\cdot Y}} \exp \zeta_t(\omega,\tilde{\omega}) \qquad (5.32)$$

where as in Eq. (5.18),

*The numerator of Eq. (5.23) will be evaluated in the next section.

$$\zeta_t(\omega,\tilde{\omega}) = \tfrac{1}{2} \int_0^t g(x_s,s)R^{-1}(s)\, g(x_s,s)'ds + $$

$$+ \int_0^t g(x_s,s)R^{-1}(s)d\tilde{B}_s' \qquad (5.33)$$

The expressions derived in this section will be applied in Chapter 6 to the Phase Locked Loop. The formulas for $E\psi$ are obtained in subsection 5.3.2 after an integration result is derived in subsection 5.3.1.

5.3.1 Integration Result

We shall need the following standard result.

Lemma 34 (Multinomial Theorem of Algebra, 56). The general term in the expansion of $(a + bx + cx^2 + \ldots)^n$, n an integer, is

$$\frac{n!}{p!\; q!\; r!\; \ldots}\; a^p\; b^q\; c^r\; \ldots\; x^{q+2r+3s+\ldots}$$

where $p + q + r + s + \ldots = n$. The greatest coefficient in the expansion is

$$\frac{n!}{(q!)^m\; (q+1)^k} \quad \text{where } qm + k = n$$

The next lemma is an immediate consequence of the multinomial theorem and the usual rules of permutations.

Lemma 35. Consider the two expressions

$$S_1 = \left(\sum_{j=1}^{N} a_j\right)^n$$

$$S_2 = \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} \cdots \sum_{j_{\frac{n}{2}}=1}^{N} a_{j_1}^2 a_{j_2}^2 \cdots a_{j_{n/2}}^2$$

where n and N are positive integers, n is even, and N > n. Conisder the "expansion" of the expressions for S_1 and S_2 into distinct terms of the form

$$a_{i_1}^{p_1} a_{i_2}^{p_2} a_{i_3}^{p_3} \cdots a_{i_{\tilde{n}}}^{p_{\tilde{n}}}$$

where $i_1, i_2, i_3, \ldots, i_{\tilde{n}} \varepsilon \{1,2,3,\ldots,N\}$

$p_{i_1}, p_{i_2}, p_{i_3}, \ldots, p_{i_{\tilde{n}}}$ are all positive even integers

$$\sum_{i=1}^{\tilde{n}} p_i = n \ , \ \tilde{n} \leq n/2$$

Let $i_1, i_2, \ldots i_{n/2}$ be distinct but otherwise arbitrary elements of $\{1,2,\ldots,N\}$. Then:

(1) For the "expansion" of S_1:

A. The coefficient of $a_{i_1}^2 a_{i_2}^2 \cdots a_{i_{n/2}}^2$ is $\dfrac{n!}{2^{n/2}}$

B. The coefficient of $a_{i_1}^4, a_{i_2}^2 \cdots a_{i_{(n/2-1)}}^2$ is $\dfrac{n!}{6 \cdot 2^{n/2}}$

C. The maximum coefficient of all the terms in the expansion is $n!/2^{n/2}$

(2) For the expansion of S_2, the coefficient of the term $a_{i_1}^2 \, a_{i_2}^2 \ldots a_{i_{n/2}}^2$ is $\left(\frac{n}{2}\right)$!

Proof: (1-A) This follows from the first part of Lemma 34 with $x = 1, a^2 = a_{i_1}^2$, $b^2 = a_{i_2}^2$, ... (1-B) This follows from the first part of Lemma 34 with $x = 1$, $a^4 = a_{i_1}^2$, $b^2 = a_{i_2}^2$, ... (1-C) This follows from the last part of Lemma 34 with $q = 2$, $m = \frac{n}{2}$, $k = 0$.

(2) The summation in S_2 goes through all $\left(\frac{n}{2}\right)!$ permutatin of i_1, i_2, ..., $i_{n/2}$. ∎

Finally we have the following integration result.

Lemma 36.* As before let $\left\{\tilde{B}_s : s \varepsilon [0,t]\right\}$ on $(\tilde{\Omega}, \tilde{B}^t, \tilde{P})$ be a (separable) Brownian motion with variance parameter $R(t)$. Let $\left\{w_s : s \varepsilon [0,t]\right\}$ on $(\Omega, B^t, \tilde{P})$ be a separable stochastic process. Also as before let $\underline{P} = P \times P$ be the product measure on $\underline{B}^t = B^t \times \tilde{B}^t$ so that B^t and $\tilde{B}^t$ are $\underline{P}$-independent. Let n be a positive integer and $\left\{w_s : s \varepsilon [0,t]\right\}$ have finite n^{th} moments. Then the following stochastic integrals are well defined and

$$E\left[\int_0^t w_s \, dB_s\right]^n = 0 \quad \text{if } n \text{ is odd} \tag{5.34}$$

$$E\left[\int_0^t w_s \, dB_s\right]^n = \frac{n!}{(n/2)! \, 2^{n/2}} \; E\left[\int_0^t w_s^2 \, R(s) \, ds\right]^{n/2} \quad \text{if } n \text{ is even} \tag{5.35}$$

where $E = E_{\underline{P}}$ throughout.

*Wiener uses this result in Ref. (57). Since he does not give a proof (and since the author, unaware of Wiener's result, spent a considerable amount of time deriving this result), a proof is included here.

Proof: The result will be shown for simplicity for $t = 1$ but it is valid for an arbitrary t. That the stochastic integrals are well defined and that Eq. (5.34) is true are clear, the later folloiwng from a consideration of

$$E\left[\int_0^1 w_s d\tilde{B}_s\right]^n = E\left[\text{l.i.m.} \sum_{i=1}^{N} w_{s_i}\left(\tilde{B}_{s_{i+1}} - \tilde{B}_{s_i}\right)\right]^n$$

$$= \lim_{N\to\infty} E\left[\sum_{i=1}^{N} w_{s_i}\left(\tilde{B}_{s_{i+1}} - \tilde{B}_{s_i}\right)\right]^n \tag{5.36}$$

since the resulting n-fold sum will involve only odd moments of $\left(\tilde{B}_{s_{i+1}} - \tilde{B}_{s_i}\right)$. Consider now the case when n is even. Let $a_i = w_{s_i}\left(\tilde{B}_{s_{i+1}} - \tilde{B}_{s_i}\right)$. Then

$$E\left[\int_0^1 w_s d\tilde{B}_s\right]^n = \lim_{N\to\infty} E\left[\sum_{i=1}^{N} a_i\right]^n$$

$$= \lim_{N\to\infty} E \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_n=1}^{N} a_{i_1} a_{i_2} \cdots a_{i_n} \tag{5.37}$$

where the general term is of the form

$$a_{i_1} a_{i_2} \cdots a_{i_n} = w_{s_{j_1}}^{p_1} w_{s_{j_2}}^{p_2} \cdots w_{s_{j_{\tilde{n}}}}^{p_{\tilde{n}}} \left(\tilde{B}_{s_{j_{1+1}}} - \tilde{B}_{s_{j_2}}\right)$$

$$\cdot\ {}^{p_1}\left(\tilde{B}_{s_{j_{2+1}}} - \tilde{B}_{s_{j_2}}\right)^{p_2} \cdots \left(\tilde{B}_{s_{j_{\tilde{n}+1}}} - \tilde{B}_{s_{j_{\tilde{n}}}}\right)^{p_{\tilde{n}}}$$

and $i_1, i_2, \ldots i_n \in (1, 2, 3, \ldots, N)$

$$\sum_{i=1}^{\tilde{n}} p_i = n \ , \ \tilde{n} \leq n$$

$$s_{j_1}, s_{j_2}, \ldots, s_{j_{\tilde{n}}} \in [0,1], \ s_{j_k} \neq s_{j_i} \text{ for } k \neq i$$

By hypothesis and the properties of the Brownian motion the expectation of the general term is

$$E\left(a_{i_1} a_{i_2} \ldots a_{i_n}\right) = E\left(w_{s_{j_1}}^{p_1} \ w_{s_{j_2}}^{p_2} \ \ldots \ w_{s_{j_{\tilde{n}}}}^{p_{\tilde{n}}}\right) E\left(\tilde{B}_{s_{j_{1+1}}} - \tilde{B}_{s_{j_1}}\right)^{p_1}$$
$$E\left(\tilde{B}_{s_{j_{2+1}}} - \tilde{B}_{s_{j_2}}\right)^{p_2} \ldots \ E\left(\tilde{B}_{s_{j_{\tilde{n}+1}}} - \tilde{B}_{s_{j_{\tilde{n}}}}\right)^{p_{\tilde{n}}} \tag{5.38}$$

We now investigate the contributions of the various terms on the right side of Eq. (5.37) to the left side of that equation. Conisder the general term of Eq. (5.38). There are two possibilities: Either at least one of the p_i is odd, in which case the right side of Eq. (5.38) vanishes and there is no contribution; or else all the p_i are even. We separate the later (i.e., those terms in the right side of Eq. (5.37) with all the p_i even) into two groups as follows:

<u>Group 1:</u> All terms having $p_i = 2$, $i = 1, \ldots, \tilde{n} = n/2$. That is all terms of the form $E(a_i^2, \ldots a_{i_{n/2}}^2)$ with the $i_1, \ldots, i_{n/2}$ distinct. Call their contribution (i.e., the sum of all such terms on the right side of Eq. (4.31)) Π_1.

$$\Pi_1 \leq \frac{N(N-1)\ \ldots\ (N-n/2+1}{(n/2)!}\ \frac{n!}{2^{n/2}}\ \frac{1}{N^{n/2}}\ K$$

$$= \frac{N^{n/2} + o(N^{n/2})}{N^{n/2}}\ \frac{n!}{(n/2)!\ 2^{n/2}}\ K$$

$$= O\left(\frac{n!}{(n/2)!\ 2^{n/2}}\ K\right) \text{ as } N \to \infty$$

so that Π_1, in addition to remaining finite, is of order

$$\Pi_1 = O(1) \quad \text{as } N \to \infty$$

where we have used the usual Landau symbols (58, 59, 60) o and O. On the other hand we can show that the contribution of the terms of group (2) is of the order 1/N as $N \to \infty$. In fact, consider the contribution Π_2' of all terms of the form $E(a_{i_1}^4, a_{i_2}^2, \ldots a_{i_{\frac{n}{2}-1}}^2)$. In manner analogous to the computations of Π_1,

$$\Pi_2' \leq {}_N C_{\frac{n}{2}-1} \cdot \frac{n!}{6 \cdot 2^{n/2}} \cdot K \cdot \frac{3}{N^{n/2}}$$

where $n!/(6 \cdot 2^{n/2})$ is, from Lemma 35 part 1-B, the coefficient of $E\left(a_{i_1}^4 a_{i_2}^2 \ldots a_{i_{\frac{n}{2}-1}}^2\right)$; ${}_N C_{\frac{n}{2}-1}$ is the number of combinations of N things taken (n/2-1) at a time; K is as before; and $3/N^{n/2}$ comes from the moments of the Brownian motion increments. Thus

Group 2: All terms having $p_i \geq 4$ for at least one i in i_1, ..., $\tilde{n}$. An example of such a term is $a^4_{i_1} \, a^2_{i_2} \ldots a^2_{i_{\frac{n}{2}-1}}$. Call their contribution Π_2. Now as $N \to \infty$ the value of the right side of Eq. (5.37) is finite and is determined solely by terms of group (1). In fact, from Eqs. (5.37) and (5.38)

$$\Pi_1 \leq {}_N C_{n/2} \cdot \frac{n!}{2^{n/2}} \cdot K \cdot \frac{1}{N^{n/2}} \tag{5.39}$$

where $\frac{n!}{2^{n/2}}$ is, from Lemma 35 part 1-A, the coefficient of $E\left(a^2_{i_1} \ldots a^2_{i_{n/2}}\right)$ in the expansion of Eq. (5.37); ${}_N C_{n/2}$ is the number of combinations of N things taken n/2 at a time,

$${}_N C_{n/2} = \frac{N!}{(N-n/2)! \; (n/2)!} \; ; \tag{5.40}$$

K is a constant (whose existence is assured by hypothesis) such that (see Eq. (5.38))

$$E\left(w^2_{s_{i_1}} \cdots w^2_{s_{i_{n/2}}}\right) \leq K < \infty;$$

and, lumping without loss of generality the variance parameter R(s) into the corresponding w^2_s, $1/N^{n/2} = E\left(\tilde{B}_{s_{i_1}} - \tilde{B}_{s_{i_1}}\right)^2 \ldots E\left(\tilde{B}_{s_{i_{\frac{n}{2}+1}}} - \tilde{B}_{s_{i_{\frac{n}{2}}}}\right)^2$. Substituting Eq. (5.40) into (5.39) we have

$$\Pi_2' \le \frac{N(N-1)\ \ldots\ (N- n/2 + 2)}{(n/2 - 1)!}\ \ 3K\ \ \frac{n!}{6 \cdot 2^{n/2}}$$

$$= \frac{N^{n/2 - 1} + o(N^{n/2 - 1})}{N^{n/2}} \cdot \frac{3K\ n!}{(n/2 - 1)!\ 6 \cdot 2^{n/2}}$$

$$= O(1/N) \text{ as } N \to \infty$$

Thus Π_2' is not only bounded but $\Pi_2' = O(1/N)$ as $N \to \infty$. It is clear then from Lemma 35 part 1-C and the definition of ${}_NC_n$ that the contributions of other "types" of terms in Group 2 (e.g., $a_{i_1}^6 a_{i_2}^2 \ldots a_{i_{\frac{n}{2}-2}}$, etc.) are o(1/N). Hence, since the number of such "types" is finite and depends only on n (and not on N), we have as desired

$$\Pi_2 \le \frac{K'}{N} \quad \text{and } \Pi_2 = O(1/N) \quad \text{as } N \to \infty$$

so that in the right side of Eq. (5.37) we need only consider terms of Group 1. Consider now

$$\sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_{n/2}=1}^{N} E\left(a_{i_1}^2\ a_{i_2}^2\ \ldots\ a_{i_{n/2}}^2\right)$$

We claim

$$\sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_n=1}^{N} E(a_{i_1} a_{i_2} \ldots a_{i_n}) = \frac{n!}{(n/2)!\ 2^{n/2}} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_{n/2}=1}^{N} E(a_{i_1}^2 a_{i_2}^2 \ldots a_{i_{n/2}}^2)$$

$$+ O(1/N) \text{ as } N \to \infty \qquad (5.41)$$

This follows from the fact that the equality is certainly true if the coefficients of each of the terms $E(a_{i_1}^2 a_{i_2}^2 \ldots a_{i_{n/2}}^2)$ of Group 1 match on both sides of the equation, the O(1/N) accounting for terms of Group 2. Now by Lemma 35 parts 1-A and 2 the coefficient of $E(a_{i_1}^2 a_{i_2}^2 \ldots a_{i_{n/2}}^2)$ on the left and right summations in Eq. (5.41) are $n!/2^{n/2}$ and $(n/2)!$ respectively so that Eq. (5.41) is indeed correct. Substituting Eq. (5.41) into Eq. (5.37) we have

$$E\left[\int_0^1 w_s \, d\tilde{B}_s\right]^n = \lim_{N\to\infty}\left\{\frac{n!}{(n/2)!\,2^{n/2}} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_{n/2}=1}^{N} E\left(a_{i_1}^2 a_{i_2}^2 \ldots a_{i_{n/2}}^2\right) + O(1/N)\right\}$$

$$= \lim_{N\to\infty}\left\{\frac{n!}{(n/2)!\,2^{n/2}} \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_{n/2}=1}^{N} E\left(\frac{w_{s_{i_1}}^2}{N} \frac{w_{s_{i_2}}^2}{N} \cdots \frac{w_{s_{i_{n/2}}}^2}{N}\right)\right\}$$

$$= \frac{n!}{(n/2)!\,2^{n/2}} \lim_{N\to\infty} E\left\{\prod_{k=1}^{n/2} \sum_{i_k=1}^{N} w_{s_{i_k}}^2 \frac{1}{N}\right\}$$

$$= \frac{n!}{(n/2)!\,2^{n/2}} E\left\{\prod_{k=1}^{n/2} \underset{N\to\infty}{\mathrm{l.i.m.}} \sum_{i_k=1}^{N} w_{s_{i_k}}^2 \frac{1}{N}\right\}$$

$$= \frac{n!}{(n/2)!\,2^{n/2}} E\left[\int_0^1 w_s^2 R(s)\, ds\right]^{n/2}$$

where the variance parameter R(s) has been restored in the last step. The above proves the lemma for a specific t and by the usual separability arguments (e.g.(3)) this can be made to apply to all rational and real t.

5.3.2 Denominator Formulas

We are now in a position to prove Theorem 13 below. Theorem 13 has four parts which provide both an exact expression (part 1) as well as an upper bound (part 2) for the denominator term D(t) of Eq. (5.32) and yield respective MSE lower bounds (parts 3 and 4). While in general the bound on D(t) is easier to evaluate than the exact expression, we will find in Chapter 6 that the exact expression (Eq. (5.42) below) can also be easily evaluated for the phase locked loop.

<u>Theorem 13.</u> Hypothesis of Theorem 11. Recall from Eq. (5.29) that $D(t) = E_{P_{X\cdot Y}} \exp \zeta_t$ where ζ_t is as in Eqs. (5.18). Then, with $E = E_{P_{X\cdot Y}}$ for short:

(1) $$D(t) = \sum_{r=0,2,4,\ldots}^{\infty} \sum_{n=r}^{\infty} \frac{1}{(r/2)!\,(n-r)!\,2^{n-r/2}}\, E\left[\int_0^t g(x_s,s)' R^{-1}(s)\, g(x_s,s)ds\right]^{n-r/2} \tag{5.42}$$

(2) $D(t) \leq D_{UB}(t)$ where

$$D_{UB}(t) = \sum_{r=0,2,4,\ldots}^{\infty} \sum_{n=r}^{\infty} \frac{t^{n-r/2-1}}{(r/2)!\,(n-r)!\,2^{n-r/2}} \int_0^t E\left[g(x_s,s)' R^{-1}(s)\, g(x_s,s)ds\right]^{n-r/2} ds$$

$$\triangleq D_{UB}(t) \tag{5.43}$$

(3) $$\varepsilon^*(t) \geq \frac{1}{2\pi e} \exp\{2H(t)\} \left\{\frac{\exp N(t)}{D(t)}\right\}^2 \tag{5.44}$$

where D(t) is given by Eq. (5.42);

(4) $$\varepsilon^*(t) \geq \frac{1}{2\pi e} \exp\{2H(t)\} \left\{\frac{\exp N(t)}{D_{UB}(t)}\right\}^2 \tag{5.45}$$

where $D_{UB}(t)$ is given by Eq. (5.43), N(t) is as in Eqs.

(5.28), and the terminology and stationarity comment of Theorem 11 apply.

Proof: First we note that if Eq. (5.42) is true then the upper bound of Eq. (5.43) and Eqs. (5.44) and (5.45) follow from Lemma 28C and from Eq. (5.23) of Theorem 11. Thus we only need to provie Eq. (5.42). By a formal expansion, noting that by hypothesis the expectation exists,

$$E(\exp \zeta_t) = \sum_{n=0}^{\infty} E\left(\frac{\zeta_t^n}{n!}\right) \tag{5.46}$$

We now derive an expression for $E(\zeta_t^n/n!)$. It is seen that for n=0, $E(\zeta_t^n/n!) = 1$; and for n=1, $E(\zeta_t^n/n!) =$

$$= \tfrac{1}{2} \int_0^t E(g(x_s,s)'R^{-1}(s)\ g(x_s,s))ds.$$

More generally, for arbitrary n (and with $g_s = g(x_s,s)$ for short),

$$E\left(\frac{\zeta_t^n}{n!}\right) = \frac{1}{n!} E\left\{\tfrac{1}{2} \int_0^t g(x_s,s)'R^{-1}(s)\ g(x_s,s)ds + \int_0^t g(x_s,s)R^{-1}(s)d\tilde{B}_s\right\}^n$$

$$= \frac{1}{n!} E\left\{\frac{1}{2^n}\left[\int_0^t g_s'R^{-1}(s)g_s\ ds\right]^n + \frac{n}{2^{n-1}}\left[\int_0^t g_s'R^{-1}(s)g_s\ ds\right]^{n-1}\left[\int_0^t g_s'R^{-1}(s)d\tilde{B}_s\right]\right\}$$

$$+ \frac{n\ (n-1)}{2^{n-2}\ 2!}\left[\int_0^t g_s'R^{-1}(s)g_s\ ds\right]^{n-2}\left[\int_0^t g_s'R^{-1}(s)d\tilde{B}_s\right]^2 +$$

$$+ \frac{n\ (n-1)(n-2)}{2^{n-3}\ 3!}\left[\int_0^t g_s'R^{-1}(s)g_s\ ds\right]^{n-3}\left[\int_0^t g_s'R^{-1}(s)d\tilde{B}_s\right]^3 +$$

$$+ \dots +$$

$$+ \frac{n(n-1)(n-2)\ \dots\ (n-r+1)}{2^{n-r}\ r!} \left[\int_0^t g_s' R^{-1}(s) g_s\ ds\right]^{n-r} \left[\int_0^t g_s' R^{-1}(s) d\tilde{B}_s\right]^{r} +$$

$$+ \frac{n(n-1)(n-2)\ \dots\ (n-r)}{2^{n-r+1}\ (r+1)!} \left[\int_0^t g_s' R^{-1}(s) g_s\ ds\right]^{n-r-1} \left[\int_0^t g_s' R^{-1}(s) d\tilde{B}_s\right]^{r+1} +$$

$$+ \dots +$$

$$+ \frac{n!}{2\ (n-1)!} \left[\int_0^t g_s' R^{-1}(s) g_s\ ds\right] \left[\int_0^t g_s' R^{-1}(s) d\tilde{B}_s\right]^{n-1}$$

$$+ \left[\int_0^t g_s' R^{-1}(s) d\tilde{B}_s\right]^{n} \qquad (5.47)$$

Suppose r is even and consider the r and (r+1) term in the above expression. For the $(r+1)^{th}$ term, let

$$w_s = \left[\int_0^t g_r' R^{-1}(r) g_r\ dr\right]^{\frac{n-r-1}{r+1}} \cdot g_s' R^{-1}(s)$$

so that

$$E\left[\int_0^t g_s' R^{-1}(s) g_s\ ds\right]^{n-r-1} \left[\int_0^t g_s' R^{-1}(s) d\tilde{B}_s\right]^{r+1} = E\left[\int_0^t w_s d\tilde{B}_s\right]^{r+1}$$
$$= 0 \qquad (5.48)$$

where the last equality follows from Lemma 36 since clearly w_s as defined here satisfies the hypothesis of this lemma. Similarly for the r^{th} term let

$$w_s = \left[\int_0^t g_r' R^{-1}(r) g_r\ dr\right]^{\frac{n-r}{r}} \cdot g_s' R^{-1}(s)$$

so that

$$
\begin{aligned}
E\left[\int_0^t g_s' R^{-1}(s) g_s \, ds\right]^{n-r}\left[\int_0^t g_s' R^{-1}(s) d\tilde{B}_s\right]^r &= E\left[\int_0^t w_s d\tilde{B}_s\right]^r \\
&= \frac{r!}{(r/2)!\, 2^{r/2}} E\left[\int_0^t w_s R(s) w_s' ds\right]^{r/2} \\
&= \frac{r!}{(r/2)!\, 2^{r/2}} E\left[\int_0^t g_r' R^{-1}(r) g_r \, dr\right]^{n-r}\left[\int_0^t g_s' R^{-1}(s) R(s) R^{-1}(s) g_s \, ds\right]^{r/2} \\
&= \frac{r!}{(r/2)!\, 2^{r/2}} E\left[\int_0^t g_s' R^{-1}(s) g_s \, ds\right]^{n-r/2} \qquad (5.49)
\end{aligned}
$$

where the second equality follows from Lemma 36 since again the independence hypothesis is clearly satisfied. Eqs. (5.48) and (5.49) can now be substituted into Eq. (5.47) and this equation can be substituted into Eq. (5.46). The net result can be cast into a double series by "summing across" as illustrated by the following diagram,

	n=0	n=1	n=2	n=3	n=4	n=5	
r=0	*	*	*	*	*	*	...
r=1		0	0	0	0	0	...
r=2			*	*	*	*	...
r=3				0	0	0	...
r=4					*	*	...
r=5						0	...

where 0 indicates a vanishing term corresponding to Eq. (5.48) and * denotes a term obtained from applying Eq. (5.49). Thus, the coefficient of the general non-vanishing term is (5.47),

$$
\frac{r!}{(r/2)!\, 2^{r/2}} \frac{1}{n!} \frac{n(n-1)\, \ldots\, (n-r+1)}{2^{n-r}\, r!}, \quad r: \text{even}, \ n: \text{arbitrary}
$$

so that as desired

$$D(t) = \sum_{r-0,2,4,..}^{\infty} \sum_{n=r}^{\infty} \frac{1}{(r/2)!(n-r)!2^{n-r/2}} \cdot$$

$$\cdot E\left[\int_0^t g(x_s,s)'R^{-1}(s)\, g(x_s,s)ds\right]^{n-r/2} \blacksquare$$

Summarizing, we have produced in this section two computable expressions -- an exact formula and a bound -- for the denominator D(t) of the main formula of this chapter, Eq. (5.23) of Theorem 11. While these expressions are in the form of infinite series, they are rapidly convergent (roughly as $1/(n!)^2$ for the phase locked loop of Chapter 6). The numerator N(t) of the main formula is now considered in the next section.

5.4 Evaluation of Main Formula: Numerator

The objective of this section is to obtain machinery for the numerical evaluation of the numerator N(t) of the main formula of this chapter (Eq. (5.23) of Theorem 11), that is for the evaluation of

$$N(t) = v(t) - \tfrac{1}{2}\, u(t) \tag{5.50}$$

where, from Eqs. (5.23), (5.31)

$$v(t) \triangleq E_{P_{X\cdot Y}} V_t = E_{P_{X\cdot Y}} \int_0^t \frac{E_{P_X}(g_s' R^{-1}(s) g_s\ \psi_s)}{E_{P_X}(\psi_s)}\, ds \tag{5.51}$$

$$u(t) = E_{P_{X \cdot Y}} \langle U_t^c, U_t^c \rangle = \int_0^t \frac{E_{P_X}(\psi_s g_s') R^{-1}(s)\, E_{P_X}(\psi_s g_s)}{(E_{P_X} \psi_s)^2}\, ds \tag{5.52}$$

and as before

$$g_s = g(x_s, s)$$
$$\psi_t = \exp \zeta_t$$
$$\zeta_t = \tfrac{1}{2} \int_0^t g_s' R^{-1}(s) g_s\, ds + \int_0^t g_s R^{-1}(s)\, d\tilde{B}_s \tag{5.53}$$

By "evaluation of N(t)" we mean the procurement of a <u>lower-bound</u> to N(t) in terms of the moments of the g_s process since such a lowerbound will provide us with an MSE lowerbound when used in Theorem 11.

The evaluation of N(t) is more complex than that of D(t) as is evident from a comparison of Eqs. (5.32), (5.50), (5.51) and (5.52). To simplify the notation we make the following assumptions and simplifications:

(1) $g_s = g(x_s, s)$ is assumed to be a scalar

(2) R(s) is assumed to be a constant

(3) $E_{P_{X \cdot Y}}$ and E_{P_X} are written as E_{xy} and E_x, respectively.

While the case of assumptions (1) and (2) will be sufficient for the application considered in Chapter 6, it is clear that no loss in generality is suffered assuming R constant since R can be lumped into g_s (Eqs. (5.51), (5.52)). Similarly it will become apparent that the case of vector g_s can be handled analogously to that of scalar g_s although the notational complexity is naturally increased.

This section is divided into three subsections. First in subsection 5.4.1 the basic nature of the difficulties involved in evaluating the numerator is idscussed and elementary solutions are presented. Subsections 5.4.2 to 5.4.6 contain a more sophisticated evaluation method parts of which will be used in Chapter 6. In particular lowerbounds on the numerator are derived in subsection 5.4.2 while the machinery to evaluate these bounds is developed in 5.4.3 to 5.4.6.

5.4.1 Basic Problem and Elementary Solution

Inspection of Eqs. (5.50), (5.51), and (5.52) shows that the basic problem in the computation of the numerator is the evaluation of the expectation of a quotient of random variables of the form

$$E_{xy}\left\{\frac{A(\tilde{\omega})}{B(\tilde{\omega})}\right\} = E_{xy}\left\{\frac{E_x\ \alpha(\omega,\tilde{\omega})}{E_x\ \beta(\omega,\tilde{\omega})}\right\} \tag{5.54}$$

where ω and $\tilde{\omega}$ are as in the definition of the NLFP (Section 2.1). Further, in the evaluation of the denominator considered in Section 5.3, E_{xy} could "operate on" the $\tilde{\omega}$-stochastic integrals involved (see for example Eqs. 5.48 and 5.49) while here it cannot. Finally, we note that since we desire a lower bound on N(t) we must find a lower bound on v(t) and an upper bound on u(t).

An elementary approach to the evaluation of N(t) is then to use standard inequalities in lower and upper bounding v(t) and u(t) respectively.

Lemma 37. Consider N(t), v(t), and u(t) as defined in Eqs. (5.50) through (5.52). Then

$$u(t) \le \int_0^t (E_x\, g_s^2) \sqrt{E_{xy}\, \psi_s^4} \sqrt{E_{xy}\, \psi_s^{-4}}\; R^{-1}\, ds \qquad (5.55a)$$

$$u(t) \le \int_0^t \sqrt{E_{xy}\, g_s^4\, \psi_s^2} \sqrt{E_{xy}\, \psi_s^{-2}}\; R^{-1}\, ds \qquad (5.55b)$$

$$u(t) \le \int_0^t \sqrt{E_{xy}\, g_s^4\, \psi_s^4} \sqrt{E_{xy}\, \psi_s^{-4}}\; R^{-1}\, ds \qquad (5.55c)$$

$$N(t) \ge \tfrac{1}{2} \int_0^t E_{xy}(g_s^2\, R^{-1})\, ds \qquad (5.55d)$$

$$v(t) \ge \int_0^t \frac{\left[E_{xy}(g_s^2\, \psi_s)\right]^2}{\sqrt{E_{xy}(g_s^4\, \psi_s^2)\, E_{xy}(\psi_s^2)}}\; R^{-1}\, ds \qquad (5.55e)$$

Proof: Equations (5.55a) through (5.55e) follow from applying Holder's and Jensen's inequality to the integrand in the definition of u(t), Eq. (5.52). Eq. (5.55d) follows upon applying the convex Jensen's inequality (Lemma 28B) to N(t) as defined in the statement of Theorem 11. Finally, Eq. (5.55e) follows by noting that for all $K \in (0,\infty]$

$$\frac{1}{x} \geq \frac{-1}{K^2} x + \frac{2}{K}$$

optimizing over K (see proof of Lemma 39), and using Holder's inequality.∎

We note that the net effect of this lemma has been to bring E_{xy} -- rather than just E_x -- to act on all quantities which are defined in terms of $\tilde{\omega}$-stochastic integrals. Once this objective has been achieved exact formulas and bounds for the resulting E_{xy} expectations can be obtained by methods analogous to those used in deriving the denominator expressions of Theorem 13. Specifically we have the following formulas for the E_{xy} expectations appearing on Eqs. (5.55a) through (5.55f).

<u>Lemma 38.</u> Consider Eqs. (5.55) of Lemma 37. Expressions and bounds for the equations appearing in these equations are given by:

$$E_{xy}\, \psi_t^m = \sum_{r=0,2,4,\ldots}^{\infty} \sum_{n=r}^{\infty} \frac{m^n}{(n-r)!(r/2)!2^{n-r/2}} E_{xy} \left\{ \left[\int_0^t g_s^2 R^{-1}\, ds \right]^{n-r/2} \right\} \tag{5.56a}$$

$$\leq \sum_{r=0,2,4,\ldots}^{\infty} \sum_{n=r}^{\infty} \frac{m^n}{(n-r)!(r/2)!2^{n-r/2}} \int_0^t E_{xy} \left\{ \left[g_s^2 R^{-1} \right]^{n-r/2} \right\} ds \tag{5.56b}$$

$$E_{xy}\, \psi_t^{-m} = \sum_{r=0,2,4,\ldots}^{\infty} \sum_{n=r}^{\infty} \frac{(-m)^n}{(n-r)!(r/2)!2^{n-r/2}} E_{xy} \left\{ \left[\int_0^t g_s^2 R^{-1}\, ds \right]^{n-r/2} \right\} \tag{5.56c}$$

$$\leq \sum_{r=0,2,4,\ldots}^{\infty} \sum_{n=r,r+2,r+4,\ldots}^{\infty} \frac{m^n}{(n-r)!(r/2)!2^{n-r/2}} \cdot \left\{ t^{n-r/2-1} \int_0^t E_{xy} \left\{ (g_s^2 R^{-1})^{n-r/2} \right\} ds - \frac{m}{2(n+1-r)} \left[\int_0^t E_{xy} (g_s^2 R^{-1}) ds \right]^{n+1-r/2} \right\} \tag{5.56d}$$

$$E_{xy}\left\{g_t^k\ \psi_t^m\right\} = \sum_{r=0,2,4,\ldots}^{\infty}\ \sum_{n=r}^{\infty} \frac{m^n}{(n-r)!(r/2)!2^{n-r/2}}\ E_{xy}\left\{\left[\int_0^t \frac{k}{g_t^{n-r/2}}\ g_s^2\ R^{-1}\ ds\right]^{n-r/2}\right\} \tag{5.56e}$$

$$E_{xy}\left\{g_t^k\ \psi_t^m\right\} \leq \sum_{r=0,2,4,\ldots}^{\infty}\ \sum_{n=r}^{\infty} \frac{m^n}{(n-r)!(r/2)!2^{n-r/2}} \left[\frac{t^{n-r/2-1}}{R^{n-r/2}}\right] \int_0^t E_{xy}\left\{g_t^k\ g_s^{2n-r}\right\} ds \tag{5.56f}$$

$$E_{xy}\left\{g_t^k\ \psi_t^m\right\} \geq \sum_{r=0,2,4,\ldots}^{\infty}\ \sum_{n=r}^{\infty} \frac{m^n}{(n-r)!(r/2)!2^{n-r/2}} \left[\int_0^t E_{xy}\ g_t^{\frac{k}{n-r/2}}\ g_s^2\ R^{-1} ds\right]^{n-r/2} \tag{5.56g}$$

where

$m = 0,\ 1,\ 2,\ 3,\ \ldots$

$k = 0,\ 2,\ 4,\ 6,\ \ldots$

and as before

$$g_t = g(x_t, t)$$

$$\psi_t = \exp\ \zeta_t = \exp\left\{\tfrac{1}{2} \int_0^t g_s^2\ R^{-1}\ ds + \int_0^t g_s\ R^{-1}\ d\tilde{B}_s\right\}$$

Proof: The identities in Eqs. (5.56a), (5.56c), (5.56e) can be proved by using Lemma 36 in manner analogous to the proof of Theorem 13. The inequalities in Eqs. (5.56b), (5.56d), (5.56f), (5.56g) can be obtained by using the Holder and Jensen inequalities of Lemma 28 in the respective identities (Eqs. (5.56a, c, e)). ∎

The results of Lemmas 37, 38 are simple to apply* and may yield satisfactory results in some problems. Unfortunately

*In fact, for the phase locked loop of Chapter 6 there is no need to use the bounds (Eqs. (5.56b, d, f, g)) in Lemma 38 since the exact expressions (Eqs. (5.56a, c, e)) can be easily evaluated.

these formulas proved to be inadequate for the phase locked loop of Chapter 6. In the next two subsections we develop better although more complex methods to evaluate the numerator N(t).

5.4.2 Numerator Bound

In the previous subsection the technique was "to bound until"

(1) the quotient form of Eq. (5.54) was bypassed, and

(2) all E_x had been eliminated so that only E_{xy} expectations remained.

In this subsection we tolerate the presence of the E_x in the interest of bound tightness and at the expense of increased computational complexity. The quotient form of Eq. (5.54), however, must still be bypassed and this is the objective of the following lemma.

<u>Lemma 39.</u> Consider N(t), v(t), u(t) as defined in Eqs. (5.50) to (5.52).

A. In general we have:

$$v(t) \geq \int_0^t \frac{\left\{E_{xy}\left[g_s^2 \ \psi_s\right]\right\}^2}{E_{xy}\left\{E_x(g_s^2 \ \psi_s) \ E_x(\psi_s)\right\}} R^{-1} \, ds \tag{5.57}$$

$$u(t) \leq \int_0^t E_{xy}\left\{(E_x \ g_s \ \psi_s)(E_x \ g_s \ \psi_s)(E_x \ \psi_s^{-2})\right\} R^{-1} \, ds \tag{5.58}$$

so that

$$N(t) \geq \int_0^t \left\{ \frac{\left\{E_{xy}\left[g_s^2 \ \psi_s\right]\right\}^2}{E_{xy}\left\{E_x(g_s^2 \ \psi_s) \ E_x(\psi_s)\right\}} - \tfrac{1}{2} \ E_{xy}\left\{(E_x \ g_s \ \psi_s)(E_x \ g_s \ \psi_s)(E_x \psi_s^{-2})\right\} \right\} R^{-1} \ ds$$

B. Consider the special case* where for a positive constant A

$$v(t) = \frac{A}{R} \ t \tag{5.59}$$

and where there exists a function $\Xi(t)$ such that

$$E_x(\psi_t) \geq \Xi(t) \ \tilde{\omega} - \text{a.s.} \tag{5.60}$$

Then

$$N(t) \geq \left\{\frac{A}{R} \ t - \frac{1}{2R} \int_0^t \frac{E_{xy}\left\{E_x(g_s \ \psi_s) \ E_x(g_s \ \psi_s)\right\}}{\Xi(s)^2} \ ds\right\} \tag{5.61}$$

<u>Proof:</u> From Eqs. (5.50) to (5.52) it is clear that we need only prove Eqs. (5.57) and (5.58). Let

$$f(K) = \frac{-1}{K^2} \ A \ B + \frac{2}{K} \ A \ , \ K\varepsilon(0,\infty]$$

where A and B are two positive constant. Clearly for all $K\varepsilon(0,\infty]$,

$$\frac{A}{B} \geq f(K) \tag{5.62}$$

---

*This will indeed be the case in Chapter 6.

Letting A, B be a.s. positive random variables on $(\tilde{\Omega},\tilde{B},\tilde{P})$ we note that the inequality in Eq. (5.62) holds a.s. so that

$$E_{xy}\left\{\frac{A}{B}\right\} \geq E_{xy}\ f(K) = \frac{-1}{K^2}\ E_{xy}(A\ B) + \frac{2}{K}\ E(A),\ K\varepsilon(0,\infty] \quad (5.63)$$

Choosing K in Eq. (5.63) such that $\partial E_{xy}\ f(K)/\partial K = 0$ and letting $A(\tilde{\omega}) = E_x(g_s^2\ \psi_s)$, $B(\tilde{\omega}) = E_x(\psi_s)$ we have

$$E_{xy}\left\{\frac{E_x(g_s^2\ \psi_s)}{E_x(\psi_s)}\right\} \geq \frac{\left\{E_{xy}\left[g_s^2\ \psi_s\right]\right\}^2}{E_{xy}\left\{E_x(g_s^2\ \psi_s)\ E_x(\psi_s)\right\}}$$

so that Eq. (5.57) is indeed true. By the convex Jensen inequality (Lemma 28B) we know that

$$\frac{1}{(E_x\ \psi_s)^2} \leq E_x\ \psi_s^{-2}$$

Hence

$$E_{xy}\left\{\frac{(E_x\ g_s\ \psi_s)^2}{(E_x\ \psi_s)^2}\right\} \leq E_{xy}\left\{(E_x\ g_s\ \psi_s)(E_x\ g_s\ \psi_s)(E_x\ \psi_s^{-2})\right\}$$

from which Eq. (5.58) follows. Eq. (5.61) follows directly from Eq. (5.59), (5.60), and (5.50) to (5.52).

The next theorem summarizes in Part A the general development in this chapter and contains in the special case of Part B the specific formula for the MSE lowerbound that will be applied in Chapter 6 to the phase locked loop. The importance of the formulas of Theorem 14 is that -- as we shall see in the remaining of this section -- they provide MSE lower-bounds that can be evaluated exactly provided the moments of the g_s process are available.

Theorem 14. Hypothesis of Theorem 10. A lowerbound on $\varepsilon^*(t)$ is given by

$$\varepsilon^*(t) \geq \left\{\frac{1}{2\pi e} \exp[2H(t)]\right\} \left\{\frac{\exp\left\{N_{LB}(t)\right\}}{D(t)}\right\}^2 \tag{5.64}$$

where

$$D(t) = E_{xy}\{\psi_t\} \tag{5.65}$$

is as given in Eq. (5.42) of Theorem 13; $H(t)$, g_s, ψ_s have the same meaning as in Theorems 10 and 11; and:

A. In general $N_{LB}(t)$ can be taken as

$$N_{LB}(t) = \int_0^t \frac{\left\{E_{xy}\left[g_s^2\ \psi_s\right]\right\}^2}{E_{xy}\left\{E_x(g_s^2\ \psi_s)\ E_x(\psi_s)\right\}} - \tfrac{1}{2}\, E_{xy}\left\{(E_x\ g_s\ \psi_s)(E_x\ g_s\ \psi_s)(E_x\ \psi_s^{-2})\ R^{-1}\right\} ds \tag{5.66}$$

B. In the special case for which Eqs. (5.59), (5.60) of Lemma 39 hold, $N_{LB}(t)$ can be taken as

$$N_{LB}(t) = \left\{\frac{A}{R}\, t - \frac{1}{2R} \int_0^t \frac{E_{xy}\left\{E_x(g_s\ \psi_s)\ E_x(g_s\ \psi_s)\right\}}{\Xi(s)}\, ds\right\} \tag{5.67}$$

As in Theorems 10 and 11, if the processes defined by the NLFP are stationary then $\varepsilon^*(t)$ is constant and bounded by the supremum over all t of the right hand side of Eq. (5.64).

Proof: Eqs. (5.64), (5.66) and (5.67) follow upon substituting in Theorem 13 the results of Lemma 39. ■

The next four subsections provide the machinery for handling the added computational complexity arising from the presence of the E_x expectations in Eqs. (5.66) and (5.67).

5.4.3 Integration and Series Results

The objective of the present subsection is to express the expectations appearing in Eqs. (5.66) and (5.67) in terms of products of stochastic integrals. In Eqs. (5.48) to (5.50) all such stochastic integrals will be reduced to ordinary integrals of the moments of the g_s process. The net result will be the evaluation of the MSE lowerbound of Eq. (5.64) both in the general case as well as in the special case considered in Chapter 6.

Of all the E_{xy} expectations appearing in Theorem 14, formulas for $E_{xy}(\psi_t)$ and $E_{xy}(g_t^2\ \psi_t)$ have already been derived in Theorem 13 and Lemma 38. Thus, in order to evaluate Eq. (5.64) what remains is to obtain computable expressions for

$$E_{xy}\left\{E_x(g_t^2\ \psi_t)\ E_x(\psi_t)\right\} \tag{5.68}$$

$$E_{xy}\left\{E_x(g_t\psi_t)\ E_x(g_t\ \psi_t)\ E_x(\psi_t^{-2})\right\} \tag{5.69}$$

$$E_{xy}\left\{E_x(g_t\psi_t)\ E_x(g_t\ \psi_t)\right\} \tag{5.70}$$

where

$$g_t = g(x_t,\ t)$$
$$\psi_t = \exp \zeta_t$$
$$\zeta_t = \tfrac{1}{2} \int_0^t g_s^2 \ R^{-1}\ ds + \int_0^t g_s \ R^{-1}\ d\tilde{B}_s \qquad (5.71)$$

We first note the "product of infinite series nature" of statements (5.68), (5.69) and (5.70). In fact, let

$$a_i = \frac{1}{i!} E_x(g_t \ \zeta_t^k) \ , \ b_j = \frac{1}{j!} E_x(\zeta_t^j)$$

$$c_k = \frac{1}{k!} E_x(g_t \ \zeta_t^k) \ , \ d_m = \frac{1}{m!} E_x(g_t \ \zeta_t^m)$$

$$e_n = \frac{(-2)^n}{n!} E_x(\zeta_t^n) \qquad (5.72)$$

Then from Eqs. (5.68), (5.69), and (5.71) we see that

$$E_{xy}\left\{E_x(g_t^2 \ \psi_t) \ E_x(\psi_t)\right\} = E_{xy}\left\{\sum_{i=0}^{\infty} a_i \cdot \sum_{j=0}^{\infty} b_j\right\} \qquad (5.73)$$

$$E_{xy}\left\{E_x(g_t \ \psi_t) \ E_x(g_t \ \psi_t) \ E_x(\psi_t^{-2})\right\} = E_{xy}\left\{\sum_{k=0}^{\infty} c_k \cdot \sum_{m=0}^{\infty} d_m \cdot \sum_{n=0}^{\infty} e_n\right\} \qquad (5.74)$$

$$E_{xy}\left\{E_x(g_t \ \psi_t) \ E_x(g_t \ \psi_t)\right\} = E_{xy}\left\{\sum_{k=0}^{\infty} c_k \cdot \sum_{m=0}^{\infty} d_m\right\} \qquad (5.75)$$

We will need the following result from the algebra of infinite series.

Lemma 40. (Cauchy (61), (62), (63), (64), (65)). Let $\{a_i\}$, $\{b_i\}$, $\{c_i\}$, $\{d_i\}$, $\{e_i\}$, i = 0, 1, 2, ... be real sequences with corresponding Euclidean norm convergent infinite series Σa_i, Σb_i, Σc_i, Σd_i, Σe_i. Let for j = 0, 1, 2, ...

$$\alpha_j = a_o b_j + a_1 b_{j-1} + \ldots + a_{j-1} b_1 + a_j b_o \quad (5.76)$$

$$\gamma_j = c_o d_j + c_1 d_{j-1} + \ldots + c_{j-1} d_1 + c_j d_o \quad (5.77)$$

$$\beta_j = \gamma_o e_j + \gamma_1 e_{j-1} + \ldots + \gamma_{j-1} e_1 + \gamma_j e_o \quad (5.78)$$

Then the Cauchy products

$$\sum_{j=0}^{\infty} \alpha_j \ , \ \sum_{j=0}^{\infty} \gamma_j \ , \ \sum_{j=0}^{\infty} \beta_j$$

converge in the Euclidean norm and furthermore

$$\sum_{k=0}^{\infty} \alpha_k = \sum_{i=0}^{\infty} a_i \cdot \sum_{j=0}^{\infty} b_j \quad (5.79)$$

$$\sum_{j=0}^{\infty} \gamma_j = \sum c_k \cdot \sum d_m \quad (5.80)$$

$$\sum_{j=0}^{\infty} \beta_j = \sum_{k=0}^{\infty} c_k \cdot \sum_{m=0}^{\infty} d_m \cdot \sum_{n=0}^{\infty} e_n \quad (5.81)$$

From the preceding lemma we can immediately write the following expressions for the expectations in statement (5.68) and (5.69).

Lemma 41. Hypothesis of Theorem 11.* Let a_i, b_i, c_i, d_i, e_i, α_i, γ_i, β_i be as defined in Eqs. (5.72), (5.76), (5.77), (5.78) respectively. Then

*We recall that the relevant part of the hypothesis of Theorems 10 and 11 is that the expectations that define N(t) and D(t) exist and are finite. We also recall that by Lemma 30 a sufficient condition for this hypothesis to be true is that g_S be a.s. bounded as indeed is the case for the phase locked loop of Chapter 6. See the discussion in subsection 5.2.3.

A. $\sum_{j=0}^{\infty} \alpha_j$, $\sum_{j=0}^{\infty} \gamma_j$, $\sum_{j=0}^{\infty} \beta_j$ converge $\underline{P}$ - a.s.;

B. $\sum_{j=0}^{\infty} E_{xy}\alpha_j$, $\sum_{j=0}^{\infty} E_{xy}\gamma_j$, $\sum_{j=0}^{\infty} E_{xy}\beta_j$ converge in the Euclidean norm:

C. $$E_{xy}\ E_x(g_t^2\ \psi_t)\ E_x(\psi_t) = \sum_{j=0}^{\infty} E_{xy}\alpha_j \tag{5.82}$$

$$E_{xy}\ (E_x\ g_t\ \psi_t)(E_x\ g_t\ \psi_t)(E_x\ \psi_t^{-2}) = \sum_{j=0}^{\infty} E_{xy}\beta_j \tag{5.83}$$

$$E_{xy}\ E_x(g_t\ \psi_t)\ E_x(g_t\ \psi_t) = \sum_{j=0}^{\infty} E_{xy}\gamma_j \tag{5.84}$$

Proof: If A is true then clearly, making use of Lemma 40, so are B and C. But by hypothesis and Lemma 30 ζ_t as defined in Eqs. (5.71) is $\underline{P}$-a.s. finite so that indeed -- since the exponential series is absolutely convergent on the entire real line -- $\Sigma\alpha_j$, $\Sigma\gamma_j$, $\Sigma\beta_j$ converge for almost all $\underline{\omega}$.∎

What remains, then, is to find expressions for the α_j, β_j, and γ_j appearing in the Cauchy products on the right hand side of Eqs. (5.82), (5.83), and (5.84). The philosophy of the approach here is not to produce one single two-page-long uncomprehensible formula for α_j, β_j, or γ_j, but rather to find an expression for the typical terms in α_j, β_j, and γ_j. Once an expression for the typical term has been derived, a digital computer can be used to efficiently perform the pertinent evaluations and sums. For example, once we find a procedure to evaluate $E_{xy}a_m b_n$, a computer can evaluate $E_{xy}a_n b_k$ for different n, k and perform the sum in Eq. (5.76) to evaluate $E_{xy}\alpha_j$. The resulting procedure for the evaluation of the α_j, β_j, γ_j will then be easier done than said.

In the next three subsections we consider respectively the evaluation of the α_j, β_j, γ_j from their present form involving sums of products of stochastic integrals* to sums of products of ordinary integrals of the g_s process.

5.4.4 Integration and Series Results: The α_j

Consider first the α_j. From Eqs. (5.76), (5.72) it is clear that the typical term in the evaluation of α_j is

$$E_{xy}(a_n\ b_k) = \frac{1}{n!}\ \frac{1}{k!}\ E_{xy}\left\{E_x(g_t^2\ \zeta_t^n)\ E_x(\zeta_t^k)\right\} \tag{5.85}$$

Furthermore, from Eqs. (5.71) we see that the typical term in Eq. (5.85) is composed of constants multiplying integrals of the form

$$I_1(r_1,r_2,m_1,m_2) \triangleq E_{xy}\left\{E_x\left\{g_t^2\left[\int_0^t g_s^2\ ds\right]^{r_1}\left[\int_0^t g_s\ d\tilde{B}_s\right]^{r_2}\right\}\ \cdot \right.$$
$$\left. \cdot\ E_x\left\{\left[\int_0^t g_s^2\ ds\right]^{m_1}\left[\int_0^t g_s\ d\tilde{B}_s\right]^{m_2}\right\}\right\} \tag{5.86}$$

where r_1, r_2, m_1, m_2 are non-negative integers. We observe that:

---

*The presence of the stochastic integrals in the α_j can be seen from Eqs. (5.71), (5.72), and (5.76) and similarly for the β_j, γ_j.

(1) Once we find an expression for I_1 of Eq. (5.86) we then have a well defined computer amenable procedure for evaluating α_J and $E_{xy}\left\{E_x(g_t^2\ \psi_t)\ E_x(\psi_t)\right\}$ of statement (5.68).

(2) In order to evaluate I_1 of Eq. (5.68) we must first eliminate all stochastic integrals and replace them by deterministic integrals. Once this is done and if we know the moments of the g_s process (as is the case for the phase locked loop of Chapter 6) we then have an explicit computable expression for I_1.

Before proceding to evaluate I we need the following lemma due to Wiener.

<u>Lemma 42.</u> (Wiener (57)). Let r_2, m_2 be two non-negative integers with $m_2 \geq r_2$. Let $F:[0,t]^{r_2} \to R$ and $G:[0,t]^{m_2} \to R$ be two functions symmetric in their arguments in the sense that*

$$F(s_1,\ s_2,\ \ldots,\ s_{r_2}) = F(s_{i_1},\ s_{i_2},\ \ldots,\ s_{i_{r_2}})$$

where $\{i_1,\ i_2,\ \ldots,\ i_{r_2}\}$ is a rearrangement of $\{1,\ 2,\ \ldots,\ r_2\}$ and similarly for G. Let $\tilde{B}_s$ be Brownian motion as in the definition of the NLFP. Then if r_2, m_2 are either both odd or both even,

---

*For example, $F(s_1,\ldots,s_{r_2}) = E_x(g_{s_1}\ldots\ g_{s_{r_2}})$.

$$E_{xy}\left\{\int_{s_1=0}^{t}\int_{s_2=0}^{t}\cdots\int_{s_{r_2}=0}^{t} F(s_1, s_2, \ldots, s_{r_2})d\tilde{B}_{s_1}d\tilde{B}_{s_2}\ldots d\tilde{B}_{s_{r_2}}\cdot\right.$$

$$\left.\cdot\int_{\sigma_1=0}^{t}\int_{\sigma_2=0}^{t}\cdots\int_{\sigma_{m_2}=0}^{t} G(\sigma_1, \sigma_2, \ldots, \sigma_{m_2})d\tilde{B}_{\sigma_1}d\tilde{B}_{\sigma_2}\ldots d\tilde{B}_{\sigma_{m_2}}\right\}$$

$$= \left[\binom{m_2}{2\nu}(2\nu-1)(2\nu-3)\ \ldots\ 5\cdot 3\cdot 1\right]R^{\nu}\int_{\lambda_1=0}^{t}\int_{\lambda_2=0}^{t}\cdots\int_{\lambda_\nu=0}^{t}\cdot$$

$$\cdot E_{xy}\left\{\int_{s_1=0}^{t}\int_{s_2=0}^{t}\cdots\int_{s_{r_2}=0}^{t} F(s_1, s_2, \ldots, s_{r_2})d\tilde{B}_{s_1}d\tilde{B}_{s_2}\ldots d\tilde{B}_{s_{r_2}}\cdot\right.$$

$$\left.\cdot\int_{\sigma_1=0}^{t}\int_{\sigma_2=0}^{t}\cdots\int_{\sigma_{m_2}=0}^{t} G(\sigma_1, \sigma_2, \ldots, \sigma_{r_2}, \lambda_1, \lambda_1, \ldots, \lambda_\nu, \lambda_\nu)d\tilde{B}_{\sigma_1}d\tilde{B}_{\sigma_2}\ldots d\tilde{B}_{\sigma_{r_2}}\right\} \tag{5.87}$$

where

$$\nu = \frac{m_2 - r_2}{2}$$

$$\binom{m_2}{2\nu} = \frac{m_2!}{(m_2-2\nu)!(2\nu)!}$$

If m_2, r_2 are not either both odd or both even the left side of Eq. (5.87) vanishes.

We observe that the net effect of this lemma is to eliminate (m_2-r_2) of the Brownian motions so that both integrals on the right hand side of Eq. (5.87) have r_2 Brownian motions. This result is useful in obtaining an expression for I of Eq. (5.86) in the following Theorem.*

Theorem 15. Conisder $I_1(r_1, r_2, m_1, m_2)$ as defined in Eq. (5.86) where r_1, m_1 are arbitrary non-negative integers;

*The principles used in the proofs of Theorems 15, 16 and 17 are described by Wiener (57) in a more elementary context.

r_2, m_2 are even integers; and $r_2 \leq m_2$.* Then

$$I_1(r_1, r_2, m_1, m_2) = \left[\binom{m_2}{2\nu}(2\nu-1)(2\nu-3)\ldots 5\cdot 3\cdot 1\right] R^{\frac{m_2+r_2}{2}} \cdot$$

$$\cdot \sum_{\eta=0}^{r_2/2} \left\{ \left[\binom{r_2}{2\eta}(2\eta-1)(2\eta-3)\ldots 5\cdot 3\cdot 1\right]^2 (r_2-2\eta)! \int_{\tilde{s}_1=0}^{t} \cdots \int_{\tilde{s}_{I1(\eta)}=0}^{t} \int_{\tilde{\sigma}_1=0}^{t} \cdots \int_{\tilde{\sigma}_{J1(\eta)}=0}^{t} \cdot \right.$$

$$\cdot \int_{s_1=0}^{t} \cdots \int_{s_{I2(\eta)}=0}^{t} E_x\left\{g_t^2\, g_{\tilde{s}_1}^2 \cdots g_{\tilde{s}_{I1(\eta)}}^2\, g_{s_1} \cdots g_{s_{I2(\eta)}}\right\} \cdot$$

$$\left. \cdot E_x\left\{g_{\tilde{\sigma}_1}^2 \cdots g_{\tilde{\sigma}_{J1(\eta)}}^2\, g_{s_1} \cdots g_{s_{I2(\eta)}}\right\} d\tilde{s}_1 \ldots d\tilde{s}_{I1(\eta)}\, d\tilde{\sigma}_1 \ldots d\tilde{\sigma}_{J1(\eta)}\, ds_1 \ldots ds_{I2(\eta)} \right\} \tag{5.88}$$

where

$$(2\eta-1)(2\eta-3)\ldots 5\cdot 3\cdot 1 \Big|_{\eta=0} \stackrel{\Delta}{=} 1$$

$$\binom{M}{N} = \frac{M!}{(M-N)!\; N!}$$

$$\nu = \frac{m_2 - r_2}{2}$$

$$I1(\eta) = r_1 + \eta$$

$$J1(\eta) = m_1 + \nu + \eta$$

$$I2(\eta) = r_2 - 2\eta$$

<u>Proof:</u> First we note that $I(r_1, r_2, m_1, m_2)$ of Eq. (5.86) can be written as (upon using Fubini):

---

*In Chapter 6 we will only need the case where r_2 and m_2 are even since $E(g_{s_1} \ldots g_{s_N}) = 0$ for N odd. An identical formula results for the case where r_2, m_2 are both odd except that $(r_2-1)/2$ is the upper limit of the sum over η in Eq. (5.88). For r_2, m_2 not either both odd or both even, I vanishes. An analogous formula can be derived for the $m_2 < r_2$ case simply by interchanging m_2 and r_2

$$
\begin{aligned}
I_1(r_1, r_2, m_1, m_2) = & \int_{\tilde{s}_1=0}^{t} \cdots \int_{\tilde{s}_{r_1}=0}^{t} \int_{\tilde{\sigma}_1=0}^{t} \cdots \int_{\tilde{\sigma}_{m_1}=0}^{t} \cdot \\
& \cdot E_{xy}\left\{ \int_{s_1=0}^{t} \cdots \int_{s_{r_2}=0}^{t} E_x\left\{ g_t^2 \, g_{\tilde{s}_1}^2 \cdots g_{\tilde{s}_{r_1}}^2 \, g_{s_1} \cdots g_{s_{r_2}} \right\} d\tilde{B}_{s_1} \cdots d\tilde{B}_{s_{r_2}} \cdot \right. \\
& \left. \cdot \int_{\sigma_1=0}^{t} \cdots \int_{\sigma_{m_2}=0}^{t} E_x\left\{ g_{\tilde{\sigma}_1}^2 \cdots g_{\tilde{\sigma}_{m_1}}^2 \, g_{\sigma_1} \cdots g_{\sigma_{m_2}} \right\} d\tilde{B}_{\sigma_1} \cdots d\tilde{B}_{\sigma_{m_2}} \cdot \right\} \\
& \cdot d\tilde{s}_1 \cdots d\tilde{s}_{r_1} \, d\tilde{\sigma}_1 \cdots d\tilde{\sigma}_{m_1}
\end{aligned}
\tag{5.89}
$$

Next we apply Lemma 42 to the $E_{xy}\{\ \}$ in Eq. (5.89) obtaining

$$
\begin{aligned}
I_1(r_1, r_2, m_1, m_2) = & \left[\binom{m_2}{2\nu} (2\nu-1) \ldots 5 \cdot 3 \cdot 1 \right] \int_{\tilde{s}_1=0}^{t} \cdots \int_{\tilde{s}_{r_1}=0}^{t} \int_{\tilde{\sigma}_1=0}^{t} \cdots \int_{\tilde{\sigma}_{m_1}=0}^{t} \\
& \cdot \int_{\lambda_1=0}^{t} \cdots \int_{\lambda_\nu=0}^{t} E_{xy}\left\{ \int_{s_1=0}^{t} \cdots \int_{s_{r_2}=0}^{t} E_x\left\{ g_t^2 \, g_{\tilde{s}_1}^2 \cdots g_{\tilde{s}_{r_1}}^2 \, g_{s_1} \cdots g_{s_{r_2}} \right\} d\tilde{B}_{s_1} \cdots d\tilde{B}_{s_{r_2}} \right. \\
& \left. \cdot \int_{\sigma_1=0}^{t} \cdots \int_{\sigma_{r_2}=0}^{t} E_x\left\{ g_{\tilde{\sigma}_1}^2 \cdots g_{\tilde{\sigma}_{m_1}}^2 \, g_{\lambda_1}^2 \cdots g_{\lambda_\nu}^2 \, g_{\sigma_1} \cdots g_{\sigma_{r_2}} \right\} d\tilde{B}_{\sigma_1} \cdots d\tilde{B}_{\sigma_{r_2}} \right\} \\
& \cdot d\lambda_1 \cdots d\lambda_\nu \, d\tilde{s}_1 \cdots d\tilde{s}_{r_1} \, d\tilde{\sigma}_1 \cdots d\tilde{\sigma}_{m_1}
\end{aligned}
\tag{5.90}
$$

where $\nu = (m_2 - r_2)/2$.

Now there are many ways to pairing the Brownian motions appearing in Eq. (5.90). Suppose that pair 2η of the Brownian motions appearing in the first integral among themselves. Clearly $0 \leq \nu \leq r_2/2$. Such a pairing can be done in

$$
\binom{r_2}{2\eta} (2\eta-1)(2\eta-3) \ \ldots \ 5 \cdot 3 \cdot 1 \tag{5.91}
$$

ways insce there are $\binom{r_2}{2\eta}$ ways of selecting 2η items from r_2 items and these 2ν items can be paired in $(2\eta-1) \ldots 5 \cdot 3 \cdot 1$ ways.

The pairing of 2η of the Brownian motions appearing in the first integral among themselves requires the pairing of 2η of the Brownian motions in the second integral among themselves (else some of the Brownian motions will remain unpaired and the E_{xy} expectation of the integral will vanish). Such a pairing in the second integral can also be done in

$$\binom{r_2}{2\eta}(2\eta-1)(2\eta-3)\ \ldots\ 5\cdot3\cdot1 \tag{5.92}$$

ways.

Finally observe that there are $(r_2 - 2\eta)$ unpaired Brownian motions in both integrals. They can be paired (pairing one Brownian motion of one integral with one of the other) in

$$(r_2 - 2\eta)! \tag{5.93}$$

ways.

Thus, for each η, $0 \le \eta \le r_2/2$, the pairing process will give (multiplying statements (5.91), (5.92), (5.93),

$$\left[\binom{r_2}{2\eta}(2\eta-1)\ \ldots\ 5\cdot3\cdot1\right]^2 (r_2 - 2\eta)!$$

identical terms. Substituting these results in Eq. (5.90) and summing over all possible η gives

$$I_1(r_1, r_2, m_1, m_2) = \left[\binom{m_2}{2\nu} (2\nu-1)\ldots 5\cdot 3\cdot 1\right] R^{\frac{m_2+r_2}{2}}$$

$$\sum_{\eta=0}^{r_2/2} \left\{\left[\binom{r_2}{2\eta} (2\eta-1)\ldots 5\cdot 3\cdot 1\right]^2 (r_2 - 2\eta)!\right.$$

$$\int_{\tilde{s}_1=0}^{t} \cdots \int_{\tilde{s}_{r_1+\eta}=0}^{t} \int_{\tilde{\sigma}_1=0}^{t} \cdots \int_{\tilde{\sigma}_{m_1+\eta+\nu}=0}^{t} \int_{s_1=0}^{t} \cdots \int_{s_{r_2-2\eta}=0}^{t}$$

$$E_x\left\{g_t^2\, g_{\tilde{s}_1}^2 \cdots g_{\tilde{s}_{r_1+\eta}}^2\, g_{s_1} \cdots g_{s_{r_2-2\eta}}\right\}$$

$$E_x\left\{g_{\tilde{\sigma}_1}^2 \cdots g_{\tilde{\sigma}_{m_1+\eta+\nu}}^2\, g_{s_1} \cdots g_{s_{r_2-2\eta}}\right\}$$

$$\left. d\tilde{s}_1 \ldots d\tilde{s}_{r_1+\eta}\, d\tilde{\sigma}_1 \ldots d\tilde{\sigma}_{m_1+\eta+\nu}\, ds_1 \ldots ds_{r_2-2\eta}\right\} \qquad (5.94)$$

Equations (5.88) and (5.94) differ only in the notation. ■

Comparison of Eqs. (5.86) and (5.88) shows that as desired the net effect of Theorem 15 is to produce an expression for $I_1(r_1, r_2, m_1, m_2)$ without stochastic integrals and in terms of the moments of the g_s process. We have therefore achieved a procedure for evaluating α_j -- and consequently $E_{xy}\, E_x(g_s^2\, \psi_s)\, E_x(\psi_s)$ -- that can be easily implemented in a digital computer.

5.4.5 Integration and Series Results: The β_j

Consider now the β_j arising in Eq. (5.83) of Lemma 41. The approach to be followed and objectives to the attained are analogous to those in the evaluation of β_j just completed. Specifically, we seek to isolate the typical term in β_j and find an expression for it without Brownian motions and in terms of the moments of the g_s process. Computer loops can then be "built around" such an expression to evaluate β_j and consequently $E_{xy}\ (E_x\ g_t\ \psi_t)(E_x\ g_t\ \psi_t)(E_x\ \psi_t^{-2})$ of statement (5.69).

From Eqs. (5.72), (5.77), (5.78) it is clear that the typical term in the evaluation of β_j is

$$E_{xy}\left\{c_n d_m e_k\right\} = \frac{1}{n!}\,\frac{1}{m!}\,\frac{(-2)^k}{k!}\,E_{xy}\left\{(E_x\ g_t\ \psi_t)(E_x\ g_t\ \psi_t)(E_x \zeta_t^k)\right\} \tag{5.95}$$

Furthermore, from Eq. (5.71) we see that the typical term in Eq. (5.95) is composed of constants multiplying integrals of the form

$$I_2(i_1, i_2, j_1, j_2, k_1, k_2) = E_{xy}\left\{E_x\left\{g_t\left[\int_0^t g_s^2\,ds\right]^{i_1}\left[\int_0^t g_s\,d\tilde{B}_s\right]^{i_2}\right\}\cdot\right.$$
$$\left.E_x\left\{g_t\left[\int_0^t g_s^2\,ds\right]^{j_1}\left[\int_0^t g_s\,d\tilde{B}_s\right]^{j_2}\right\} E_x\left\{\left[\int_0^t g_s^2\,ds\right]^{k_1}\left[\int_0^t g_s\,d\tilde{B}_s\right]^{k_2}\right\}\right\} \tag{5.96}$$

where i_1, j_1, i_2, j_2, k_1, k_2 are non-negative integers. It is clear that I_2 will vanish unless

$i_2 =$	$j_2 =$	$k_2 =$
odd	odd	even
even	even	even
odd	even	odd
(even	odd	odd)*

since whenever $(i_2 + j_2 + k_2)$ is odd the E_{xy} expectation of the resulting stochastic integral will vanish. While in Chapter 6 we shall only need the odd-odd-even case (since $E(g_{s_1} \cdots g_{s_N}) = 0$ for N odd), formulas analogous to that of the following theorem can be derived for the other cases.

Theorem 16. Consider $I_2(i_1, i_2, j_1, j_2, k_1, k_2)$ as defined in Eq. (5.96) where i_1, j_1, k_1 are arbitrary non-negative integers and i_2 is odd, j_2 is odd, k_2 is even. Then

$$I_2(i_1, i_2, j_1, j_2, k_1, k_2) =$$

$$\sum_{\nu_1=0}^{\frac{i_2-1}{2}} \sum_{\nu_2=0}^{\frac{j_2-1}{2}} \sum_{\nu_3=0}^{k_2/2} \sum_{\eta_{23}=0}^{k_2-2\nu_3} K(\nu_1, \nu_2, \nu_3, \eta_{12}, \eta_{13}, \eta_{23}) \cdot$$

$$\{\nu_1, \nu_2, \nu_3, \eta_{12}, \eta_{13}, \eta_{23}\} \varepsilon P$$

$$\cdot \int_{\tilde{s}_1=0}^{t} \cdots \int_{\tilde{s}_{i_1+\nu_1}=0}^{t} \int_{\tilde{\sigma}_1=0}^{t} \cdots \int_{\tilde{\sigma}_{j_1+\nu_2}=0}^{t} \int_{\tilde{\pi}_1=0}^{t} \cdots \int_{\tilde{\pi}_{k_1+\nu_3}=0}^{t}$$

$$\int_{s_1=0}^{t} \cdots \int_{s_{\eta_{12}}=0}^{t} \int_{\sigma_1=0}^{t} \cdots \int_{\sigma_{\eta_{13}}=0}^{t} \int_{\pi_1=0}^{t} \cdots \int_{\pi_{\eta_{23}}=0}^{t}$$

*This case is identical to the odd-even-odd case since $c_n \equiv d_n$.

$$E_x\left\{g_t\ g^2_{\tilde{s}_1}\cdots g^2_{\tilde{s}_{i_1+\nu_1}}\ g_{s_1}\cdots g_{s_{\eta_{12}}}\ g_{\sigma_1}\cdots g_{\sigma_{\eta_{13}}}\right\}$$

$$E_x\left\{g_t\ g^2_{\tilde{\sigma}_1}\cdots g^2_{\tilde{\sigma}_{j_1+\nu_2}}\ g_{s_1}\cdots g_{s_{\eta_{12}}}\ g_{\pi_1}\cdots g_{\pi_{\eta_{23}}}\right\}$$

$$\cdot\ E_x\left\{g^2_{\tilde{\pi}_1}\cdots g^2_{\tilde{\pi}_{k_1+\nu_3}}\ g_{\sigma_1}\cdots g_{\sigma_{\eta_{13}}}\ g_{\pi_1}\cdots g_{\pi_{\eta_{23}}}\right\}$$

$$d\tilde{s}_1\cdots d\tilde{s}_{i_1+\nu_1}\ d\tilde{\sigma}_1\cdots d\tilde{\sigma}_{j_1+\nu_2}\ d\tilde{\pi}_1,,,d\tilde{\pi}_{k_1+\nu_3}$$

$$ds_1\cdots ds_{\eta_{12}}\ d\sigma_1\cdots d\sigma_{\eta_{13}}\ d\pi_1\cdots d\pi_{\eta_{23}} \tag{5.97}$$

where:

$$P = \left\{\nu_1\ \nu_2,\nu_3,\eta_{12},\eta_{13},\eta_{23} \geq 0:\ \begin{aligned} i_2 &= 2\nu_1 + \eta_{12} + \eta_{13} \\ j_2 &= 2\nu_2 + \eta_{23} + \eta_{12} \\ k_2 &= 2\nu_3 + \eta_{13} + \eta_{23} \end{aligned}\right\}$$

$$\eta_{12} = j_2 - 2\nu_2 - \eta_{23} \tag{5.98}$$

$$\eta_{13} = i_2 - 2\nu_1 - \eta_{12} \tag{5.99}$$

$$\begin{aligned} K(\nu_1,\nu_2,\nu_3,\eta_{12},\eta_{13},\eta_{23}) = &\left[{}_{k_2}C_{2\nu_3}(2\nu_3-1)(2\nu_3-3)\ldots5\cdot3\cdot1\right]\cdot \\ &\cdot\left[{}_{j_2}C_{2\nu_2}(2\nu_2-1)(2\nu_3-3)\ldots5\cdot3\cdot1\right]\left[{}_{i_2}C_{2\nu_1}(2\nu_1-1)(2\nu_1-3)\ldots5\cdot3\cdot1\right]\cdot \\ &\cdot\left[{}_{(k_2-2\nu_3)}C_{\eta_{23}}(\eta_{23}!)\right]\left[{}_{j_2-2\nu_2}C_{\eta_{23}}(\eta_{12}!)\right]\cdot \\ &\cdot\left[{}_{(i_2-2\nu_1)}C_{\eta_{12}}(\eta_{13}!)\right]\cdot R^{\frac{i_2+j_2+k_2}{2}} \end{aligned} \tag{5.100}$$

Proof: First we note that I_2 in Eq. (5.96) can be written as (upon using Fubini):

$$
\begin{aligned}
&I_2(i_1,i_2,j_1,j_2,k_1,k_2) = \\
&\quad E_{xy}\Bigg\{\int_{\tilde{s}_1=0}^{t}\cdots\int_{\tilde{s}_{i_1}=0}^{t}\ \int_{\tilde{\sigma}_1=0}^{t}\cdots\int_{\tilde{\sigma}_{j_1}=0}^{t}\ \int_{\tilde{\pi}_1=0}^{t}\cdots\int_{\tilde{\pi}_{k_1}=0}^{t}\ \int_{s_1=0}^{t}\cdots\int_{s_{i_2}=0}^{t}\ \int_{\sigma_1=0}^{t}\cdots\int_{\sigma_{j_2}=0}^{t}\ \int_{\pi_1=0}^{t}\cdots\int_{\pi_{k_2}=0}^{t} \\
&\quad E_x(g_t\, g_{\tilde{s}_1}^2\cdots g_{\tilde{s}_{i_1}}^2\ g_{s_1}\cdots g_{s_{i_2}}) \\
&\quad E_x(g_t\, g_{\tilde{\sigma}_1}^2\cdots g_{\tilde{\sigma}_{j_1}}^2\ g_{\sigma_1}\cdots g_{\sigma_{i_2}}) \\
&\quad E_x(g_{\tilde{\pi}_1}^2\cdots g_{\tilde{\pi}_{k_1}}^2\ g_{\pi_1}\cdots g_{\pi_{k_2}}) \\
&\quad d\tilde{s}_1\cdots d\tilde{s}_{i_1}\ d\tilde{\sigma}_1\cdots d\tilde{\sigma}_{j_1}\ d\tilde{\pi}_1\cdots d\tilde{\pi}_{k_1}\ d\tilde{B}_{s_1}\cdots d\tilde{B}_{s_{i_2}}\ d\tilde{B}_{\sigma_1}\cdots d\tilde{B}_{\sigma_{j_2}}\ d\tilde{B}_{\pi_1}\cdots d\tilde{B}_{\pi_{k_2}}\Bigg\}
\end{aligned} \tag{5.101}
$$

As in Theorem 15 the objective is to account for all possible pairings of the Brownian motion increments

$$d\tilde{B}_{s_1}\cdots d\tilde{B}_{s_{i_2}}\quad d\tilde{B}_{\sigma_1}\cdots d\tilde{B}_{\sigma_{j_2}}\quad d\tilde{B}_{\pi_1}\cdots d\tilde{B}_{\pi_{k_2}} \tag{5.102}$$

Call the $1,\ldots,i_2$ group of Brownian increment in (5.102) the "first group" and similarly the $1,\ldots,j_2$ and $1,\ldots,k_2$ increments the "second" and "third" groups respectively. Suppose we make

ν_1 pairs within the first group, $0 \le \nu_1 \le (i_2-2)/2$;
ν_2 pairs within the second group, $0 \le \nu_2 \le (j_2-2)/2$;
ν_3 pairs within the third group, $0 \le \nu_3 \le k_2/2$. (5.103)

Such pairings can be done in

$$\left[{}_{k_2}C_{2\eta_3}(2\eta_3-1)\ldots5\cdot3\cdot1\right]\cdot\left[{}_{j_2}C_{2\nu_2}(2\nu_2-1)\ldots5\cdot3\cdot1\right]\cdot$$

$$\cdot\left[{}_{i_2}C_{2\nu_1}(2\nu_1-1)\ldots5\cdot3\cdot1\right] \quad (5.104)$$

ways where as usual ${}_NC_K = \frac{N!}{(N-K)!\ N!}$

Now let

η_{23} = pairings between groups 2 and 3, $0 \le \eta_{23} \le (k_2-2\nu_3)$;

η_{12} = pairings between groups 1 and 2, $\eta_{12} = j_2-2\nu_2-\eta_{23}$;

η_{13} = pairings between groups 1 and 3, $\eta_{13} = i_2-2\nu_1-\eta_{12}$.

Such pairings can be done in

$$\left[{}_{(k_2-2\nu_3)}C_{\eta_{23}}(\eta_{23}!)\right]\cdot\left[{}_{(j_2-2\nu_2)}C_{\eta_{23}}(\eta_{12}!)\right]\cdot$$

$$\cdot\left[{}_{(i_2-2\nu_1)}C_{\eta_{12}}(\eta_{13}!)\right] \quad (5.105)$$

Furthermore we must require that every selection of ν_1, ν_2, ν_3, η_{23}, η_{12}, η_{13} according to (5.103) and (5.105) obeys the equations

$$i_2 = 2\nu_1 + \eta_{12} + \eta_{13}$$

$$j_2 = 2\nu_2 + \eta_{23} + \eta_{12}$$

$$k_2 = 2\nu_3 + \eta_{13} + \eta_{23} \quad (5.106)$$

From Eqs. (5.101) to (5.106) it is now clear that Eqs. (5.97) to (5.100) are indeed true. In fact, the summation in (5.97) arises from the allowed values in (5.103) and (5.105) and membership in P is equivalent to satisfaction of Eqs. (5.106). Further K(.,.,.,.,.,.,.) appearing in (5.97) and defined in (5.100) arises by multiplying (5.104) and (5.106). For any particular ν_1, ν_2, ν_3, η_{23} in the sum in (5.97), η_{12} and η_{13} are indeed defined as in (5.98) and (5.99) as is evident from Eqs. (5.106). Finally the correctness of the integrand of (5.97) follows from the definition of the ν_1, ν_2, ν_3, η_{12}, η_{13}, and η_{23}.■

Comparison of Eqs. (5.96) and (5.97) shows that as desired the net effect of Theorem 16 is to produce an expression for $I_2(i_1, i_2, j_1, j_2, k_1, k_2)$ without stochastic integrals and in terms of the moments of g_s process. We have therefore achieved a procedure for evaluating the β_j -- and consequently $E_{xy}\left\{(E_x\, g_t\, \psi_t)(E_x\, g_t\, \psi_t)(E_x\, \psi_t^{-2})\right\}$ -- that can be easily implemented* in a digital computer once the moments of the g_s process have been obtained.

5.4.6 Integration and Series Results: The γ_j

Finally consider the γ_j. From Eqs. (5.72), (5.78) it is clear that the typical term in the evaluation of γ_j is

$$E_{xy}(a_n b_k) = \frac{1}{n!}\,\frac{1}{k!}\,E_{xy}\left\{E_x(g_t\, \zeta_t^n)\; E_x(g_t\, \zeta_t^k)\right\} \qquad (5.107)$$

Furthermore, from Eqs. (5.71) we see that the typical term in Eq. (5.107) is composed of constants multiplying integrals of the form

*E.g., in (5.97) the sum over P is trivial to computer implement.

$$I_3(r_1,r_2,m_1,\ m_2) \triangleq E_{xy}\left\{E_x\left\{g_t\left[\int_0^t g_s^2\ ds\right]^{r_1}\left[\int_0^t g_s\ d\tilde{B}_s\right]^{r_2}\right\}\cdot\right.$$

$$\left.\cdot\ E_x\left\{g_t\left[\int_0^t g_s^2\ ds\right]^{m_1}\left[\int_0^t g_s\ d\tilde{B}_s\right]^{m_2}\right\}\right\} \quad (5.108)$$

where r_1, r_2, m_1, m_2 are non-negative integers.

The situation here is analogous to that of the α_j and $I_1(.,.,.,.,.)$ defined by Eq. (5.86) and thus we can proceed immediately to the counterpart of Theorem 15.

<u>Theorem 17.</u> Consider $I_3(r_1,r_2,\ m_1,\ m_2)$ as defined in Eq. (5.108) where r_1, m_1 are arbitrary non-negative integers; r_2, m_2 are odd integers; and $r_2 \leq m_2$.* Then

$$I_3(r_1,r_2,m_1,m_2) = \left[\binom{m_2}{2\nu}(2\nu-1)(2\nu-3)\ldots5\cdot3\cdot1\right]R^{\frac{m_2+r_2}{2}}\cdot$$

$$\cdot\sum_{\eta=0}^{(r_2-1)/2}\left\{\left[\binom{r_2}{2\eta}(2\eta-1)(2\eta-3)\ldots5\cdot3\cdot1\right]^2(r_2-2\eta)!\int_{\tilde{s}_1=0}^{t}\cdots\int_{\tilde{s}_{I1(\eta)}=0}^{t}\int_{\tilde{\sigma}_1=0}^{t}\cdots\int_{\tilde{\sigma}_{J1(\eta)}=0}^{t}\cdot\right.$$

$$\cdot\int_{\tilde{s}_1=0}^{t}\cdots\int_{\tilde{s}_{I2(\eta)}=0}^{t}E_x\left\{g_{\tilde{s}_1}^2\cdots g_{\tilde{s}_{I1(\eta)}}^2\ g_t\ g_{s_1}\cdots g_{s_{I2(\eta)}}\right\}\cdot$$

$$\left.\cdot\ E_x\left\{g_{\tilde{\sigma}_1}^2\cdots g_{\tilde{\sigma}_{J1(\eta)}}^2\ g_t\ g_{s_1}\cdots g_{s_{I2(\eta)}}\right\}ds_1\ldots ds_{I2(\eta)}\ d\tilde{\sigma}_1\ldots d\tilde{\sigma}_{J1(\eta)}\ d\tilde{s}_1\ldots d\tilde{s}_{I1(\eta)}\right\}$$

$$(5.109)$$

---

*In Chapter 6 we only need the case where r_2 and m_2 are odd since $E(g_{s_1}\ldots g_{s_N}) = 0$ for N odd. An identical formula results for the case where r_2, m_2 are both even except that the upper limit of the sum over in Eq. (5.109) is $r_2/2$. For r_2, m_2 not either both odd or both even, I vanishes. An analogous formula can be derived for the $m_2 < r_2$ case simply by interchanging m_2 and r_2.

where: $\left[(2\eta-1)(2\eta-3)\ldots5\cdot3\cdot1\right|_{\eta=0} \triangleq 1$

$$\binom{M}{N} = \frac{M!}{(M-N)!\ N!}$$

$$I1(\eta) = r_1 + \eta$$

$$J1(\eta) = m_1 + \nu + \eta$$

$$I2(\eta) = r_2 - 2\eta$$

Proof: Analogous to the proof of Theorem 15. ∎

Comparison of Eqs. (5.108) and (5.109) shows that as desired the net effect of Theorem 17 is to produce an expression for $I_3(r_1, r_2, m_1, m_2)$ without stochastic integrals and in terms of the moments of the g_s process. We have therefore achieved a procedure for evaluating γ_g -- and consequently $E_{xy}\left\{E_x(g_t\ \psi_t)\ E_x(g_t\ \psi_t)\right\}$ -- that can be easily implemented in a digital computer.

Summarizing Section 5.4, the objective has been the evaluation of the numerator N(t) of the main MSE lowerbound formula given in Eq. (5.23) of Theorem 11. The basic nature of the difficulties involved was first discussed and elementary solutions presented. More sophisticated methods involving Cauchy products of infinite series were then developed. Theorem 14 of this section is the main computational result of Chapter 5 since together with Theorem 14 and the theory of subsections (5.4.3) to (5.4.6) it allows the computation of a lowerbound on optimum MSE for the continuous time NLFP in terms of moments of the g_s process, or more precisely, in terms of integrals of moments of the g_s process. In Chapter 6 we will see that these integrals and consequently their corresponding Cauchy products can be computed exactly (i.e., with no approximation or further bounding) in the case of the phase locked loop. Before proceding to this example we briefly give in the next section the discrete time version of the bound and bring attention to the direct applicability of all the material in this chapter to non-MSE distortion measures.

5.5 The Discrete Bound and Non-MSE Distortion Measures

A lowerbound on optimum filtering MSE can now be derived for discrete time problems in fashion analogous to the foregoing continuous time bound. The similarity hinges on the fact that the Bucy-Mortensen-Duncan representation theorem -- unlike Kailath's likelihood ratio formula -- has a discrete time counterpart as given in Lemma 2 of Chapter 2 which is identical in form to the continuous version (compare Lemmas 1 and 2). In deriving the continuous time bound we used Lemma 1 of Chapter 2 and Theorem 7C of Chapter 3 to prove Theorem 10 in Chapter 5. Here we simply use Lemma 2 instead of Lemma 1 and Theorem 7 instead of Theorem 7C to prove the following Theorem.

Theorem 18. Consider the discrete time NLFP of Section 2.1. Let ε_k^* be the associated optimum MSE. Then a lowerbound on ε_k^* is given by

$$\varepsilon_k^* - \frac{1}{2\pi e}\exp\{2H(k)\}\left\{\frac{\exp E_{P_{X\cdot Y}}\log E_{P_X}(\exp \zeta_k)}{E_{P_{X\ Y}}(\exp \zeta_k)}\right\}^2 \quad (5.110)$$

where the notation is that of the NLFP and

$$\zeta_k = \zeta_k(x_j,\ y_j,\ j\varepsilon[0,k])$$

$$= \sum_{j=0}^{k} y_j'\ R^{-1}(j)\ g(x_j,j) - \tfrac{1}{2}\sum_{j=0}^{k} g(x_j,j)'R^{-1}(j)\ g(x_j,j)$$

$$= \tfrac{1}{2}\sum_{j=0}^{k} g(x_j,j)'R^{-1}(j)\ g(x_j,j) + \sum_{j=0}^{k} g(x_j,j)'R^{-1}(j)\ \tilde{B}_j \quad (5.111)$$

H(k): Entropy of x_k

provided the necessary expectations exist. (If the processes defined by the NLFP are stationary then ε_k^* is constant in k and lowerbounded by the supremum over 111 k of the right hand side of Eq. (5.110)).

<u>Proof:</u> By Lemma 4,

$$I(x_k;\ y_0^k) = E_{P_{X\cdot Y}}\log\frac{dP_{X\cdot Y}}{dP_{X\times Y}} = E_{P_{X\cdot Y}}\log\frac{dP_X^{\mathcal{Y}_k}}{dP_X}$$

where the probability measures are as in the definition of the discrete NLFP and Lemma 2 of Section 2.1 and $\mathcal{Y}_k = \sigma\{y_j, j\varepsilon[0,k]\}$. Using Lemma 2 (discrete Bucy-Mortensen-Duncan representation),

$$I(x_k;\ y_0^k) = E_{P_{X\cdot Y}} \log \left\{ \frac{E_{P_X}^{\sigma\{x_k\}}(\exp\ \zeta_k)}{E_{P_X}(\exp\ \zeta_k)} \right\}$$

$$= E_{P_{X\cdot Y}} \log E_{P_X}^{\sigma\{x_k\}}(\exp\ \zeta_k) - E_{P_{X\cdot Y}} \log E_{P_X}(\exp\ \zeta_k) \tag{5.112}$$

where ζ_k is as given by Eqs. (5.111). By Jensen's inequality (Lemma 28B) we have

$$E_{P_{X\cdot Y}} \log E_{P_X}^{\sigma\{x_k\}}(\exp\ \zeta_k) \leq \log E_{P_{X\cdot Y}} E_{P_X}^{\sigma\{x_k\}}(\exp\ \zeta_k)$$

$$= \log E_{P_{X\cdot Y}}(\exp\ \zeta_k) \tag{5.113}$$

where use was made of the smoothing property of the conditional expectation. Substituting (5.113) into (5.112) gives

$$I(x_k;\ y_0^k) \leq \log E_{P_{X\cdot Y}}(\exp\ \zeta_k) - E_{P_{X\cdot Y}} \log E_{P_X}(\exp\ \zeta_k)$$

which when substituted into Eq. (3.31) of Theorem 7 (Section 3.4) gives (5.110). The stationarity part follows from reasoning analogous to that of Zakai-Ziv presented in Section 5.1. ∎

While an evaluation of the formula of Theorem 18 could proceed along lines similar to those of the continuous bound it is not pursued in the present context since the specific application we have a mind in this study, the phase locked loop of Chapter 6, is a continuous time problem. The importance of the discrete time NLFP (and hence that of Theorem 18) cannot be overemphasized since, as is widely accepted, discrete time problems are at least as important in applications as continuous problems.

Finally we note that the entire development of this chapter in both continuous and discrete time is directly and immediately applicable to non-MSE distortion measures. In fact, the central problem in Chapter 5 has been the evaluation of $I(x_t;\ y_0^t)$ by means of an upper bound. Once this evaluation is performed it can be substituted in Eq. (3.33) of Theorem 7A in Chapter 3 to yield a lowerbound on filtering distortion for the desired (non-MSE) distortion measure. The generality of Shannon information and the fact that it is an adequate measure of "information" is -- as was emphasized in Chapter 3 -- evident in the lower bound computation context.

CHAPTER 6

THE PHASE LOCKED LOOP PROBLEM*

6.1 Preliminaries

The objective of this chapter is to apply the formulas derived in Chapter 5 to obtain a lowerbound on the steady state optimum filtering MSE associated with the phase locked loop problem (PLLP) defined below.

6.1.1 The PLLP

The importance of the PLLP in communications is well known and amply discussed in most text books in this area (e.g., Van Trees (66,42), Sakrison (17,67), Viterbi (68)). In recent research Bucy (69), Gustafson (69), Gustafson-Speyer (70) and Willsky (71) have successfully applied modern system theory to produce suboptimal filters that in Monte Carlo simulations perform significantly better than the classical loop in the important below threshold (nonlinear) region of receiver operation. The question that naturally arises is how well do these suboptimal filters perform when compared to the Kushner filter, that is, how much room for improvement is left. Since the conditional mean and associated optimal estimation error

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*The computations in this chapter were enormously expedited by the use of MACSYMA, MIT's Project MAC Symbolic Manipulation system developed by Prof. J. Moses, et.al. The kind assistance of Prof. Moses in this matter is gratefully acknowledged.

are impossible to compute, a tight lowerbound on this error sheds considerable light in the performance limitation question.

For our purposes it suffices to define the PLLP as the subclass of the NLFP of Section 2.1 with message model and observations described by

$$dx_t = dB_t \ , \quad (dB_t)^2 = dt/\tau \ , \quad t \geq 0 \tag{6.1}$$

$$dy_t = \sqrt{2A}\,\sin(\omega t + x_t + \phi) + d\tilde{B}_t, \ (d\tilde{B}_t) = \frac{N_o}{2}\,dt \tag{6.2}$$

ϕ : uniformly distributed over $[-\pi,\pi]$ and independent of B_t, $\tilde{B}_t$

where ϕ represents the effect of a uniform initial condition on x_t and A, N_o, τ, ω are positive constants. Thus, in the nomenclature of Section 2.1 used in the formulas of the previous chapters we have that for the PLLP considered here

$$a(x_t,t) = 0 \ , \quad b(x_t,t) = 1$$

$$g(x_t,t) = \sqrt{2A}\,\sin(\omega t + x_t + \phi) \tag{6.3}$$

$$Q(t) = 1/\tau \tag{6.4}$$

$$R(t) = N_o/2 \tag{6.5}$$

The problem can be naturally parameterized in terms of the normalized time t' and noise to signal ratio P^2

$$t' = \frac{t}{\tau} \tag{6.6}$$

$$p^2 = \frac{N_o}{2A\tau} \qquad (6.7)$$

Throughout this chapter we shall also use the notation

$$\theta_t = \omega t + x_t + \phi \qquad (6.8)$$

$$E_x = E_{x\phi}$$

where the last statement emphasizes that expectation over x_t should include its effective initial condition ϕ.

The basic procedure followed in this chapter is to substitute Eqs. (6.3), (6.4), (6.5) into the relevant equations of Chapter 5 and then to evaluate the resulting expressions. It is important to emphasize that the formulas from Chapter 5 that we shall use here will be evaluated exactly for the PLLP and there is no further bounding in the present chapter*.

6.1.2 Outline of the Chapter

The remainder of this chapter is organized as follows. The evaluation of the steady state MSE lowerbound for the PLLP is executed in Section 6.4 entirely "from scratch": We depart from Theorem 14 of Chapter 5 and arrive at specific equations for the desired lowerbound which are then summarized in Theorem 20. The numerical results are presented in Section 6.5 where the bound of Theorem 20 is compared for different noise to

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*A minor exception to this statement is included in Section 6.3.

signal ratios with the Zakai-Ziv bound (38) described in Section 5.1 and with the Snyder-Rhodes bound (77).

Sections 6.2 and 6.3 contain the evaluation for the PLLP of many of the formulas derived in Chapter 5. Only a few of these results are actually used in the development of Section 6.4 -- namely those that proved to give the tighter bound -- and the rest are presented merely to illustrate the relative tightness of some of the bounds as well as the straightforward nature of the computations involved for the PLLP. Consequently, most, if not all, of Sections 6.2, 6.3 may be skipped on a first reading.

6.2 Denominator Evaluation; Elementary Numerator Formulas*

The objective in this section is to obtain computable expressions for Eqs. (5.56) of Lemma 38 of Chapter 5. These expressions will enable us to compute the denominative D(t) of Eq. (5.42) of Theorem 13 as well as the E_{xy} expectations appearing in the elementary numerator formulas of Lemma 37. The desired formulas follow quickly from the next lemma which also contains all the basic techniques that will be used over and over throughout this chapter.

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*Of all the development in this section only Lemma 43 and Eq. (6.32) of Theorem 19 will be needed in Section 6.4.

Lemma 43. Let x_t, ϕ, $g_t = g(x_t, t)$ be as in Eqs. (6.1), (6.2), (6.3), (6.8) so that $g_t = \sqrt{2A}\sin\theta_t$, $\theta_t = \omega t + x_t + \phi$. Then:

A. For j, k integers and k nonzero:

$$E_x \sin j\phi = E_x \cos k\phi = 0 \tag{6.9}$$

$$E_x \sin j\theta_t = E_x \cos k\theta_t = 0 \tag{6.10}$$

B. For $0 \le s_i, r_j \le t < \infty$ for all i, j,

$$E_x \sin(\theta_{s_1} + \dots + \theta_{s_m} - \theta_{r_1} - \dots - \theta_{r_n}) = 0 \tag{6.11}$$

for all m,n

$$E_x \cos(\theta_{s_1} + \dots + \theta_{s_m} - \theta_{r_1} - \dots - \theta_{r_n}) = 0 \tag{6.12}$$

if $m \neq n$

$$E_x \cos(\theta_{s_1} + \dots + \theta_{s_m} - \theta_{s_{m+1}} - \dots - \theta_{s_{2m}}) = \cos\Omega \exp\left\{\frac{1}{2\tau} X\right\} \tag{6.13}$$

where

$$\Omega = \sum_{i=1}^{m} \omega(s_i - s_{m+i}) \tag{6.14}$$

$$X = \sum_{i=1}^{2m} a_i\, s_i \tag{6.15}$$

and the a_i are integers that depend only on the order of the s_i. In particular

$$E_x \sin(\theta_{s_1} - \theta_{s_2}) = 0 \tag{6.16}$$

$$E_x \cos m(\theta_{s_1} - \theta_{s_2}) = \cos m(\omega s_1 - \omega s_2) \exp \left\{ \frac{-m^2}{2\tau} |s_1 - s_2| \right\} \tag{6.17}$$

C. For n: even,

$$E_x g_t^n = (2A)^{n/2} \left[\frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots \quad n} \right] \tag{6.18}$$

D. For n: odd and $s_i \varepsilon [0,t]$, $1 \le i \le n$, $t < \infty$,

$$E_x(g_{s_1} g_{s_2} \cdots g_{s_n}) = 0 \tag{6.19}$$

E. For n: even and ω arbitrarily large

$$\int_{s_1=0}^{t} \int_{s_2=0}^{t} \cdots \int_{s_n=0}^{t} E_x(g_{s_1} g_{s_2} \cdots g_{s_n}) ds_1 \, ds_2 \cdots ds_n$$

$$= 0\left(\frac{1}{\omega\tau}\right) \tag{6.20}$$

F. For n a non-negative integer and ω arbitrarily large

$$E_x \left[\int_0^t g_s^2 \, ds \right]^n = \left[2A \cdot \frac{t}{2} \right]^n + 0\left(\frac{1}{\omega\tau}\right) \tag{6.21}$$

Proof:

A. Eq. (6.9) is clear since for $k \neq 0$

$$E_x \sin k\phi = \int_{-\pi}^{\pi} \frac{1}{2\pi} \sin k\phi \, d\phi = 0$$

$$E_x \cos k\phi = \int_{-\pi}^{\pi} \frac{1}{2\pi} \cos k\phi \, d\phi = 0$$

Further

$$E_x \sin k\theta_t = E_x \sin k(\omega t + x_t)\, E_x \cos k\phi + E_x \cos k(\omega t + x_t)\, E_x \sin k\phi = 0$$

where use has been made of the independence of x_t, ϕ and of Eq. (6.9) similarly

$$E_x \cos k\theta_t = 0$$

B. Consider Eq. (6.11),

$$E_x \sin(\theta_{s_1} + \ldots + \theta_{s_m} - \theta_{r_1} - \ldots - \theta_{r_n})$$

$$= E_x \sin \alpha \, E_x \cos \beta + E_x \cos \alpha \, E_x \sin \beta$$

where

$$\alpha = \sum_{x=1}^{m} (\omega s_i + x_{s_i}) - \sum_{i=1}^{n} (\omega r_i - x_{r_i})$$

$$\beta = (m-n)\,\phi \qquad (6.22)$$

Now $E_x \sin \alpha = 0$ since α is a Gaussian random variable and $E_x \sin \beta = 0$ from Part A so that (6.11) is indeed true. Consider Eq. (6.12). If $m \neq n$,

$$E_x \cos(\theta_{s_1} + \ldots + \theta_{s_m} - \theta_{r_1} - \ldots - \theta_{r_n})$$

$$= E_x \cos \alpha \, E_x \cos \beta - E_x \sin \alpha \, E_x \sin \beta$$

where α and β are as in (6.22). Eq. (6.12) now follows by noting from Part A that $E_x \cos \beta = E_x \sin \beta = 0$ since $m \neq n$. Consider Eq. (6.13).

$$E_x \cos(\theta_{s_1} + \ldots + \theta_{s_m} - \theta_{s_{m+1}} - \ldots - \theta_{s_{2m}})$$

$$= \cos \Omega \, E_x \cos \beta - \sin \Omega \, E_x \sin \beta \qquad (6.23)$$

$$= \cos \Omega \, E_x \cos \beta$$

where now $\beta = x_{s_1} + \ldots + x_{s_m} - x_{s_{m+1}} - \ldots - x_{s_{2m}}$ and the fact that β is a Gaussian rv has been used. Let β have variance σ^2. Since

$$E_x \cos \beta = \sum_{n=0}^{\infty} (-1)^n \frac{E_x \beta^{2n}}{(2n)!}$$

and $E_x \beta^{2n} = 1 \cdot 3 \cdot 5 \ldots (2n-1)\sigma^{2n}$, we have

$$E_x \cos \beta = \exp \left\{ \frac{-1}{2} \sigma^2 \right\} \qquad (6.24)$$

Now we determine σ^2. Clearly

$$\sigma^2 = [1]' \left[\frac{1}{\tau} \min (s_i, s_j)\right] [1] \qquad (6.25)$$

where [1] is the vector formed by m ones followed by m negative ones and $E_x x_{s_i} x_{s_j} = \min (s_i, s_j)/\tau$ $i,j = 1, \ldots, 2m$ are the entries of the 2m x 2m matrix. Thus from Eqs. (6.23), (6.24), (6.25) it is clear that for any given $s_1, \ldots, s_{2m}$ we can find integers a_i, $i = 1, \ldots, 2m$ such that (6.13) holds. Finally (6.16) follows from (6.11) and (6.17) from Eqs. (6.23), (6.24), (6.25).

C. For n even the expansion of $\sin^n \alpha$ for any α is

$$\sin^n \alpha = W_n + \sum_{k=1}^{\infty} a_k \cos k\alpha$$

where a_k are the Fourier coefficients and

$$W_n = \frac{1}{2\pi} \int_0^{2\pi} \sin^n \alpha \, d\alpha = \frac{1 \cdot 3 \cdot 5 \ldots (n-1)}{2 \cdot 4 \cdot 6 \ldots n} \qquad (6.26)$$

Hence using Part A,

$$E_x g_t^n = (2A)^{n/2} \left[W_n + \sum_{k=1}^{\infty} a_k E_x \cos k\theta_t\right]$$

$$= (2A)^{n/2} W_n$$

as desired.

D. Consider Eq. (6.19),

$$E_x(g_{s_1} \ldots g_{s_n}) = (2A)^{n/2}\, E_x(\sin \theta_{s_1} \ldots \sin \theta_{s_n})$$

For n odd the sines in the above expression can be combined using repeatedly the trigonometric identity $\sin \alpha \sin \beta = (\cos(\alpha-\beta) - \cos(\alpha+\beta))/2$ into a sum of terms of the form

$$E_x \cos\left(\sum_{i=1}^{n} (a_i\, \omega s_i + b_i\, x_{s_i}) + k\phi\right) \tag{6.27}$$

where the a_i, b_i are either +1 or -1. Since $k \neq 0$ because n is odd all these terms vanish and Eq. (6.19) is obtained.

E. Consider Eq. (6.20). By an argument similar to that of Part D we conclude that if the integrand in Eq. (6.20) is expanded using repeatedly the identity $\sin \alpha \sin \beta = (\cos(\alpha-\beta) - \cos(\alpha+\beta))/2$, only terms of the form

$$E_x \cos\left(\theta_{s_{i_1}} + \ldots + \theta_{s_{i_{n/2}}} - \theta_{s_{i_{\frac{n}{2}+1}}} - \ldots - \theta_{s_{i_n}}\right) \tag{6.28}$$

(where $i_1, \ldots, i_n$ is a reordering of $1, \ldots, n$) will not vanish since only such terms will give a $k=0$ in Eq. (6.27). From Eqs. (6.13), (6.14), (6.15) of Part B the integral of Eq. (6.28) is of the form (replacing $i_1, \ldots, i_n$ with $1, \ldots, n$ for simplicity of notation)

$$\int_{s_1=0}^{t} \ldots \int_{s_n=\theta}^{t} \cos(\omega s_1 + \ldots \omega s_{n/2} - \omega s_{\frac{n}{2}+1} - \ldots - \omega s_n) \exp\left\{\frac{1}{2\tau}\sum_{i=1}^{n} a_i s_i\right\} ds_1 \ldots ds_n \tag{6.29}$$

where the a_i are integers function of the s_i. Taking ω arbitrarily large we can invoke the Riemann-Lebesgue lemma (58) to conclude that (6.29) and therefore (6.20) are indeed $0(\frac{1}{\omega\tau})$.

F. Consider Eq. (6.21).

$$E_x\left[\int_0^t g_s^2\, ds\right]^n = \int_{s_1=0}^{t} \cdots \int_{s_n=0}^{t} (2A)^n\, E_x(\sin^2\theta_{s_1} \ldots \sin^2\theta_{s_n})\, ds_1 \ldots ds_n$$

$$= (2A)^n \int_{s_1=0}^{t} \cdots \int_{s_n=0}^{t} \frac{1}{2^n} E_x\left\{(1 - \cos 2\theta_{s_1}) \ldots (1 - \cos 2\theta_{s_n})\right\} ds_1 \ldots ds_n$$

$$= (2A)^n \frac{t^n}{2^n} + 0(\frac{1}{\omega\tau})$$

where the last expression follows by arguments analogous to those of Part E. ∎

We note that in Parts E and F of the previous lemma what enables us to obtain simple expressions like Eqs. (6.20), (6.21) is the fact that we can choose ω as large as we please so that all ω dependent terms can be ignored. This will also be the case in all equations to follow in this chapter. It is important to emphasize that there is no approximation whatever involved in this assumption since the optimal processor does not depend on the carrier frequency ω.

The computable expressions for Eqs. (5.56) of Lemma 38 are given in the following theorem.

Theorem 19. Consider $E_{xy}\, \psi_t^m$, $E_{xy}\, g_t^k\, \psi_t^m$ with k even and m an arbitrary integer for the PLLP of Eqs. (6.1) through (6.5) where as usual $\psi_t = \exp L_t$ and L_t is as in Eq. (5.18). Then:

$$E_{xy}\, \psi^m_{t'} = \sum_{r=0,2,4,\ldots}^{\infty} \quad \sum_{n=r}^{\infty} \frac{m^n}{(n-r)!(r/2)!\; 2^{n-r/2}} \left[\frac{t'}{P^2}\right]^{n-r/2} \tag{6.30}$$

$$E_{xy}\, g^k_{t'}\, \psi^m_{t'} = (2A)^{k/2} \frac{1 \cdot 3 \cdot 5 \cdots (k-1)}{2 \cdot 4 \cdot 6 \cdots k} \sum_{r=0,2,4\ldots}^{\infty} \quad \sum_{n=r}^{\infty} \frac{m^n}{(n-r)!(r/2)!\; 2^{n-r/2}} \left[\frac{t'}{P^2}\right]^{n-r/2} \tag{6.31}$$

so that

$$D(t') = \sum_{r=0,2,4,\ldots}^{\infty} \quad \sum_{n=r}^{\infty} \frac{1}{(n-r)!(r/2)!\; 2^{n-r/2}} \left[\frac{t'}{P^2}\right]^{n-r/2} \tag{6/32}$$

where the normalized time t' and the noise to signal ratio P^2 are as in Eqs. (6.6), (6.7)*.

<u>Proof</u>: Consider first Eq. (6.30). Substituting Eq. (6.21) of Lemma 43F in the typical term of the sum in Eq. (5.56a) of Lemma 38 we get

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*Observe from Eqs. (6.30), (6.31), and Lemma 43C that $E_{xy}\, g^k_{t'}\, \psi^m_{t'} = E_{xy}\, g^k_{t'} \cdot E_{xy}\, \psi^m_{t'}$. This is reasonable since as ω becomes large we would expect, in view of the definition of ψ_t, $g^k_{t'}$ and $\psi^m_{t'}$ to become uncorrelated as indeed they are.

$$\frac{m^n}{(n-r)!(r/2)!\ 2^{n-r/2}} E_{xy} \left\{ \left[\int_0^t g_s^2 R^{-1}\, ds \right]^{n-r/2} \right\}$$

$$= \frac{m^n}{(n-r)!(r/2)!\ 2^{n-r/2}} \frac{2^{n-r/2}}{N_0^{n-r/2}} \left[2A \frac{t}{2} \frac{\tau}{\tau} \right]^{n-r/2}$$

$$= \frac{m^n}{(n-r)!(r/2)!\ 2^{n-r/2}} \left[\frac{t'}{p^2} \right]^{n-r/2}$$

as desired.

Consider now Eq. (6.31). From the typical term in the sum of Eq. (5.56e), Lemma 38, with $R = N_0/2$

$$\frac{m^n}{(n-r)!(r/2)!\ 2^{n-r/2}} E_{xy} \left\{ g_t^k \left[\int_0^t g_s^2 R^{-1}\, ds \right]^{n-r/2} \right\}$$

$$= \frac{m^n}{(n-r)!(r/2)!\ 2^{n-r/2}} \left(\frac{2}{N_0} \right)^{n-r/2} \int_{s_1=0}^{t} \cdots \int_{s_{n-r/2}=0}^{t} E_{xy}\left(g_t^k\, g_{s_1}^2 \cdots g_{s_{n-r/2}}^2 \right) ds_1 \cdots ds_{n-r/2} \tag{6.33}$$

As in the proof of Lemma 43C we know that for k even

$$g_t^k = (2A)^{k/2} \left[W_k + \sum a_k \cos k\theta_t \right]$$

where as before a_k are the Fourier coefficients in the expansion of $\sin^k \alpha$, $\theta_t = \omega t + x_t + \phi$, and W_k is as given in Eq. (6.26). Thus,

$$E_{xy}\left(g_t^k\, g_{s_1}^2 \cdots g_{s_{n-r/2}}^2\right) = (2A)^{k/2 + n-r/2}\, E_{xy}\left\{\left(W_k + \sum_{k=1}^{\infty} a_k \cos k\theta_t\right)\right.$$

$$\left. \cdot \frac{1}{2^{n-n/2}} (1 - \cos 2\theta_{s_1}) \ldots (1 - \cos 2\theta_{s_{n-r/2}})\right\}$$

$$= \frac{(2A)^{k/2 + n-r/2}}{2^{n-r/2}}\, E_{xy}\left\{W_k + a_k \cos k\theta_t + f(\omega s_1, \ldots, \omega s_{n-r/2})\right\}$$

$$= \frac{(2A)^{k/2 + n-r/2}}{2^{n-r/2}}\left\{W_k + E_{xy}\, f(\omega s_1, \ldots, \omega s_{n-r/2})\right\} \quad (6.34)$$

where Lemma 43A has been used and f is a sum of a finite number of terms all of which depend non-trivially on $\omega s_1, \ldots, \omega s_{n-r/2}$ and all of which are of the form of Eqs. (6.11) to (6.15) of Lemma 43B. Upon substituting Eq. (6.34) in the integral of Eq. (6.33) we see that by reasoning as in Lemma 43E and F the integral of E_{xy} f is $O(\frac{1}{\omega\tau})$ so that Eq. (6.31) is indeed true. □

Summarizing, in this section we have obtained in Theorem 19 exact expressions for the denominator D(t) as well as for the E_{xy} expectations appearing in the elementary numerator formulas of Lemmas 37 and 38. In the process of obtaining these results we have presented in Lemma 43 the basic concepts that are to be used in the numerator evaluation of the following section.

6.3 Numerator Evaluation*

From Theorem 14 of Chapter 5 it is evident that in order to evaluate the numerator N(t) we need to compute

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*Only the development of subsection 6.3.1 will be needed in Section 6.4.

$$E_{xy}\ \{E_x(g_t\ \psi_t)\ E_x(g_t\ \psi_t)\}$$

$$E_{xy}\ \{E_x(g_t^2\ \psi_t)\ E_x(\psi_t)\}$$

$$E_{xy}\ \{E_x(g_t\ \psi_t)\ E_x(g_t\ \psi_t)\ E_x(\psi_t^{-2})\}$$

These three expectations are considered respectively in the following three subsections.

6.3.1 $I_3(.,.,.,.)$ of Theorem 17

In subsection 5.4.6 a procedure for evaluating $E_{xy}\ \{E_x(g_t\ \psi_t)\ E_x(g_t\ \psi_t)\}$ was given in terms of the expression for $I_3(r_1,\ r_2,\ m_1,\ m_2)$ given in Eq. (5.109) of Theorem 17. The objective of the present subsection is to produce compatible formulas for $I_3(r_1,\ r_2,\ m_1,\ m_2)$ for pertinent r_1, r_2, m_1, m_2. The derivation can be divided into three steps.

Step 1. The first step is contained in the following lemma.

Lemma 44. Consider $I_3(r_1,\ r_2,\ m_1,\ m_2)$, defined in Eq. (5.108), and given by Eq. (5.109) of Theorem 17 where without loss of generality we take r_1, m_1 to be arbitrary non-negative integers; r_2, m_2 even integers; and $r_2 \le m_2$. Then for the PLLP $I_3(.,\ .,\ .,\ .)$ reduces to

$$R^{-(r_1+r_2+m_1+m_2)}\, I_3(r_1,\, r_2,\, m_1,\, m_2)$$

$$= (2A)\left[\binom{m_2}{2v}(2v-1)(2v-3)\ldots 5\,.\,3\,.\,1\right]2^{r_2}\ \frac{(t')^{r_1+m_1+\frac{m_2-r_2}{2}}}{(P^2)^{r_1+m_1+\frac{r_2}{2}+\frac{m_2}{2}}}$$

$$\sum_{\eta=0}^{(r_2-1)/2}\left\{\left[\binom{r_2}{2\eta}(2\eta-1)(2\eta-3)\ldots 5\,.\,3\,.\,1\right]^2 (r_2-2\eta)!\right.$$

$$\left.\left[\frac{t'}{2}\right]^{2\eta}\int_{s_1'=0}^{t'}\cdots\int_{s_{r_2-2\eta}'=0}^{t'}\left[E_x\left\{\sin\theta_{t'}\,\sin\theta_{s_1'}\ldots\sin\theta_{s_{r_2-2\eta}'}\right\}\right]^2 ds_1'\ldots ds_{r_2-2\eta}'\right\}$$

(6.35)

where the $R^{-(\ )}$ factor multiplying the I_3 arises naturally from the R^{-1} omitted from each of the integrals in Eq. (5.108) (see Eq. (5.71)), $v = (m_2-r_2)/2$; P^2, s_i', t' are the noise to signal ratio and normalized times as in Eqs. (6.6), (6.7) and $\theta_{s'}$ is the normalized version of θ_s in Eq. (6.8).

<u>Proof</u>: Substituting $g_s = \sqrt{2A}\sin\theta_s$ in the integral in Eq. (5.109), normalizing time and using the identity $\sin^2\alpha = (1-\cos 2\alpha)/2$, we get

$$\frac{(2A)(2A\tau)^{I1+J1+I2}}{2^{I1+J1}}\int_{\tilde{s}_1'=0}^{t'}\cdots\int_{\tilde{s}_{I1}'=0}^{t'}\int_{\tilde{\sigma}_1'=0}^{t'}\cdots\int_{\tilde{\sigma}_{J1}'=0}^{t'}\int_{s_1'=0}^{t'}\cdots\int_{s_{I2}'=0}^{t'}$$

$$E_x\left\{(1-\cos 2\theta_{\tilde{s}_1'})\ldots(1-\cos 2\theta_{\tilde{s}_{I1}'})\sin\theta_{t'}\sin\theta_{s_1'}\ldots\sin\theta_{s_{I2}'}\right\}$$

$$E_x\left\{(1-\cos 2\theta_{\tilde{\sigma}_1'})\ldots(1-\cos 2\theta_{\tilde{\sigma}_{J1}'})\sin\theta_{t'}\sin\theta_{s_1'}\ldots\sin\theta_{s_{I2}'}\right\}$$

$$d\tilde{s}_1'\ldots d\tilde{s}_{I1}'\, d\tilde{\sigma}_1'\ldots d\tilde{\sigma}_{J1}\, ds_1'\ldots ds_{I2}'$$

(6.36)

where as before $I1(\eta) = r_1 + \eta$, $J1(\eta) = m_1 + v + \eta$, $I2(\eta) = r_2 - 2\eta$.

When the products in the integrand of Eq. (6.36) are multiplied, we can eliminate by arguments analogous to those in Lemma 43E, F all terms that are not of zero frequency so that

$$\text{Eq. (6.36)} = \frac{(2\Lambda)(2\Lambda\tau)^{I1+J1+I2}}{2^{I1+J1}} \int_{\tilde{s}_1'=0}^{t'} \cdots \int_{\tilde{s}_{I1}'=0}^{t'} \int_{\tilde{\sigma}_1'=0}^{t'} \cdots \int_{\tilde{\sigma}_{J1}'=0}^{t'} \int_{s_1'=0}^{t'} \cdots \int_{s_{I2}'=0}^{t'}$$

$$\left[E_x(\sin\theta_{t'}\sin\theta_{s_1'}\cdots\sin\theta_{s_{I2}'})\right]^2 d\tilde{s}_1' \cdots\cdots\cdots ds_{I2}'$$

$$= \frac{(2\Lambda)(2\Lambda\tau)^{I1+J1+I2}}{2^{I1+J1}} (t')^{I1+J1} \int_{s_1'=0}^{t'} \cdots \int_{s_{I2}'=0}^{t'}$$

$$\left[E_x(\sin\theta_{t'}\sin\theta_{s_1'}\cdots\sin\theta_{s_{I2}'})\right]^2 ds_1' \cdots ds_{I2}' \tag{6.37}$$

Eq. (6.36) can now be obtained by substituting Eq. (6.37) into Eq. (5.109), using the definition of P^2 and regrouping constants. ∎

<u>Step 2</u>. The net effect of Lemma 44 and Step 1 has been to reduce the entire computation of $E_{xy}\{E_x(g_t\psi_t)E_x(g_t\psi_t)\}$ to the evaluation of

$$\tilde{I}_3(j_2) = \int_{s_1=0}^{t} \cdots \int_{s_{j_2}=0}^{t} \left[E_x(\sin\theta_t \sin\theta_{s_1} \cdots \sin\theta_{s_{j_2}})\right]^2 ds_1 \cdots ds_{j_2}$$

$$j_2 = 1, 3, 5, \ldots \tag{6.38}$$

where to avoid further clutter in the notation the facts that we are dealing with normalized times (e.g., s_i') and that $\tilde{I}_3$ is also a function of t′ are not explicit. The reason that $\tilde{I}_3$ does not vanish from Eq. (6.35) is that as we shall see it possesses a zero frequency component.

Exact evaluation of $\tilde{I}_3$ is straightforward though tedious. Rather than produce general formulas (which is perfectly possible) we shall illustrate the evaluation process in Steps 2 and 3 by considering the case $j_2=3$ in Eq. (6.38).

The particular objective of Step 2 is to reduce Eq. (6.38) to "simple integrals" -- that is, to integrals that can be found in the table of integrals of just about any engineering or scientific handbook (e.g. (73), (74), (75)).

Consider then $\tilde{I}_3(j_2=3)$. As most of the development of this chapter, the following relies heavily on the material included in the statement and proof of Lemma 43. From Eq. (6.38),

$$\tilde{I}_3(3) = \int_{s_1=0}^{t} \int_{s_2=0}^{t} \int_{s_3=0}^{t} \left[E_x(\sin\theta_t \sin\theta_{s_1} \sin\theta_{s_2} \sin\theta_{s_3})\right]^2 ds_1\, ds_2\, ds_3 \tag{6.39}$$

Now making use of trigonometric identities such as

$$\sin\alpha \sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$$

and of Lemma 43 we can show that

$$E_x(\sin\theta_t \sin\theta_{s_1} \sin\theta_{s_2} \sin\theta_{s_3})$$

$$= \frac{1}{8} E_x \Big\{ \cos(\theta_t - \theta_{s_1} - \theta_{s_2} + \theta_{s_3}) + \cos(\theta_t - \theta_{s_1} + \theta_{s_2} - \theta_{s_3})$$

$$+ \cos(\theta_t + \theta_{s_1} - \theta_{s_2} - \theta_{s_3}) \Big\} \tag{6.40}$$

Substituting Eq. (6.40) in (6.38) we see that, up to $O(\frac{1}{\omega\tau})$,

$$\tilde{I}_3(3) = \frac{1}{64} \int\int\int \Big[E_x \cos(\theta_t - \theta_{s_1} - \theta_{s_2} + \theta_{s_3})$$

$$+ E_x \cos(\theta_t - \theta_{s_1} + \theta_{s_2} - \theta_{s_3})$$

$$+ E_x \cos(\theta_t + \theta_{s_1} - \theta_{s_2} - \theta_{s_3}) \Big]^2 ds_1\, ds_2\, ds_3$$

$$= \frac{1}{64} \int\int\int \Big\{ \Big[E_x \cos(\theta_t - \theta_{s_1} - \theta_{s_2} + \theta_{s_3}) \Big]^2$$

$$+ \Big[E_x \cos(\theta_t - \theta_{s_1} + \theta_{s_2} - \theta_{s_3}) \Big]^2$$

$$+ \Big[E_x \cos(\theta_t + \theta_{s_1} - \theta_{s_2} - \theta_{s_3}) \Big]^2 \Big\} ds_1\, ds_2\, ds_3$$

$$= \frac{3}{64} \int_{s_1=0}^{t} \int_{s_2=0}^{t} \int_{s_3=0}^{t} \Big[E_x \cos(\theta_t + \theta_{s_1} - \theta_{s_2} - \theta_{s_3}) \Big]^2 ds_1\, ds_2\, ds_3$$

$$= \frac{3}{128} \int_{s_1=0}^{t} \int_{s_2=0}^{t} \int_{s_3=0}^{t} \Big[E_x \cos(x_t + x_{s_1} - x_{s_2} - x_{s_3}) \Big]^2 ds_1\, ds_2\, ds_3 \tag{6.41}$$

Parenthetically we mention that we can show that Eq. (6.41) generalizes to

$$\tilde{I}_3(j_2) = \frac{1}{2^{2(j_2+1)}} (j_2+1)^{C} \frac{j_2+1}{2} \int_{s_1=0}^{t} \cdots \int_{s_{j_2}=0}^{t} \left[E \cos\left(x_t + x_{s_1} + \ldots + x_{s_{\frac{j_2-1}{2}}} - x_{s_{\frac{j_2+1}{2}}} - \ldots - x_{s_{j_2}} \right) \right]^2 ds_1 \ldots ds_{j_2} \tag{6.42}$$

We can write an expression for the integrand of Eq. (6.41) provided there is a definite order between the t, s_1, s_2, s_3. Thus the integral in Eq. (6.41) can be broken up into 3! = 6 integrals since R^3 can be partitioned, up to a set of Lebesque measure zero, into 3! regions where a definite order exists between the variables. Fortunately most of the resulting integrals will turn out to be identical so that we need only evaluate 2 integrals rather than 3!. For j_2=5, 5! = 120 regions result but we need only evaluate less than 5 different integrals. For j_2=7, 7! = 5040 regions result but we need only evaluate less than 14 different integrals.

Naturally it is necessary to determine beforehand the different types of integrals (rather than to compute j_2! integrals and then see which are the same). This can be accomplished by the following two methods.

The first method is based on observing that

$$\tilde{I}_3(3) = \sum_{\substack{i_1, i_2, i_3 \in \{1,2,3\} \\ i_1 \neq i_2 \neq i_3}} \int_{s_{j_3}=0}^{t} \int_{s_{j_2}=0}^{t} \int_{s_{j_1}=0}^{t} \left[E \cos\left(x_{s_{j_1}} + x_{s_{j_2}} - x_{s_{j_3}} - x_t \right) \right]^2 ds_{j_1}\, ds_{j_2}\, ds_{j_3} \tag{6.43}$$

where the sum is over all possible reorderings of 1, 2, 3. Two types of identical terms arise: Terms of the form

$$\pm \left(x_{s_{j_1}} - x_{s_{j_2}}\right) \pm \left(x_{s_{j_3}} - x_t\right) \tag{6.44}$$

and terms of the form

$$\pm \left(x_{s_{j_1}} + x_{s_{j_2}}\right) \mp \left(x_{s_{j_3}} + x_t\right) \tag{6.45}$$

where it is assumed that $s_{j_1} < s_{j_2} < s_{j_3} < t$

Terms of the form of (6.44) can be selected in

$$2 \cdot 1 \cdot 1 + 1 \cdot 2 \cdot 1 = 4$$

ways which yield integrals which are identical up to a change of variables. Similarly terms of the form (6.45) can be chosen in

$$2 \cdot 1 \cdot 1$$

ways. Thus Eq. (6.43) becomes

$$\begin{aligned} \tilde{I}_3(3) = 4 \int_{s_3=0}^{t} \int_{s_2=0}^{s_3} \int_{s_1=0}^{s_2} \left[E \cos\left(x_{s_1} - x_{s_2} - x_{s_3} + x_t\right)\right]^2 ds_1\, ds_2\, ds_3 \\ + 2 \int_{s_3=0}^{t} \int_{s_2=0}^{s_3} \int_{s_1=0}^{s_2} \left[E \cos\left(x_{s_1} + x_{s_2} - x_{s_3} - x_t\right)\right]^2 ds_1\, ds_2\, ds_3 \end{aligned} \tag{6.46}$$

Naturally 4 + 2 = 3!. Now for $s_1 < s_2 < s_3 < t$,

$$E \cos(x_{s_1} - x_{s_2} - x_{s_3} + x_t) = \exp\left\{\frac{-1}{2}(-s_1 + s_2 - s_3 + t)\right\}$$

$$E \cos(x_{s_1} + x_{s_2} - x_{s_3} - x_t) = \exp\left\{\frac{-1}{2}(-s_1 - 3s_2 + 3s_3 + t)\right\}$$

Substituting these results into Eq. (6.46) we have

$$\tilde{I}_3(3) = 4 \int_{s_3=0}^{t} \int_{s_2=0}^{s_3} \int_{s_1=0}^{s_2} \exp(s_1 - s_2 + s_3 - t)\, ds_1\, ds_2\, ds_3$$

$$+ 2 \int_{s_3=0}^{t} \int_{s_2=0}^{s_3} \int_{s_1=0}^{s_2} \exp(s_1 + 3s_2 - 3s_3 - t)\, ds_1\, ds_2\, ds_3 \qquad (6.47)$$

Clearly the integrals in Eq. (6.47) can indeed be evaluated with the aid of an ordinary table of integrals.

A second and perhaps more convenient and reliable way of arriving at Eq. (6.47) is to have a digital computer go through the $j_2!$ possible orders of $s_1, \ldots, s_{j_2}$ checking which yield identical integrands. Such a program, producing as output the coefficient of each integral (i.e., number of repetitions of each integral) as well as the parenthesis of the exponential in each integrand, was satisfactorily implemented in this study.

Finally we note that if necessary $\tilde{I}_3(3)$ can be upper bounded by

$$\tilde{I}_3(3) \leq 3! \int_{s_3=0}^{t} \int_{s_2=0}^{s_3} \int_{s_1=0}^{s_2} \exp(s_1 - s_2 + s_3 - t)\, ds_1\, ds_2\, ds_3 \tag{6.48}$$

and

$$\tilde{I}_3(3) \leq t^3 \tag{6.49}$$

as is evident from Eqs. (6.47) and (6.39) respectively. Generalizing, $\tilde{I}_3(j_2)$ of Eq. (6.38) can be similarly upper bounded by

$$\tilde{I}_3(j_2) \leq j_2! \int_{s_{j_2}=0}^{t} \int_{s_{j_2-1}=0}^{s_{j_2}} \int_{s_1=0}^{s_2} \exp(s_1 - s_2 + s_3 - \ldots + s_{j_2} - t)\, ds_1 \ldots ds_{j_2} \tag{6.50}$$

$$\tilde{I}_3(j_2) \leq t^{j_2} \tag{6.51}$$

These bounds were useful in evaluating the "tail" of the series where negligible numerical improvement is obtained by evaluating $\tilde{I}_3(j_2)$ exactly.

For the PLLP it was necessary to obtain exact expressions for $\tilde{I}_3(j_2)$ for $j_2 = 1,3,5,7$. For $j_2 \geq 9$ a combination of the bounds of Eqs. (6.50), (6.51) was used (Eq. (6.50) for $j_2 = 9, \ldots, 21$ and Eq. (6.51) for $j_2 = 23, \ldots, 51$). The presence or absence of all the $j_2 \geq 9$ "tail" terms made a difference only in the fifth significant digit (or fourth on rare occasions).

Step 3. Once an expression of the form of Eq. (6.47) is obtained, what remains to be done is the evaluation of the resulting integrals. We note that as evident from Lemma 43B the integrals are always of the form

$$\int_{s_{j_2}=0}^{t} \int_{s_{j_2-1}}^{s_{j_2}} \cdots \int_{s_1=0}^{s_2} \exp \left\{ -t + \sum_{i=1}^{j_2} a_i \, s_i \right\} ds_1 \ldots ds_{j_2}$$

where the a_i are integers. While the necessary integrations could be performed by anybody with a knowledge of elementary integral calculus, more efficient and reliable ways can be used. Observe that numerical integration is not an attractive alternative* since (1) the integration is in j_2 - dimensional Euclidean space and (2) the integrands are quite simple. The approach that seems ideally suited to this problem is the use of a symbolic (non-numerical) integration program. The particular integral used in this study was MACSYMA developed by Prof. J. Moses et al at MIT's Project MAC. Thus for $j_2 = 3$, Eq. (6.47) evaluates to

$$\tilde{I}_3(3) = 4\left\{t'e^{-t'} + 2e^{-t'} + t'-2\right\}$$

$$+ 2\left\{-t'e^{-t'}/3 - 2e^{-t'}/9 - e^{-4t'}/36 + 1/4\right\}$$

*Attention has been given in the literature to the evaluation of multiple integrals. See for example (76).

As is evident, all the $\tilde{I}_3(j_2)$ turn out to be, as is reasonable to expect, linear combinations of $t^m e^{-nt}$ for m, n non-negative integers.

Summarizing, what has been accomplished in this subsection is to reduce the computation of the typical term in the Canchy product expression for $E_{xy} \{E_x(g_t\psi_t) E_x(g_t\psi_t)\}$ as given in Eq. (5.109) of Theorem 17 to the computation of elementary integrals.

6.3.2 $I_1(., ., ., .)$ of Theorem 15

In subsection 5.4.4 a procedure for evaluating $E_{xy} \{E_x(g_s^2\psi_s) E_x(\psi_s)\}$ was given in terms of the expression for $I_1(r_1, r_2, m_1, m_2)$ given in Eq. (5.88) of Theorem 15. The objective of this subsection is to produce computable formulas for $I_1(r_1, r_2, m_1, m_2)$ for pertinent r_1, r_2, m_1, m_2. Since this development is not needed in the bound evaluation of Section 6.4 -- only the results of subsection 6.3.1 are necessary -- this subsection is continued in Appendix B to Chapter 6 with the sole purpose of illustrating the ease of evaluation of the formulas of Chapter 5.

6.3.3 $I_2(., ., ., ., ., .)$ of Theorem 16

In subsection 5.4.5 a procedure for evaluating $E_{xy} \{(E_xg_t\psi_t)(E_xg_t\psi_t)(E_x\psi_t^{-2})\}$ was given in terms of the expression for $I_2(i_1, i_2, j_1, j_2, k_1, k_2)$ given in Eq. (5.97) of Theorem 16. The objective of this subsection is to produce computable formulas for $I_2(i_1, i_2, j_1, j_2, k_1, k_2)$ for pertinent $i_1, i_2, j_1, j_2, k_1, k_2$. Since this development is not needed in the bound evaluation of Section 6.4 -- only the results of subsection 6.3.1 are necessary -- this subsection is continued in Appendix C to Chapter 6 with the sole purpose of illustrating the ease of evaluation of the formulas of Chapter 5.

6.4 Bound Evaluation for the PLLP

In this section we derive formulas for the steady state filtering MSE lowerbound almost "from scratch": We begin with Theorem 14B and, using a few of the results of Sections 6.2, 6.3 arrive at the desired formulas which we summarize in Theorem 20. Since we are interested in the steady state error we can in effect consider Eqs. (6.1), (6.2) in the infinite time interval $t \varepsilon (-\infty, \infty)$ so that prior to any time t we have an infinite observation interval. We can then use the Zakai-Ziv reasoning outlined in Section 5.1 (i.e., defining errors ε_A, ε_B, ε_C, etc.) to allow us to take the supremium over t > 0 of Eq. (5.62) of Theorem 14 so that a lower bound on the steady state error ε^*_{ss} is given by

$$\varepsilon^*_{ss} \geq \sup_{t'>0} \left\{ t' \, \frac{\exp\{2 \, N_{LB}\,(t')\}}{D(t')^2} \right\} \qquad (6.52)$$

where we have made use of the fact that since x_t is a Gaussian source,

$$\frac{1}{2\pi e} \exp[2H(t)] = t'$$

t' is the normalized time (see Eq. (6.6)); $D'(t)$ is the normalized version of Eq. (5.63); and $N_{LB}(t')$ is the normalized version of Eq. (5.67) <u>provided</u> the hypothesis of Part B of Theorem 14 is valid -- that is provided Eqs. (5.59) and (5.60) hold for the PLLP. Verifying these hypothesis is therefore the first order of business.

In regards to Eq. (5.60) we simply observe that from the convex Jensen inequality (Lemma 28A) we surely have for the PLLP

$$\begin{aligned} E_x\psi_t &\equiv E_x \exp\left\{\frac{1}{2}\int_0^t g_s^2 R^{-1}\, ds + \int_0^t g_s R^{-1}\, d\tilde{B}_s\right\} \\ &= E_x \exp\left\{\frac{1}{P^2}\int_0^{t'} \sin^2\theta_{s'}\, ds' + \frac{4A}{N_o}\int_0^t \sin\theta_s\, d\tilde{B}_s\right\} \\ &\geq \exp\left\{\frac{t'}{2P^2}\right\} \end{aligned} \quad (6.53)$$

where Lemma 43A and the nomenclature of Eqs. (6.3) to (6.8) has been used. Thus Eq. (5.60) holds with

$$\Xi(t) = \exp\left\{\frac{t}{2P^2}\right\} \quad (6.54)$$

Equation (5.59) is clear from an intuitive point of view since upon substituting in the expression for v(t) the identity $\sin^2 \alpha = (1 - \cos 2\alpha)/2$ we have

$$v(t) = E_{xy} \int_0^t \frac{E_x(g_s^2 \psi_s)}{E_x(\psi_s)} R^{-1} \, ds$$

$$= \frac{A}{R} E_{xy} \int_0^t \left\{ 1 - E_x \left[\cos 2(\omega s + x_s + \phi) \exp \{\zeta_s\} \right] \right\} ds$$

and choosing ω arbitrarily large, the second term in the integrand becomes $0(\frac{1}{\omega\tau})$.

Lemma 45. For the PLLP Eq. (5.59) holds. That is,

$$v(t) = E_{xy} \int_0^t \frac{E_x(g_s^2 \psi_s)}{E_x \psi_s} R^{-1} \, ds = \frac{A}{R} t \tag{6.55}$$

so that

$$v(t) = \frac{t'}{p^2} \tag{6.56}$$

where again the nomenclature of Eqs. (6.3) to (6.8) is used.

Proof: Clearly (6.55) implies (6.56). That (6.55) is true is shown in Appendix A. ∎

Having verified the hypothesis of Theorem 14B we can use Eq. (5.67) for $N_{LB}(t')$ appearing in Eq. (6.52) so that using the usual normalized terminology,

$$N_{LB}(t') = \frac{1}{P^2}\left\{t' - \frac{1}{2A}\int_0^{t'} \frac{E_{xy}\left\{E_x(g_{s'}\psi_{s'})\ E_x(g_{s'}\psi_{s'})\right\}}{\exp\left\{s'/P^2\right\}}\, ds'\right\}$$

(6.57)

where we have made use of Eq. (6.54). We recall that $E_{xy}\ \{E_x(g_s\psi_s)\ E_x(g_s\psi_s)\}$ can be expressed, by using Eqs. (5.84), (5.77), (5.72) in terms of $I_3(.,\ .,\ .,\ .)$ of Eq. (6.35) which in turn can be expressed in terms of $\tilde{I}_3(\cdot)$ which can be evaluated by the procedure of subsection 6.3.1. We can now summarize the previous development in the following theorem.

Theorem 20. Consider the PLLP. A lowerbound on the optimum steady state filtering MSE is given by the following equations where the nomenclature of Eqs. (6.3) to (6.8) is used. From Eq. (6.52):

$$\varepsilon^*_{ss} \geq \sup_{t'>0}\ t'\ \frac{\exp\{2\ N_{LB}\ (t')\}}{D(t')^2} \tag{6.58}$$

From Eq. (6.30):

$$D(t') = \sum_{r=0,2,4...}^{\infty}\ \sum_{n=r}^{\infty} \frac{1}{(n-r)!(r/2)!\ 2^{n-r/2}} \left[\frac{t'}{P^2}\right]^{n-r/2}$$

(6.59)

From Eq. (6.57):

$$N_{LB}(t') = \frac{1}{P^2}\left\{t' - \int_0^{t'} \frac{\frac{1}{2A}\ E_{xy}\left\{E_x(g_{s'}\psi_{s'})\ E_x(g_{s'}\psi_{s'})\right\}}{\exp\left\{s'/P^2\right\}}\, ds'\right\}$$

(6.60)

From Eq. (5.84):

$$\frac{1}{2A} E_{xy} \left\{ E_x(g_{t'} \psi_{t'}) \; E_x(g_{t'} \psi_{t'}) \right\} = \sum_{j=0}^{\infty} \frac{1}{2A} E_{xy} (\gamma_j) \quad (6.61)$$

From Eq. (5.77):

$$\frac{1}{2A} E_{xy}(\gamma_j) = \sum_{i=0}^{j} \frac{1}{2A} E_{xy} (c_i \; d_{j-i}) \quad (6.62)$$

From Eqs. (5.72), (5.71):

$$c_n = E_x(g_{t'} \zeta_{t'}^n) \quad (6.63)$$

$$d_k = E_x(g_{t'} \zeta_{t'}^k) \quad (6.64)$$

$$\zeta_t = \frac{1}{2} \int_0^{t'} g_{s'}^2 \, R^{-1} \, ds' + \int_0^{t'} g_{s'} \, R^{-1} \, d\tilde{B}_{s'} \quad (6.65)$$

$$\frac{1}{2A} E_{xy}(c_n d_k) = \sum_{r_1=1,3,5,\ldots}^{n} \; \sum_{r_2=1,3,5,\ldots}^{k} \Lambda(r_1, r_2) \quad (6.66)$$

where if $r_1 \leq r_2$:

$$\Lambda(r_1, r_2) = \frac{1}{(n-r_1)!(k-r_2)!} \cdot \frac{2^{r1}}{2^{n-r1} \, 2^{k-r2}} \cdot \frac{1}{2^{\frac{r_2 - r_1}{2}} \left(\frac{r_2-r_1}{2}\right)!}$$
$$\cdot \left[\frac{1}{P^2}\right]^{n+k - \frac{r_1 + r_2}{2}} \left[t'\right]^{n+k - \frac{3r_1 + r_2}{2}}$$
$$\cdot \sum_{\eta=0}^{(r_1-1)/2} \left\{ \frac{1}{(r_1-2\eta)!} \; \frac{1\cdot3\cdot5 \ldots (2\eta-1)}{2\cdot4\cdot6 \ldots 2\eta} \; \frac{1}{2^{2\eta}(2\eta)!} \, (t')^{2\eta} \cdot \tilde{I}_3(r_1-2\eta) \right\} \quad (6.67)$$

and if $r_1 > r_2$,

$$A(r_1, r_2) = \frac{1}{(n-r_1)!(k-r_2)!} \; \frac{2^{r_2}}{2^{n-r_1}\, 2^{k-r_2}} \; \frac{1}{2^{\frac{r_1 - r_2}{2}} \left(\frac{r_1-r_2}{2}\right)!}$$

$$\cdot \left[\frac{1}{p^2} \right]^{n+k - \frac{r_1 + r_2}{2}} \left[t' \right]^{n+k - \frac{r_1 + 3r_2}{2}}$$

$$\cdot \sum_{\eta=0}^{(r_2-1)/2} \left\{ \frac{1}{(r_2-2\eta)!} \; \frac{1 \cdot 3 \cdot 5 \; \dots \; (2\eta-1)}{2 \cdot 4 \cdot 6 \; \dots \; 2\eta} \; \frac{1}{2^{2\eta}(2\eta)!} \, (t')^{2\eta} \, \tilde{I}_3(r_2 - 2\eta) \right\} \tag{6.68}$$

where

$$\left. \frac{1 \cdot 3 \cdot 5 \; \dots \; (2\eta-1)}{2 \cdot 4 \cdot 6 \; \dots \; 2\eta} \right|_{\eta=0} \triangleq 1 \tag{6.69}$$

From Eq. (6.38) for j_2 odd:

$$\tilde{I}_3(j_2) = \int_{s_1'=0}^{t'} \dots \int_{s_{j_2}'}^{t'} \left[E_x \left(\sin \theta_{t'} \sin \theta_{s_1'} \dots \sin \theta_{s_{j_2}'} \right) \right]^2 ds_1' \dots ds_{j_2}' \tag{6.70}$$

which can be evaluated by the method of Section 6.3.1 E.O.T.

Since $\tilde{I}_3(j_2)$ reduces to a linear combination of $t^n \exp(-mt)$ what has been accomplished is to reduce the evaluation of a lowerbound on the steady state optimum filtering MSE* to such a linear combination. In particular we note that there is only one numerical integration that needs to be performed -- the simple scalar integration in Eq. (6.60).

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*As noted in Chapter 5 all the preceding development (whose object was to obtain an upper bound on information) applies for other non-MSE distortion measures.

6.5 Numerical Results

The bound of Theorem 20 is presented in Fig. 6.1 as compared with the Zakai-Ziv bound of Section 5.1 and as compared with the Snyder-Rhodes bound (77). We can make the following observations.

(1) All the bounds shown in Fig. 6.1 are lower bounds on $E(x_t-\hat{x}_t^*)^2$, for $t \to \infty$, where as usual $\hat{x}_t^*$ is the Kushner estimate. They are not mod 2π quantities which naturally is the more relevant quantity for the phase modulation problem.

(2) Large Noise to Signal Ratio P. As initially conjectured in Subsection 5.2.1, the bound of Theorem 20 exhibits its best behavior relative to the Zakai-Ziv bound in the high noise to signal ratio (highly nonlinear) region of operation. Thus, as evident from Fig. 6.1, the bound of Theorem 20 is tighter than the Zakai-Ziv bound for approximately $P > 1.1$. On the other hand the ease with which the Zakai-Ziv bound can be evaluated is certainly a valuable feature.

(3) Small Noise to Signal Ratio P. The behavior of the bound of Theorem 20 in the smaller P region is disappointing and may be traced to the technique used in eliminating the $\tilde{B}_t$ from the denominator in the integrand of u(t), Eq. (5.52). This technique involved lowerbounding $E_x\psi_t$ by $\Xi(t)$ as evident in Eqs. (5.67), (5.60), (6.54):

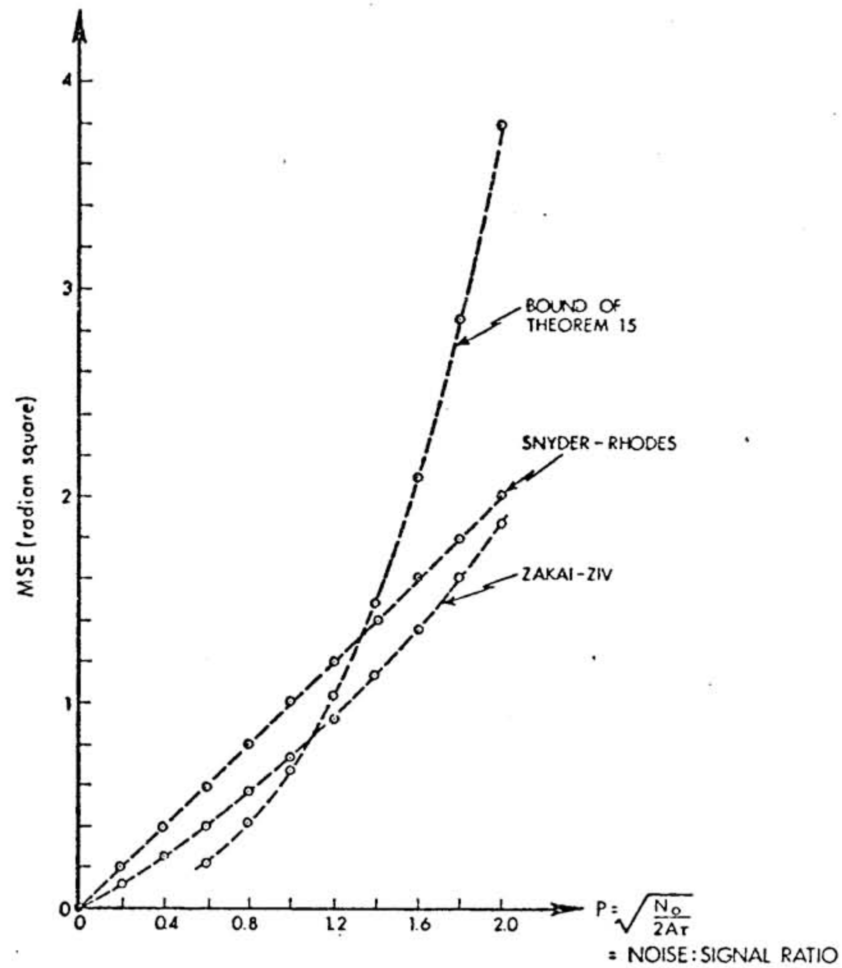

Figure 6.1 Comparison of Bound of Theorem 15 with Zakai-Ziv and Snyder-Rhodes Bounds

$$\Xi(t) = \exp\left(\frac{t'}{2P^2}\right) \leq E_x \psi_{t'}$$

However $E_{xy}\psi_{t'} = \exp(t'/P^2)$ which suggests that tightness is lost here. Thus the difficulty may lie in the evaluation technique rather than being an intrinsic limitation of the bound.

(4) The Snyder-Rhodes bound (Fig. 6.1) is not based on rate distortion theory but on the Cramer-Rao inequality and the Karbunen-Loeve expansion. Again the bound of Theorem 20 behaves better in the high noise region perhaps because, as is well known, the Cramer-Rao inequality is tightest in the low observation noise almost linear region. As in the case of the Zakai-Ziv bound the simplicity of evaluation of the Snyder-Rhodes bound should be noted as well as its remarkable performance in the significant low P region.

(5) Finally we mention that the lowerbound developed in Chapter 5 (specifically in Theorem 11) can always be evaluated by Monte Carlo simulation -- the usual drawbacks of simulation being reliability and computation time. In terms of computation time, the bound of Theorem 20 takes an average of 0.55 minute per point (i.e., per noise to signal ratio) in an IBM 360 although naturally program development time is considerably more.

BIBLIOGRAPHY

1. H.J. Kushner, "On the Differential Equations Satisfied By Conditional Probability Densities of Markov Processes With Applications," SIAM J. on Control, 2, 1, 1962.

2. P.A. Frost and T. Kailath, "An Innovations Approach to Least Squares Estimation--Part III," IEEE Trans. Auto. Control, Vol. AC-16, June 1971.

3. T.E. Duncan, "Probability Densities for Diffusion Processes," Ph.D. Thesis, Stanford University, 1967.

4. R.S. Bucy and P.D. Joseph, Filtering for Stochastic Processes with Applications to Guidance, Wiley, 1968.

5. R.S. Bucy, "Nonlinear Filtering," IEEE Trans. Auto. Control, 1965, p. 198.

6. R.E. Mortensen, "Optimal Control of Continuous Time Stochastic Systems," Ph.D. Thesis, Univ. of California, Berkeley, Jan. 1966.

7. W.M. Wonham, "Some Applications of Stochastic Differential Equations to Optimal Nonlinear Filtering," SIAM J. on Control, 2, 1965.

8. G. Kallianpur and C. Striebel: Estimation of Stochastic Processes: Arbitrary System Process With Additive White Noise Observation Errors, Ann. Math. Statist. 39 (1968), 785-801.

9. G. Kallianpur and C. Striebel: Stochastic Differential Equations Occurring in the Estimation of Continuous Parameter Stochastic Processes, Theor. Probability Appl. 4 (1969), 597-622.

10. G. Kallianpur and C. Striebel: Stochastic Differential Equations in Stochastic Estimation Problems, Multivariate Analysis II, Academic Press, 1969.

11. M. Fujisaki, G. Kallianpur, and H. Kunita, "Stochastic Diff. Eq. for the Nonlinear Filt. Prob.," Osaka J. Math, 9, 1972.

12. J.M.C. Clark: The Representation of Functions of Brownian Motion by the Stochastic Integrals, Ann. Math. Statist., 41 (1970), 1282-1295.

13. A.H. Jazwinski, "Stochastic Processes and Filtering Theory," Academic Press, N.Y., 1967.

14. R.S. Bucy and K. Senne, "Realization of Optimum Discrete Time Nonlinear Estimators," First Symposium on Nonlinear Est. Theory, San Diego.

15. R.S. Bucy, "Realization of Nonlinear Filters," Second Symposium on Nonlinear Est. Theory, San Diego.

16. R.G. Gallager, Information Theory and Reliable Communication, Wiley, 1968.

17. D.J. Sakrison, Notes on Analogue Communication, Van Nostrand, 1970.

18. F. Jelinek, Probabilistic Information Theory, McGraw-Hill, 1968.

19. M.S. Pinsker, Information and Information Stability of Random Variables and Processes, Holden Day, San Francisco, 1964.

20. Toby Berger, Rate Distortion Theory, Prentice-Hall, 1971.

21. C.E. Shannon and W. Weaver, The Mathematical Theory of Communication, Univ. of Illinois Press, 1949.

22. C.E. Shannon, "Coding Theorems for a Discrete Source With a Fidelity Criterion," IRE Natl. Conv. Rec. Part 4, 1959.

23. A.B. Carlson, Communication Systems, McGraw-Hill, 1968.

24. I.M. Gelfand and A.M. Yaglom, "Calculation of the Amount of Information About a Random Function...," Am. Math. Soc. Trans., 1959.

25. A. Perez, "Notions generalisees d'incertitude,...," Trans. First Prague Conf. on Info. Th., Stat. Dec. Funct.," Publ. House, Czech. Acad. Sci., Prague, 1957.

26. T. Berger, "Rate Distortion Theory for Sources With Abstract Alphabets and Memory," Info. and Control, Vol. 13, 1968.

27. T.J. Goblick, "Limitations on Transmission from Analogue Sources," IEEE Trans. Info. Th., Oct. 1965.

28. H.L. Weidemann, E.B. Stear, "Entropy Analysis of Estimating Systems," IEEE Trans. Info. Th., Vol. IT-16, May 1970.

29. ____________, "Entropy Analysis of Feedback Control Systems", in Leondes, ed., Advances in Auto. Control.

30. T. Duncan, "On the Calculation of Mutual Information," SIAM J. Appl. Math., Vol. 19, July, 1970.

31. T. Kailath, "A General Likelihood Ratio Formula for Random Signals in Gaussian Noise," IEEE Trans. Info. Th., Vol. IT-15, No. 3, May 1969.

32. T. Kailath, "A Further Note On...," IEEE Trans. Info. Th., Vol. IT-16, No. 4, July 1970.

33. T. Kailath, "The Innovations Approach to Detection and Estimation Theory," Proc. IEEE, Vol. 58, No. 5, May 1970.

34. R.M. Gray, "Information Rates of Autoregressive Sources," Ph.D. Thesis, U.S.C., 1969.

35. R.M. Gray, "Information Rates of Autoregressive Processes," IEEE Trans. IT, Vol. IT-16, July 1970.

36. W. Toms and T. Berger, "Info. Rates of Stochastically Driven Dynamic Systems," IEEE Trans. Info. Th., Jan. 1971.

37. W. Toms, "Info. Rates of Dynamic Systems," Ph.D. Thesis, Cornell University, 1971.

38. M. Zakai and J. Ziv, "Lower and Upper Bounds on Optimal Filtering Error of Certain Diffusion Processes," IEEE Trans. Info. Th., Vol. IT-18, May 1972.

39. A.D. Wyner and J. Ziv, Bounds on the Rate Distortion Function for Stationary Sources With Memory, IEEE Trans. Info. Theory, Vol. IT-17, No. 5, Sept. 1971, pp. 508-513.

40. J.K. Wolf and J. Ziv, Transmission of Noisy Information to a Noisy Receiver with Minimum Distortion, IEEE Trans. Info. Theory, Vol. IT-16, No. 4, July 1970.

41. R.L. Dobrushin and B.S. Tsybakov, "Info. Transmission With Additional Noise," IRE Trans. Info. Th., Vol. IT-8, September 1962.

42. H.L. VanTrees, Detection Estimation and Modulation Theory, Wiley, 1968.

43. J.T. Pinkston, "Information Rates of Independent Sample Sources," M.S. Thesis, MIT, 1966.

44. E. Wong, Stochastic Processes in Information and Dynamic Systems, McGraw Hill, 1971.

45. E. Wong, "Recent Progress in Stochastic Processes," IEEE Trans.Info. Th., May 1973.

46. K. Ito, "On a Formula Concerning Stochastic Differentials," Nagoya Math. J. 3, 1951.

47. P.A. Meyer, Probability and Potentials, Blaisdell, 1966.

49. J.L. Doob, Stochastic Processes, Wiley, 1953.

50. M. Loeve, Probability Theory, VanNostrand, 1963.

51. P.A. Meyer, "A Decomposition Theorem for Supermartingales," Illinois J. Math. 6, 1972.

52. P.A. Meyer, "On the Multiplicative Decomposition of Positive Supermartingales," in J. Chover, ed., Markov Processes and Potential Theory, Wiley, 1967.

53. C. Doleans-Dade, "Quelques Applications de la Formule de Changement de Variables pour les Seminartingales," Z. Wahr., Geb. 16, 1970.

54. ———— and P.A. Meyer, "Integrals Stochastiques par Rapoort Aux Martingales Locales," Sem. Prob. IV, Univ. de Strasbourg, Lec. Notes Math, Springer.

55. K. Ito and S. Watanabe, "Transformation of Markov Professes by Multiplicative Functionals," Ann. Inst. Fourier, Grenoble, 15, 1965.

56. G.S. Carr, Formulas and Theorems in Pure and Applied Mathematics, 2nd edition, 1920, Chelsea Publishing Co., New York.

57. N. Wiener, Nonlinear Problems in Random Theory, MIT Technology Press and John Wiley, N.Y., 1958.

58. J. Dettman, Applied Complex Variables, Macmillan, 1965.

59. R. Bartle, The Elements of Real Analysis, Wiley, 1964.

60. E. Kreyzig, Advanced Engineering Mathematics, Wiley, 1967.

61. K. Knopp, Theory and Application of Infinite Series, Hafner Pub. Co., N.Y. 1949.

62. E.D. Rainville, Infinite Series, Macmillan, 1967.

63. O.E. Stanaitis, An Introduction to Sequences, Series and Improper Integrals, Holden-Day, 1967.

64. A.I. Markushevich, Infinite Series, Heath, Boston, 1967.

65. I.I. Hirschman, Infinite Series, Holt, Rinehart and Winston, 1962.

66. H.L. VanTress, Detection, Estimation, and Modulation Theory Part II, Wiley, 1971.

67. D.J. Sakrison, Communication Theory: Transmission of Waveforms and Digital Information, Wiley, 1968.

68. A.J. Viterbi, Principles of Coherent Communication, McGraw-Hill, 1966.

69. R.S. Bucy, Realization of Nonlinear Filters," Proc. 2nd Symp. on Nonlinear Est., San Digeo, 1971.

70. D.E. Gustafson, On Optimal Estimation and Control of Linear Systems with State-Dependent and Control-Dependent Noise, Ph.D. Thesis, Dept. of Aero. and Astro., MIT, 1972.

71. D.E. Gustafson and J.L. Speyer, "Linear Minimum Variance Estimation in Systems with State Dependent Noise," Proc. 2nd Symp. on Nonlinear Est. Theory, San Diego, 1971.

72. A.S. Willsky, Dynamical Systems Defined on Groups: Structural Properties and Estimation, Ph.D. Thesis, Dept. Aero. and Astro., MIT, 1973.

73. R. Hudson, The Engineer's Manual, Wiley, 1967.

74. Chemical Rubber Co., Standard Mathematical Tables, 1965.

75. K. Rektorys, Survey of Applicable Mathematics, Iliffe Books Ltd., London, 1969.

77. D.L. Snyder and I.B. Rhodes, "Filtering and Control Performance Bounds with Implication on Asymptotic separation," Automatica, 1973.

78. P.R. Halmos, Measure Theory, Van Nostrand, 1950, Automatica, Vol. 8, 1972.

79. A. Gelb, ed., Applied Optimal Estimation, The MIT Press, 1974.

80. D.E. Johansen, "Optimal Control of Linear Stochastic Systems with Complexity Constraints," in Leondes, Advances in Control, Academic Press, New York.

81. M. Athans, "The Matrix Minimum Principle," Inform. Control, Vol. 11, 1968.

82. M. Athans and E. Tse, "A Direct Derivation of the Optimal Linear Filter Using the Maximum Principle," IEEE Trans. Auto. Control, Vol. AC-12, 1967.

83. C.E. Hutchinson and J.A. D'Appolito, "Minimum Variance Reduced Order Filter," 1972 IEEE Decision and Control Conference.

84. J. Center, J. D'Appolito, S. Marcus, "Reduced Order Estimators and Their Application to Aircraft Navigation," The Analytic Sciences Corp. Tech. Report, AFAL-TR-73-367, 1974.

85. P.D. Joseph, "Subopt. Linear Filtering," Space Tech. Lab. Corp. Tech. Report, Dec. 1963.

86. J.S. Meditch, "Suboptimal Linear Filtering for Continuous Dynamic Processes," Aerospace Corp. Tech. Report, July, 1964.

87. E.E. Pentecost, "Synthesis of Computationally Effective Sequential Linear Estimators," Ph.D. Thesis, UCLA, 1965.

88. M. Aoki and J.R. Huddle, "Estimation of the State Vector of a Linear Stochastic System with Constrained Estimators," IEEE Trans. Auto. Control, Vol. AC-12, 1967.

89. A.R. Stubberud and D.A. Wismer, "Suboptimum Kalman Filtering Techniques," in C. Leondes, ed., Theory and Applications of Kalman Filtering, Agardograph, Feb., 1970.

90. T.R. Damiani, "A General Partial-State Estimator," 14th Midwest Symp. on Circuit Theory, Denver, 1971.

91. P.J. McLane, "Linear Optimal Control of a Linear System With State and Control Dependent Noise," JACC, 1970.

92. P.J. McLane, "The Optimal Regulator Problem for a Stat. Linear System With State Dependent Noise," JACC, 1969.

93. R.G. Bartle, The Elements of Integration, Wiley, 1966.

94. B.J. Uttam and W.F. O'Halloran, "On Observers and Reduced Order Optimal Filters for Linear Stochastic Systems," JACC, 1972.

95. B.J. Uttam, W.F. O'Halloran, On the Computation of Optimal Stochastic Observer Gains, Sixth Annual Southeastern Symposium on System Theory, Baton Rouge, Feb. 1974.

APPENDIX A

PROOF OF LEMMA 45

The purpose of this appendix is to complete the proof of Lemma 45 of Chapter 6. Specifically we need to prove Eq. (6.55):

$$v(t) \triangleq E_{xy}\int_o^t \frac{E_x(g_s^2\psi_s)}{E_x(\psi_s)} R^{-1}\, ds = \frac{A}{R}t + O(\frac{1}{\omega\tau}) \tag{A.1}$$

where as throughout Chapter 6

$$g_s = \sin\theta_s \tag{A.2}$$

$$\theta_s = \omega s + x_s + \phi \tag{A.3}$$

$$\psi_s = \exp\left\{\tfrac{1}{2}\int_o^t g_s^2 R^{-1} ds + \int_o^t g_s R^{-1} d\tilde{\beta}\right\} \tag{A.4}$$

Now using $\sin^2\alpha = (1-\cos 2\alpha)/2$ in Eq. (A.1) we get:

$$\begin{aligned} v(t) &= \frac{2A}{2R} E_{xy}\int_o^t \left\{1 - \frac{E_x\{\cos 2\theta_s\ \psi_s\}}{E_x\psi_s}\right\} ds \\ &= \frac{A}{R}t - \int_o^t E_{xy}\left\{\frac{E_x\{\cos 2\theta_s\ \psi_s\}}{E_x\psi_s}\right\} \end{aligned} \tag{A.5}$$

Clearly Eq.(A.1) follows from Eq. (A.5) provided the second term in Eq. (A.5) is $O(\frac{1}{\omega\tau})$ as the carrier frequency $\omega\to\infty$.

First we simplify the notation somehow. We note that the first term in the exponential defining ψ_s is

$$\int_o^t g_s^2 R^{-1} = \frac{2A}{R}\left[\tfrac{1}{2}t - \int_o^t \cos 2\theta_s ds\right] \tag{A.6}$$

$$= \frac{2A}{R}\ \tfrac{1}{2}t - \tfrac{1}{2}\int_o^t \cos 2\omega s \cdot \cos 2(x_s+\phi)ds + \tfrac{1}{2}\int_o^t \sin 2\omega s\ \sin 2(x_s+\phi)ds$$

$$= \frac{A}{R}t + O\left(\frac{1}{\omega\tau}\right) \qquad \text{as } \omega\to\infty \qquad \text{a.s.} \tag{A.6}$$

by the Rieman-Lebesgue lemma since almost all of the x_s are continuous. While we could wait to let $\omega\to\infty$ at the end of the proof, it unclutters the notation if we use Eq. (A.6) to cancel $\exp\{At/R + O(1/\omega\tau)\}$ from both numerator and denominator of the integrand in Eq. (A.5). If in addition we take $2A=R=1$ we have that for the purposes of this proof we can take

$$\psi_t = \exp\left\{\int_o^t \sin\theta_s\ d\tilde{\beta}_s\right\} \tag{A.7}$$

$$\theta_s = \omega s + x_s + \phi \tag{A.8}$$

define

$$v_1(\omega,t) = E_{xy}\int_o^t \frac{E_x\{\cos 2\theta_s\ \psi_s\}}{E_x\{\psi_s\}}\ ds \tag{A.9}$$

and show that

$$\lim_{\omega\to\infty} v_1(\omega,t) = 0 \tag{A.10}$$

We take for every integer N>2 the random time of $\underline{B}_s$, τ_s^n, as the first passage time at which $\psi_s=n$. We can then speak of the stopped process $\psi_{s\wedge\tau_s^n}$ where as usual $\wedge$ means minimum (49,50). Further in what follows we let $I_{\{\ \}}$ be the indicator (characteristic) function.

We begin with a necessary strengthening of Doob's submartingale inequality.

<u>Lemma A1</u>. For every integer $k \geq 1$, $n \geq 1$

$$P\{\tau_t^n \leq t\} \leq \frac{\exp\{k^2 t/2\}}{\ell^k} \tag{A.11}$$

$$P\ \tau_t^n > t \geq 1 - \frac{\exp\{k^2 t/2\}}{\ell^k} \tag{A.12}$$

<u>Proof</u>: First observe that $E_x \psi_t$ as defined in Eq. A.7 is a submartingale of $\tilde{B}_t$ (a convex function of a martingale). Furthermore for every integer $k \geq 1$ $[E_x \psi_t]^k$ is also a $\tilde{B}_t$ submartingale (since $f(x) = x^k$, $k \geq 1$ is monotone non-decreasing convex for $x \geq 0$ and $E_x \psi_t \geq 0$). Applying Doob's (49) submartingale inequality we get for any $\alpha > 0$

$$P\left\{\max_{s \leq t} \left[E_x \psi_s\right]^k \geq \alpha\right\} \leq \frac{E_{xy}[E_x \psi_t]^k}{\alpha}$$

$$\leq \frac{E_{xy}(\psi_t^k)}{\alpha} \tag{A.13}$$

where the last step follows from Jensen's inequality (50). Now

$$P\left\{\max_{s \leq t} E_x \psi_s \geq L\right\} = P\left\{\max_{s \leq t} \left[E_x \psi_s\right]^k \geq L^k\right\} \tag{A.14}$$

Combining Eqs. (A.13), (A.14) with $\alpha = L^k$ we have

$$P\left\{\max_{s \leq t} E_x \psi_s \geq L\right\} \leq \frac{E_{xy}(\psi_t^k)}{L^k} \tag{A.15}$$

Multiplying now Eq. (A.15) by -1 and adding one to both sides we get

$$P\left\{\max_{s\le t} E_x\psi_s \le L\right\} \equiv 1-P\left\{\max_{s\le t} E_x\psi_s \ge L\right\}$$

$$\ge 1 - \frac{E_{xy}(\psi_t^k)}{L^k} \tag{A.16}$$

Finally the desired results follow from Eqs. (A.15), (A.16) upon observing that

$$P\left\{\max_{s\le t} \psi_s \ge L\right\} = P\left\{\tau_t^n \le t\right\}$$

$$P\left\{\max_{s\le t} \psi_s \le L\right\} = P\left\{\tau_t^n \ge t\right\}$$

and for ψ_t as defined in Eq. (A.7)

$$E_{xy}\psi_t^k = \exp\{k^2 t/2\} \quad \blacksquare$$

<u>Lemma A2</u>. Define for integers $j\ge 0$, $n>2$

$${}^{j}R_2^n(\omega,t) \triangleq n^j E_{xy}\left\{\int_{s=0}^{t} E_x(\cos 2\theta_s \psi_s) I_{\{\tau_s^n < s\}} ds\right\} \tag{A.17}$$

Then for every integer $k>j+2$ and $n>2$

$${}^{j}R_2^n(\omega,t) \le \frac{1}{n^{k-(j+2)}}\left\{\frac{\exp(k^2 t/2)-1}{k-2}\right\} \tag{A.18}$$

<u>Proof</u>: Clearly

$$E_{xy}\left\{E_x(\cos 2\theta_s \ \psi_s) I_{\{\tau_s^n < s\}}\right\} \le E_{xy}\left\{E_x(\psi_s) I_{\{\tau_s^n < s\}}\right\}$$

$$< \int_n^{\infty} L \ \frac{\exp\{k^2 s/2\}}{L^k} \ dL \tag{A.19}$$

$$= \frac{1}{n^{k-2}} \frac{\exp\{k^2 s/2\}}{k-2} \tag{A.19}$$

where the next to the last step follows from Eq. (A.11) of Lemma A1. Integrating Eq. (A.19) with respect to s gives Eq. (A.18) as desired. ∎

Lemma A3. Define for integers $j \geq 0$, $n > 2$

$$^jR_1(\omega,t) \triangleq E_{xy}\left\{\int_{s=0}^{t} E_x\{\cos 2\theta_s \psi_s\}[E_x\psi_s]^j ds\right\} \tag{A.20}$$

$$^jR_1^n(\omega,t) \triangleq E_{xy}\left\{\int_{s=0}^{t} E_x\{\cos 2\theta_s \psi_s\}[E_x\psi_s]^j I_{\{\tau_s^n > s\}} ds\right\} \tag{A.21}$$

Then for any integer $k > j+2$

$$^jR_1(\omega,t) - {}^jR_1^n(\omega,t) \leq \frac{1}{n^{k-(j+2)}}\left\{\frac{\exp(k^2 t/2)-1}{k-(j+2)}\right\} \tag{A.22}$$

Proof:

$$\begin{aligned} ^jR_1(\omega,t) - {}^jR_1^n(\omega,t) &= E_{xy}\int_{s=0}^{t} E_x(\cos 2\theta_s\psi_s)(E_x\psi_s)^j(1-I_{\{\tau_s^n > s\}})ds \\ &= E_{xy}\int_{s=0}^{t} E_x(\cos 2\theta_s\psi_s)(E_x\psi_s)^j I_{\{\tau_s^n < s\}} ds \\ &= E_{xy}\int_{s=0}^{t} (E_x\psi_s)^{j+1} I_{\{\tau_s^n < s\}} ds \end{aligned} \tag{A.23}$$

Now

$$P\left\{\max_{s\leq t}(E_x\psi_s)^{j+1} \geq n^{j+1}\right\} = P\left\{\max_{s\leq t} E_x\psi_s \geq n\right\} = P\{\tau_s^n < s\}$$

Hence using Eq. (A.11) of Lemma A1,

$$E_{xy}\left\{(E_x\psi_s)^{j+1} I_{\{\tau_s^n<s\}}\right\} \le \int_n^\infty L^{j+1} \frac{\exp(k^2 s/2)}{L^k}\, dL = \frac{1}{n^{k-(j+2)}} \frac{\exp(k^2 s/2)}{k-(j+2)} \quad (A.24)$$

Integrating Eq. (A.24) with respect to s and substituting this result in Eq. (A.23) gives Eq. (A.22) as desired. ∎

Lemma A4. Define as in the previous lemma

$$^{j}R_1(\omega,t) \doteq E_{xy}\left\{\int_{s=0}^{t} E_x(\cos 2\theta_s\psi_s)(E_x\psi_s)^j ds\right\} \quad (A.25)$$

where j is a non-negative integer. Then for every $\varepsilon_1>0$ and every j there exists a w>0 such that for $\omega>w$

$$^{j}R_1(\omega,t) < \varepsilon_1 \quad (A.26)$$

Proof: Substituting Eq. (A.7) in Eq. (A.25) we get

$$^{j}R_1(\omega,t) =$$

$$E_{xy}\left\{\int_{s=o}^{t} E_x\left\{\cos 2\theta s \sum_{i=o}^{\infty} \frac{1}{r!}\left[\int_{\pi=o}^{s} \sin\theta_\pi d\tilde{B}_\pi\right]^i\right\}\right.$$

$$\cdot E_x\left\{\sum_{r_1=o}^{\infty} \frac{1}{r_1!}\left[\int_{\sigma_1=o}^{s} \sin\theta_{\sigma_1} d\tilde{B}_{\sigma_1}\right]^{r_1}\right\}\cdot\cdot\cdot$$

$$\left.\cdot E_x\left\{\sum_{i_j=o}^{\infty} \frac{1}{r_j!}\left[\int_{\sigma_j=o}^{s} \sin\theta_{\sigma_j} d\tilde{B}_{\sigma_j}\right]^{r_j}\right\}\right\}$$

Making use of Lemma 43 we can eliminate some of the terms in the summations to get

$$
{}^{j}R_1(\omega,t) =
$$

$$
E_{xy}\left\{\int_{s=o}^{t} E_x\left\{\cos 2\theta_s \sum_{i=2,4,6,\ldots}^{\infty} \frac{1}{i!}\left[\int_{\pi=o}^{s} \sin\theta_\pi d\tilde{\beta}_\pi\right]^i\right\}\right. ,
$$

$$
\cdot E_x\left\{\sum_{i_1=0,2,4,\ldots}^{\infty} \frac{1}{i_1!}\left[\int_{\sigma_1=o}^{s} \sin\theta_{\sigma_1} d\tilde{\beta}_{\sigma_1}\right]^{i_1}\right\} \cdots
$$

$$
\left. \cdot \; E_x\left\{\sum_{i_j=0,2,4,\ldots}^{\infty} \frac{1}{i_j!}\left[\int_{\sigma_j=o}^{s} \sin\theta_{\sigma_g} d\tilde{\beta}_{\sigma_j}\right]^{i_j}\right\}\right\} \qquad (A.27)
$$

Clearly by the methods of Chapter 5 we can eliminate all stochatic integral from Eq. (A.27). We can then use the Rieman-Lebesgue Lemma since the resulting expression contains a sum of multiple ordinary integrals with continuous integrans containing at least one sinusoid, namely that arising from $\cos 2\theta_s = \cos(2\omega s + 2(x_s+\phi))$. Each integral containing a sinusoid introduces a $O(1/\omega)$ factor so that not only there is no "dc" component -- except one of $O(1/\omega)$ -- but also an increase in j would introduce more $O(1/\omega)$ factors and indeed for any j we can find a $w>0$ such that for $\omega>w$ the right hand side of Eq. (A.27) can be made less than any given $\varepsilon_1>0$. ∎

<u>Lemma A5</u>. Define for integers $j\geq 0$, $n>2$

$$
{}^{j}R^n(\omega,t) \triangleq \int_0^t E_{xy}\left\{E_x(\cos 2\theta_s \psi_s)(\psi_{s\wedge\tau_s^n})^j\right\} ds \qquad (A.28)
$$

Then for any j and any $\varepsilon_1>o$ we can find a $w>o$ such that for $\omega>w$

$$|{}^{j}R^{n}(\omega,t)|<\varepsilon_1 + \frac{1}{n^{k-(j+2)}}\,(\exp(k^2 t/2)-1)\left[\frac{1}{k-(j+2)} + \frac{1}{k-2}\right] \qquad (A.29)$$

for any $k>j+2$ and any $n>2$.

Proof: Recall the definitions of ${}^{j}R_2^n(\omega,t)$, ${}^{j}R_1^n(\omega,t)$, and ${}^{j}R_1(\omega,t)$ as defined in Eqs. (A.18), (A.21), (A.20) respectively. Clearly

$$ {}^{j}R^{n}(\omega,t) = {}^{j}R_1^n(\omega,t) + {}^{j}R_2^n(\omega,t)$$

since

$$\begin{aligned}
{}^{j}R^{n}(\omega,t) &= \int_{s=o}^{t} E_{xy}\left\{E_x(\cos 2\theta_s \psi_s)(\psi_{s\wedge\tau_s^n})^j I_{\{\tau_s^n>s\}}\right\}ds \\
&+ \int_{s=o}^{t} E_{xy}\left\{E_x(\cos 2\theta_s \psi_s)(\psi_{s\wedge\tau_s^n})^j I_{\{\tau_s^n<s\}}\right\}ds \\
&= \int_{s=o}^{t} E_{xy}\left\{E_x(\cos 2\theta_s \psi_s)\ \psi_s^j\ I_{\{\tau_s^n>s\}}\right\}ds \\
&+ \int_{s=o}^{t} E_{xy}\left\{E_x(\cos 2\theta_s \psi_s) n^j I_{\{\tau_s^n<s\}}\right\}ds
\end{aligned}$$

Furthermore,

$$\begin{aligned}
|{}^{j}R^{n}(\omega,t)| &\equiv |{}^{j}R_1^n(\omega,t) - {}^{j}R_1(\omega,t) + {}^{j}R_1(\omega,t) + {}^{j}R_2^n(\omega,t)| \\
&\le |{}^{j}R_1^n(\omega,t) - {}^{j}R_1(\omega,t)| + |{}^{j}R_1(\omega,t)| + |{}^{j}R_2^n(\omega,t)|
\end{aligned}$$

Eq. (A.29) now follows from Lemmas A2, A3, A4. ∎

Theorem A1. Consider

$$v_1(\omega,t) = E_{xy} \int_o^t \frac{E_x(\cos 2\theta_s \psi_s)}{E_x(\psi_s)} ds \tag{A.30}$$

where

$$\psi_s = \exp\left\{\int_o^t \sin\theta_s d\tilde{B}_s\right\}$$

Then

$$\lim_{\omega\to\infty} v_1(\omega,t) = 0 \tag{A.31}$$

Proof: Define for integers n>2

$$v_1^n(\omega,t) \triangleq E_{xy} \int_o^t \frac{E_x(\cos 2\theta_s \psi_s)}{E_x \psi_{s\wedge\tau_s^n}} ds \tag{A.32}$$

where the optional stopping time τ_s^n is as previously defined. It is clear that

$$\lim_{n\to\infty} v_1^n(\omega,t) = v_1(\omega,t)$$

so we need to show that

$$\lim_{\omega\to\infty} \lim_{n\to\infty} v_1^n(\omega,t) = 0 \tag{A.33}$$

Recall the series expansion

$$\frac{1}{z} = \sum_{i=o}^{\infty} (-1)^i \frac{(z-2n)^i}{(2n)^i} \tag{A.34}$$

for

$$o<z<2n \tag{A.35}$$

and any integer n>2.

Since $0 < E_x\psi_{s\wedge\tau_s^n} < 2n$ we have from Eqs. (A.32), (A.34), (A.35),

$$v_1^n(\omega,t) = E_{xy} \sum_{i=o}^{\infty} \frac{(-1)^i}{(2n)^i} \int_o^t E_{xy}\left\{E_x(\cos 2\theta_s\psi_s)(E_x\psi_{s\wedge\tau_s^n} - 2n)^i\right\}ds$$

$$= \sum_{i=o}^{\infty} \frac{(-1)^i}{(2n)^i} \sum_{j=o}^{i} (-1)^j \binom{i}{j} (2n)^{i-j} \int_o^t E_{xy}\left\{E_x(\cos 2\theta_s\psi_s)(E_x\psi_{s\wedge\tau_s^n})^j\right\}ds$$

Hence

$$|v_1^n(\omega,t)| \leq \sum_{i=o}^{\infty} \frac{1}{(2n)^i} \sum_{j=o}^{i} \binom{i}{j} (2n)^{i-j} |{}^jR^n(\omega,t)| \qquad (A.36)$$

where ${}^jR^n(\omega,t)$ is as defined in Eq. (A.28) of Lemma A5. Consider the i=o term of Eq. (A.36). For any given integer n>2 we can find a number w_o such that for $\omega > w_o$

$$(\text{i=o term of (A.36)}) = \left|\int_o^t E_{xy}\ E_x(\cos 2\theta_s\psi_s)\ ds\right|$$

$$\equiv |{}^oR_1(\omega,t)| < \frac{0.5}{n} \qquad (A.37)$$

for $\omega \geq w_o$

Consider the i=1 term of Eq. (A.36). From Lemmas A4, A5 for any given integer n>2 we can find a number w_1 such that for $\omega > w_1$

$$(\text{i=1 term of (A.36)}) = \left|\int_o^t E_{xy}\ E_x(\cos 2\theta_s\psi_s)ds + \frac{1}{2n}\int_o^t E_{xy}\left\{E_x(\cos 2\theta_s\psi_s)E_x(\psi_{s\wedge\tau_s^n})\right\}ds\right|$$

$$\equiv |{}^oR_1(\omega,t)| + |{}^1R^n(\omega,t)|$$

$$\leq \frac{0.125}{n} + \frac{0.125}{n} + \frac{1}{n^{k-3}}(\exp k^2 t/2-1)\left[\frac{1}{k-3} + \frac{1}{k-3}\right]$$

for any integer $k>3$. Clearly for say $k=4$ we can find an integer N_1 such that for all $n \geq N_1$ the last term is smaller than $0.25/n$ and

$$(i=1 \text{ term of (A.36)}) < \frac{0.5}{n} \tag{A.38}$$

for $\omega \geq w_1$, $n \geq N_1$.

Consider the i th term of Eq. (A.36) for $i \geq 2$. From Lemmas A4, A5 we can find, for any $\varepsilon_1 > 0$, a number w_i such that for $\omega \geq w_i$

(i th term of (A.36))

$$= |{}^{o}R_1(\omega,t)| + \frac{1}{(2n)^i} \sum_{j=1}^{i} \binom{i}{j} (2n)^{i-j} |{}^{j}R^{n}(\omega,t)|$$

$$\leq \varepsilon_1 + \frac{1}{(2n)^i} \sum_{j=1}^{i} \binom{i}{j} (2n)^{i-j} \varepsilon_1 + \tag{A.39}$$

$$+ \frac{1}{(2n)^i} \sum_{j=1}^{i} \binom{i}{j} (2n)^{i-j} \frac{1}{n^{k-(j+2)}}(\exp(k^2 t/2)-1)\left[\frac{1}{k-(j+2)} + \frac{1}{(k-2)}\right]$$

for any k j+2. In particular for any integer n, we can choose w_i to make ε_1 so small that the first two terms of (A.39) are less than $0.5/n^i$. Furthermore we can find a $k>j+2$* such that it is possible to find an integer N_i such that for $n>N_i$ the

---

*For example $k=(j+2) + (i-j) + 25$.

last term of (A.39) is less than $0.5/n^i$. Consequently we can find a real number w_i and an integer N_i such that

$$\text{(ith term of (A.36))} \quad > \frac{0.5}{n^i} + \frac{0.5}{n^i} = \frac{1}{n^i} \tag{A.40}$$

for $\omega \geq w_i$, $n \geq N_i$.

We can thus bound every term in Eq. (A.36) such that, from Eqs. (A.37), (A.38), (A.40), for $\omega \geq \sup_i w_i$, $n \geq \sup_i N_i$

$$|v_1^n(\omega,t)| \leq \frac{0.5}{n} + \frac{0.5}{n} + \sum_{i=2}^{\infty} \frac{1}{n^i}$$

$$= \frac{1}{1-\frac{1}{n}} - 1$$

$$\rightarrow 0 \quad \text{as} \quad n,\omega \rightarrow \infty \qquad \blacksquare$$

APPENDIX B

CONTINUATION OF SUBSECTION 6.3.2

As was the case with the development of subsection 6.3.1 the following derivation can be divided into three steps.

Step 1. The first step is contained in the following lemma.

Lemma B1. Consider $I,(r_1, r_2, m_1, m_2)$, defined in Eq. (5.86) and given by Eq. (5.88) of Theorem 15 where without loss of generality we take r_1, m_1 to be arbitrary non-negative intergers; r_2, m_2 even integers; and $r_2 \leq m_2$. Then for the PLLP $I,(.,.,.,.)$ reduces to

$$R^{-(r_1+r_2+m_1+m_2)}\, I(r_1,r_2,m_2,m_2) =$$

$$\frac{(2A)}{2}\, \frac{2}{p^2}^{\,r_1+m_1+r_2/2+m_2/2}\, K_2(r_2,m_2).$$

$$\sum_{n=0}^{r_{2/2}} K,(\eta,r_1,r_2,m_1,m_2)\, t'^{\,2\eta+r_1+m_1+\nu}$$

$$\int_{s_1'=0}^{t'} \cdots \int_{s'_{I2(\eta)}=0}^{t'} [E_x(\sin\theta_{s_1},\ldots\sin\theta_{s_{I2(\eta)}})]^2 ds_1' \ldots ds_{I2(\eta)} \tag{B.1}$$

where the $R^{-(\ )}$ factor multiplying $I_1(.,.,.,.,.)$ arises naturally from the R^{-1} omitted from each of the intergrals in Eq. (5.86) (see Eqs. (5.71)) and

$$I2(\eta) = r_2 - 2\eta \tag{B.2}$$

$$K_1(\eta, r_1, r_2, m_1, m_2) = \left[\binom{i_2}{2\eta} (2\eta-1)\ldots5.3.1\right]^2 \frac{(i_2-2\eta)!}{2^{2\eta+r_1+m_1+\nu}} \tag{B.3}$$

$$K_2(r_2, m_2) = \left[\binom{m_2}{2\nu} (2\nu-1)\ldots5.3.1\right] \tag{B.4}$$

$$\nu = (m_2-r_2)/2$$

P^2, s_i', t' are the noise to signal ratio and normalized times as in Eqs. (6.6), (6.7) and $\theta_{s'}$ is the normalized times as in Eqs. (6.8).

<u>Proof</u>: Substituting $g_s = \sqrt{2A}\sin\theta_s$ in the integral in Eq. (5.88), normalizing time and using the identity $\sin^2\alpha = (1-\cos 2\alpha)/2$, we get

$$\frac{(2A)\ (2A\tau)^{r_1+m_1+m_2/2+r_2/2}}{2^{1+2\eta+r_1+m_1+\nu}} \int_{\tilde{s}_1'=0}^{t}\cdots\int_{\tilde{s}_{I1(\eta)}'=0}^{t}\cdots\int_{\tilde{\sigma}_1'=0}^{t}\cdots\int_{\tilde{\sigma}_{J_1(\eta)}'=0}^{t}\cdots\int_{s_1'=0}^{t}\cdots\int_{s_{I2(\eta)}'=0}^{t}$$

$$E_x\left\{(1-\cos 2\theta_{t'})(1-\cos 2\theta_{\tilde{s}_1'})\ldots(1-\cos 2\theta_{\tilde{s}_{I1(\eta)}'})\sin\theta_{s_1'}\ldots\sin\theta_{s_{I2(\eta)}'}\right\}$$

$$E_x\left\{(1-\cos 2\theta_{\tilde{\sigma}_1'})\ldots(1-\cos 2\theta_{\tilde{\sigma}_{J1(\eta)}})\sin\theta_{s_1'}\ldots\sin\theta_{s_{I2(\eta)}'}\right\}$$

$$d\tilde{s}_1'\ldots d\tilde{s}_{I1(\eta)}'\ d\tilde{\sigma}_1'\ldots d\tilde{\sigma}_{J1(\eta)}\ ds_1'\ldots ds_{I2(\eta)}' \tag{B.5}$$

where as before $I1(\eta) = r_1 + \eta$, $J1(\nu) = m_1 + \nu + \eta$. When the products in the integrand of (B.5) are multiplied, we can eliminate by arguments analogous to those in Lemma 43E, F all terms that are not of zero frequency so that

$$\text{Eq.(B.5)} = \frac{(2A)(2A\tau)^{r_1+m_1+m_2/2+r_2/2}}{2^{1+2\ \ +r_1+m_1+\nu}} \int_{\tilde{s}_1'=0}^{t'} \cdots \int_{\tilde{s}_{I1}'=0}^{t'} \int_{\tilde{\sigma}_1'=0}^{t'} \cdots \int_{\tilde{\sigma}_{J_1}'=0}^{t'} \int_{s_1'=0}^{t'} \cdots \int_{s_{I2}'=0}^{t'}$$

$$\left[E_x(\sin\theta_{s_1'} \ldots \sin\theta_{s_{I2(\eta)}'})\right]^2 \, d\tilde{s}_1' \ldots \ldots \ldots ds_{I2(\eta)}'$$

$$= \frac{(2A)\ (2A\tau)^{r_1+m_1+m_2/2+r_2/2}}{2^{1+2\eta+r_1+m_1+\nu}} (t')^{I1(\eta)+J1(\eta)} \int_{s_1'=0}^{t'} \cdots \int_{s_{I1(\eta)}'=0}^{t'}$$

$$\left[E_x(\sin\theta_{s_1'} \ \ldots\ \sin\theta_{s_{I2(\eta)}'})\right]^2 ds_1' \ldots ds_{I2(\eta)}' \tag{B.6}$$

Eq. (B.1) can now be obtained by substituting Eq.(B.6) in Eq. (5.88) using the definition of P^2 and regrouping constants. ■

Step 2. The net effect of Lemma B1 and Step 1 has been to reduce the entire computation of $E_{xy}\{E_x(g^2\psi)E_x(\psi)\}$ to the evaluation of

$$\tilde{I}_1\ (i_2) \underset{i_2=2,4,6\ldots}{\triangleq} \int_{s_1=0}^{t} \cdots \int_{s_{i_2}=0}^{t} \left[E_x(\sin\theta_{s_1} \ \ldots\ \sin\theta_{s_{i_2}})\right]^2 ds_1 \ldots ds_{i_2} \tag{B.7}$$

where to avoid further clutter in the notation the facts that we are dealing with normalized times (e.g., s_k') and the $\tilde{I}_1$ is also a function of t' are not explicit. The reason that $\tilde{I}_1$ does not vanish from Eq. (B.1) is that as we shall see it possesses a zero frequency component.

Exact evaluation of $\tilde{I}_1$ is straightforward though tedious. Rather than produce general formulas (which is perfectly possible) we shall illustrate the evaluation process in Steps 2 and 3 by considering the case $i_2 = 4$ in Eq.(B.7).

The particular objective of Step 2 is to reduce Eq. (B.7) to "simple integrals" -- that is, to integrals that can be found in the table of integrals of just about any engineering or scientific handbook (e.g. (73), (74), (75)).

Consider then $\tilde{I}_1(i_2=4)$. As most of the development of this chapter, the following relies heavily on the material included in the statement and proof of Lemma 43. From Eq.(B.7),

$$\tilde{I}_1(4) = \int_{s_1=0}^{t} \int_{s_2=0}^{t} \int_{s_3=0}^{t} \int_{s_4=0}^{t} \left[E_x(\sin\theta_{s_1}, \sin\theta_{s_2}, \sin\theta_{s_3} \sin\theta_{s_4})\right]^2 ds_1\, ds_2\, ds_3\, ds_4 \tag{B.8}$$

Now making use of trigonometric identities such as

$$\text{sub } \alpha \sin \beta = \tfrac{1}{2} (\cos(\alpha-\beta) - \cos (\alpha+\beta)) \tag{B.9}$$

and of Lemma 43 we can show that

$$E_x(\sin\theta_{s_1} \sin\theta_{s_2} \sin\theta_{s_3} \sin\theta_{s_4}) =$$

$$= \frac{1}{8} E_x \{\cos(\theta_{s_1}-\theta_{s_2}-\theta_{s_3}+ \theta_{s_4}) + \cos(\theta_{s_1}-\theta_{s_2}+\theta_{s_3}-\theta_{s_4})$$

$$+ \cos(\theta_{s_1}+\theta_{s_2}-\theta_{s_3}-\theta_{s_4})\} \tag{B.10}$$

Substituting (B.10) in (B.8) we see that, up to O $(\frac{1}{\omega\tau})$,

$$\tilde{I}_1(4) = \frac{1}{64} \int \int \int \int \Big[E_x \cos(\theta_{s_1} - \theta_{s_2} - \theta_{s_3} + \theta_{s_4})$$

$$+ E_x \cos(\theta_{s_1} - \theta_{s_2} + \theta_{s_3} - \theta_{s_4})$$

$$+ E_x \cos(\theta_{s_1} + \theta_{s_2} - \theta_{s_3} - \theta_{s_4})\Big]^2 \, ds_1 \, ds_2 \, ds_3 \, ds_4.$$

$$= \frac{1}{64} \int \int \int \int \Big\{ \Big[E_x \cos(s_1 - s_2 - s_3 + s_4) \Big]^2$$

$$+ \Big[E_x \cos(s_1 - s_2 + s_3 - s_4 \Big]^2$$

$$+ \Big[E_x \cos(s_1 + s_2 - s_3 - s_4) \Big]^2 \Big\} \, ds_1 \, ds_2 \, ds_3 \, ds_4$$

$$= \frac{3}{64} \int_{s_1=0}^{t} \int_{s_2=0}^{t} \int_{s_3=0}^{t} \int_{s_4=0}^{t} \Big[E_x \cos(\theta_{s_1} + \theta_{s_2} - \theta_{s_3} - \theta_{s_4}) \Big]^2 ds_1 \, ds_2 \, ds_3 ds_4$$

$$= \frac{3}{128} \int_{s_1=0}^{t} \int_{s_2=0}^{t} \int_{s_3=0}^{t} \int_{s_4=0}^{t} \Big[E_x \cos(x_{s_1} + x_{s_2} - x_{s_3} - x_{s_4}) \Big]^2 ds_1 \, ds_2 \, ds_3 \, ds_4$$

(B.11)

parenthetically we mention that we can show that Eq. (B.11) generalizes to

$$\tilde{I}_1(i_2) = \frac{1}{2^{2i_2}} \, {}_{i_2}C_{i_{2/2}} \int_{s_1=0}^{t} \cdots \int_{s_{i_2}=0}^{t} \Big[E_x \cos(x_{s_1} + \ldots + x_{s_{i_{2/2}}} - x_{s_{i_{2/2}}} - \cdots - x_{s_{i_2}}) \Big]^2$$

$$ds_1 \ldots ds_{i_2} \qquad \text{(B.12)}$$

We can write an expression for the integrand of Eq. (B.11) provided there is a definite order between the S_1, S_2, S_3, S_4. Thus the integral in (B.11) can be broken up into $4! = 24$ integrals since R^4 can be partitioned, up to a set of Lebesgue measure zero, into 4! regions where a definite order exists between the variables. Fortunately most of the resulting integrals will turn out to be identical so that we need only evaluate 2 integrals rather than 4! For i_2=6, 6! = 720 regions result but we need only evaluate 4 different integrals. For i_2=8, 8! = 40320 regions result but we need only evaluate 8 different integrals.

Naturally it is necessary to determine beforehand the different types of integrals (rather than to compute i_2! integrals and then see which are the same). This can be accomplished by the following two methods.

The first method is based on observing that

$$\tilde{I}_1(4) = \sum_{\substack{j_1,j_2,j_3,j_4 \in \{1,2,3,4\} \\ j \neq \dots \neq j_4}} \int_{s_{j_4}=0}^{t} \int_{s_{j_3}=0}^{s_{j_4}} \int_{s_{j_2}=0}^{s_{j_3}} \int_{s_{j_1}=0}^{s_{j_2}} \left[E \cos(x_{s_{j_1}} + x_{s_{j_2}} - x_{s_{j_3}} - x_{s_{j_4}})\right]^2 ds_{j_1}\, ds_{j_2}\, ds_{j_3}\, ds_{j_4} \tag{B.13}$$

where the sum is over all possible reorderings of 1,2,3,4. Two types of identical terms arise, terms of the form

$$\pm (x_{s_{j_1}} - x_{s_{j_2}}) \pm (x_{s_{j_2}} - x_{s_{j_4}}) \tag{B.14}$$

and terms of the form

$$\pm (x_{s_{j_1}} + x_{s_{j_2}}) \mp (x_{s_{j_3}} + x_{s_{j_4}}) \tag{B.15}$$

where it is assumed that $s_{j_1} < \dots < s_{j_4}$. Terms of the form (B.14) can be relected in

$4 \cdot 2 \cdot 2 \cdot 1 = 16$

ways that yield integrals which are identical up to a change of variables. Similarly terms of the form (B.15) can be chosen in

$$4 \cdot 1 \cdot 2 \cdot 1 = 8$$

ways. Thus Eq. (B.13) becomes

$$I_1(4) = 16 \int_{s_4=0}^{t} \int_{s_3=0}^{s_4} \int_{s_2=0}^{s_3} \int_{s_1=0}^{s_2} \left[E \cos(x_{s_1}-x_{s_2}-x_{s_3}+x_{s_4})\right]^2 ds_1\, ds_2\, ds_3\, ds_4$$

$$+ 8 \int_{s_4=0}^{t} \int_{s_3=0}^{s_4} \int_{s_2=0}^{s_3} \int_{s_1=0}^{s_2} \left[E \cos(x_{s_1}+x_{s_2}-x_{s_3}-x_{s_4})\right]^2 ds_1\, ds_2\, ds_3\, ds_4 \tag{B.16}$$

Naturally 16+8=4!. Now for $s_1<s_2<s_3<s_4$,

$$E \cos(x_{s_1}-x_{s_2}-x_{s_3}+x_{s_4}) = \exp\left\{\frac{-1}{2}(-s_1+s_2-s_3+s_4)\right\}$$

$$E \cos(x_{s_1}+x_{s_2}-x_{s_3}-x_{s_4}) = \exp\left\{\frac{-1}{2}(-s_1-3s_2+3s_3+s_4)\right\}$$

Substituting these results into Eq. (B.16) we have

$$\tilde{I}_1(4) = 16 \int_{s_4=0}^{t} \int_{s_3=0}^{s_4} \int_{s_2=0}^{s_3} \int_{s_1=0}^{s_2} \exp(s_1-s_2+s_3-s_4)\, ds_1\, ds_2\, ds_3\, ds_4$$

$$+ 8 \int_{s_4=0}^{t} \int_{s_3=0}^{s_4} \int_{s_2=0}^{s_3} \int_{s_1=0}^{s_2} \exp(s_1+3s_2-3s_3-3s_4)\, ds_1\, ds_2\, ds_3\, ds \tag{B.17}$$

Clearly the integrals in Eq. (B.17) can indeed be evaluated with the aid of an ordinary table of integrals.

A second and perhaps more convenient and reliable of arriving at Eq. (B.17) is to have a digital computer go through the i_2! possible order of $s_1, \ldots, s_{i_2}$ checking which yield identical integrands. Such a progdram yields as output the coefficient of each intergral (i.e., the number of repetitions of each particular term) as well as the parenthesis of the exponential in each integrand. In the present evaluation both of these methods were implemented.

Step 3. Once an expression of the form of Eq. (B.17) is obtained what remains to be done is the evaluation of the resulting integrals. We note that as evident from Lemma 43B the integrals are always of the form

$$\int_{s_{i_2}=0}^{t} \int_{s_{i_2-1}=0}^{s_{i_2}} \cdots \int_{s_1=0}^{s_2} \exp\{\sum_{i=1}^{i_2} a_i s_i\}\, ds_1 \ldots ds_{i_2}$$

where a_i are integers. While the necessary integrations could be performed by anybody with a knowledge of elementary integral calculus, more efficient and reliable ways can be used. Observe that numerical integration is not an attractive alternative* since (1) the integration is in i_2-dimensional Euclidean space and (2) the integrands are so simple that they can be evaluated by hand. The approach that seems to be ideally suited to this problem is the use of a symbolic (non-numerical) integration program. The particular program used in the present evaluation was MACSYMA developed by Prof. J. Moses et al at M.I.T's Project MAC. For i_2=4 Eq. (B.17) evaluates to

*Attention has been given in the literature to the evaluation of multiple integrals. See for example (76).

$$\tilde{I}_1(4) = 16(-t\,\varepsilon^{-t} - 3\,\varepsilon^{-t} + t^2/2 - 2t + 3)$$

$$+ \; 8(t\varepsilon^{-t}/3 + 5\,\varepsilon^{-t}/9 + \varepsilon^{-4t}/144 + t/4 - 9/16)$$

As is evident, all the $\tilde{I},(i_2)$ turn out to be, as is reasonable to expect, linear combinations of $t^m\,\varepsilon^{-nt}$ for m, n non-negative integers.

Summarizing, what has been accomplished in this section is to reduce the computation of thetypical term in the Cauchy product expression for $E_{xy}\ \{E_x(g_t^2\ \psi_t)\ E_x(\psi_t)\}$ as given in Eq. (5.88) of Theorem 15 to the computaiton of elementary integrals.

APPENDIX C

CONTINUATION OF SUBSECTION 6.3.3

As was the case with the development of subsection 6.3.1 and Appendix B the following derivation can be divided into three steps.

Step 1. The first step is contained in the following lemma.

Lemma C1. Consider $I_2(i_1, i_2, j_1, j_2, k_1, k_2)$, defined in Equation (5.96), and given by Eq. (5.97) of Theorem 16 where without loss of generality we take i_1, j_1, k_1 as arbitrary non-negative integers, i_2 odd, j_2 odd, k_2 even. Then for the PLLP $I_2(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot)$ reduces to

$$R^{-(i_1+j_1+k_1)-(i_2+j_2+k_2)} I_2(i_1, i_2, j_1, j_2, k_1, k_2) = (2A).$$

$$\sum_{\nu_1=0}^{\frac{i_2-1}{2}} \sum_{\nu_2=0}^{\frac{j_2-1}{2}} \sum_{\nu_3=0}^{k_2/2} \sum_{\nu_3=0}^{k_2-2\nu_3} \sum_{n_{23}=0}^{k_2-2\nu_3} \Big\{ K'(\nu_1, \nu_2, \nu_3, n_{12}, n_{13}, n_{23}).$$

$$\{\nu_1, \nu_2, \nu_3, n_{12}, n_{13}, n_{23}\} \epsilon P$$

$$\left(\frac{2}{p^2}\right)^{i_1+j_1+k_1+\frac{i_2+j_2+k_2}{2}} (t')^{(i_1+j_1+k_1)+(\nu_1+\nu_2+\nu_3)} \frac{1}{2^{(i_1+j_1+k_1)+(\nu_1+\nu_2+\nu_3)}}$$

$$\int_{s_1'=0}^{t'} \cdots \int_{s_{n_{12}}'=0}^{t'} \int_{\sigma_1'=0}^{t'} \cdots \int_{\sigma_{n_{13}}'=0}^{t'} \int_{\pi_1'=0}^{t'} \cdots \int_{\pi_{n_{23}}'}^{t'}$$

$$E_x(\sin\theta_{t'} \sin\theta_{s_1'} \cdots \sin\theta_{s_{n_{12}}'} \sin\theta_{\sigma_1'} \cdots \sin\theta_{\sigma_{n_{13}}'})$$

$$E_x(\sin\theta_{t'} \sin\theta_{s_1'} \cdots \sin\theta_{s_{n_{12}}'} \sin\theta_{\pi_1'} \cdots \sin\theta_{\pi_{n_{23}}'})$$

$$E_x(\sin\theta_{\sigma_1'} \cdots \sin\theta_{\sigma_{n_{13}}'} \sin\theta_{\pi_1'} \cdots \sin\theta_{\pi_{n_{23}}'})$$

$$ds_1' \cdots ds_{n_{12}}' \, d\sigma_1' \cdots d\sigma_{n_{13}}' \, d\pi_1' \cdots d\pi_{n_{23}}' \Big\} \qquad (C.1)$$

where the $R^{-(\ )}$ factor multiplying I(.,.,.,.,.,.,.) arises naturally from the R^{-1} omitted from each of the integrals in Eq.

$$K'(.,.,.,.,.,.,.) = K(.,.,.,.,.,.,.)\ R^{\frac{-(i_2+j_2+k_2)}{2}}$$

K (.,.,.,.,.,.,.): as in Eq. (5.100), Theorem 16

P: as in Theorem 16

η_{12}: as in Eq.(5.98), Theorem 16

η_{13}: as in Eq.(5.99), Theorem 16

P^2, t', s_i', σ_i', π_i' are the noise to signal ratio and normalized times as in Eqs. (6.6), (6.7) and $\theta_{s'}$ is the normalized version of θ_s in Eq. (6.8).

<u>Proof:</u> Substituting $g_s = \sqrt{2A}\sin\theta_s$ in the integral in Eq. (5.97), normalizing time, and using the identity $\sin^2\alpha = (1 - \cos 2\alpha)/2$, we get

$$(2A) \sum \sum \sum \sum K'\, R^{-(i_1+j_1+k_1)-\frac{(i_j+j_2+k_2)}{2}}\,(2A)^{i_1+j_1+k_1+\frac{i_2+j_2+k_2}{2}}$$

$$\tau^{(i_1+j_1+k_1)+\frac{(i_2+j_2+k_2)}{2}}\ \frac{1}{2^{(i_1+j_1+k_1)+(\nu_1+\nu_2+\nu_3)}}$$

$$\int_{\tilde{s}_1'=0}^{t'} \cdots \int_{\tilde{s}_{i_1+\nu_1}'=0}^{t'} \int_{\tilde{\sigma}_1'=0}^{t'} \cdots \int_{\tilde{\sigma}_{j_1+\nu_2}'}^{t'} \int_{\tilde{\pi}_1'=0}^{t'} \cdots \int_{\pi_{k_1+\nu_3}'}^{t'} \quad \int_{s_1'=0}^{t'} \cdots \int_{s_\eta'=0}^{t'} \int_{\sigma_1'=0}^{t'} \cdots \int_{\sigma_{\eta_{13}}'=0}^{t'} \int_{\pi_1'=0}^{t'} \cdots \int_{\pi_{\eta_{23}}'=0}^{t'}$$

$$E_x\left\{\sin\theta_t' \ (1-\cos 2\theta_{\tilde{s}_1'})\cdots(1-\cos 2\theta_{\tilde{s}_{i_1+\nu_1}'})\sin\theta_{s_1'}\cdots\sin\theta_{s_{\eta_{12}}'}\ \sin\theta_{\sigma_1'}\cdots\sin\theta_{\sigma_{\eta_{13}}'}\right\}$$

$$E_x\left\{\sin\theta_t' \ (1-\cos 2\theta_{\tilde{\sigma}_1'})\cdots(1-\cos 2\theta_{\tilde{\sigma}'+\nu_2})\sin\theta_{s_1'}\cdots\sin\theta_{s_{\eta_{12}}}' \qquad \sin\theta_{\pi_1'}\cdots\sin\theta_{\pi_{\eta_{23}}'}\right\}$$

$$E_x\left\{ \qquad (1-\cos 2\theta_{\tilde{\pi}_1'})\cdots(1-\cos 2\theta_{\tilde{\pi}_{k_1+\nu_3}'}) \qquad \sin\theta_{\sigma_1'}\cdots\sin\theta_{\sigma_{\eta_{13}}'}\ \sin\theta_{\pi_1'}\cdots\sin\theta_{\pi_{\eta_{23}}'}\right\}$$

$$d\tilde{s}_1'\cdots d\tilde{s}_{i_1+\nu_1}' \quad d\tilde{\sigma}_1\cdots d\tilde{\sigma}_{j_1'+\nu_2} \quad d\tilde{\pi}_1'\cdots d\tilde{\pi}_{k_1+\nu_3}'$$

$$ds_1'\cdots ds_{\eta_{12}}' \quad d\sigma_1'\cdots d\sigma_{\eta_{13}}' \quad d\pi_1'\cdots d\pi_{\eta_{23}}'$$

where we have used the fact that (see definition of P in Theorem 16) that

$$(i_2+j_2+k_2)/2 = \nu_1+\nu_2+\nu_3+\eta_{12}+\eta_{13}+\eta_{23}.$$

When the products in the integrand of (C.2) are multiplied, we can eliminate by arguments analogous to those in Lemmas 43E,F all terms that are not of zero frequency so that, recalling $R=N_o/2$

$$\text{Eq.(C.2)} = (2A) \sum \sum \sum \sum \Bigg\{ K' \frac{1}{2^{i_1+j_1+k_1+\nu_1+\nu_2+\nu_3}}$$

$$\left(\frac{2A\tau}{N_o/2}\right)^{i_1+j_1+k_1+\frac{(i_2+j_2+k_2)}{2}} (t')^{i_1+j_1+k_1+\nu_1+\nu_2+\nu_3}$$

$$\int_{s_1'=0}^{t'} \cdots \int_{s_{\eta_{12}}'=0}^{t'} \quad \int_{\sigma_1'=0}^{t'} \cdots \int_{\sigma_{\eta_{13}}'=0}^{t'} \quad \int_{\pi_1'=0}^{t'} \cdots \int_{\pi_{\eta_{23}}'=0}^{t'}$$

$$E_x(\sin\theta_{t'} \quad \sin\theta_{s_1'} \cdots \sin\theta'_{s_{\eta_{12}}} \quad \sin\theta_{\sigma_1'} \cdots \sin\theta_{\sigma_{\eta_{13}}'} \quad)$$

$$E_x(\sin\theta_{t'} \quad \sin\theta_{s_1'} \cdots \sin\theta_{s_{\eta_{12}}'} \quad \sin\theta_{\pi_1'} \cdots \sin\theta_{\pi_{\eta_{23}}'})$$

$$E_x(\quad \sin\theta_{\sigma_1'} \cdots \sin\theta_{\sigma_{\eta_{13}}'} \quad \sin\theta_{\pi_1'} \cdots \sin\theta_{\pi_{\eta_{23}}'})$$

$$ds_1' \ldots ds_{\eta_{12}}' \quad d\sigma_1' \ldots d\sigma_{\eta_{13}}' \quad d\pi_1' \ldots d\pi_{\eta_{23}}' \Bigg\} \qquad \text{(C.3)}$$

Eq. (C.1) can now be obtained from (C.3) by simply using the definition of P^2. ∎

Step 2. The net effect of Lemma C1 has been to reduce the entire computation of $E_{xy}\{(E_x g_t \psi_t)(E_x g_t \psi_t)(E_x \psi_t^{-2})\}$ to the evaluation of

$$
\begin{aligned}
\tilde{I}_2(\eta_{12},\eta_{12},\eta_{23}) = \int_{s_1=0}^{t} \cdots \int_{s_{\eta_{12}}}^{t} \int_{\sigma_1=0}^{t} \cdots \int_{\sigma_{\eta_{13}}=0}^{t} \int_{\pi_{\eta_{23}}=0} \\
E_x(\sin\theta_t \sin\theta_{s_1} \cdots \sin\theta_{s_{\eta_{12}}} \sin\theta_{\sigma_1} \cdots \sin\theta_{\sigma_{\eta_{13}}} \qquad\qquad\qquad\qquad) \\
E_x(\sin\theta_t \sin\theta_{s_1} \cdots \sin\theta_{s_{\eta_{12}}} \qquad\qquad\qquad\qquad \sin\theta_{\pi_1} \cdots \sin\theta_{\pi_{\eta_{23}}}) \\
E_x(\qquad\qquad\qquad\qquad \sin\theta_{\sigma_1} \cdots \sin\theta_{\sigma_{\eta_{13}}} \sin\theta_{\pi_1} \cdots \sin\theta_{\pi_{\eta_{23}}}) \\
ds_1 \cdots ds_{\eta_{12}} \; d\sigma_1 \cdots d\sigma_{\eta_{13}} \; d\pi_1 \cdots d\pi_{\eta_{23}} \qquad (C.4)
\end{aligned}
$$

where, as before, to avoid further clutter in the notation the facts that we are dealing with normalized times (e.g., s_i') and that $\tilde{I}_2$ is also a function of t' are not explicit. $\tilde{I}_2$ contains the zero frequency components in $I_2(.,.,.,.,.,.)$ and consequently does not vanish for arbitrarily large carrier frequency ω.

Exact evaluation of $\tilde{I}_2$ is straightforward but tedious. As in the previous subsection we stop at this point the general treatment and illustrate the evaluation process by means of an example. Also as in the previous subsection the specific objective of Step 2 is to reduce Eq.(C.4) to "simple integrals" -- that is integrals that can be found in the tables of any engineering or scientific handbook.

The first question that arises is what set of arguments $(\eta_{12}, \eta_{13}, \eta_{23})$ of $\tilde{I}_2$ in Eq. (C.4) are needed. We recall from Lemma 41 of Chapter 5 that we are computing

$$
E_{xy} \left\{ (E_x g\psi)'(E_x g\psi) \; (E \; \psi^{-2}) \right\} = \sum_{j=0}^{\infty} E_{xy} \left\{ \beta_j \right\} \qquad (C.5)
$$

where as we have shown $E_{xy}\{\beta_j\}$ can be expressed in terms of $\tilde{I}(\eta_{12},\eta_{13},\eta_{23})$ as defined in Eq. (C.4). The first question is then what specific $\tilde{I}_2(\eta_{12}, \eta_{13}, \eta_{23})$ as defined in Eq. (C.4). The first question is then what specific $\tilde{I}_2(\eta_{12}, \eta_{13}, \eta_{23})$ need to be computed for a given partial sum $\sum_{j=0}^{J}$ of Eq. (C.5). A program that implements the computation of the $E_{xy}\{\beta_j\}$ including the sum in Eq. (5.97) of Theorem 16 naturally provides a list of necessary $(\eta_{12}, \eta_{13}, \eta_{23})$. Thus:

For J=2, need $\eta_{12} = 1$, $\eta_{13} = 0$, $\eta_{23} = 0$

	η_{12}	η_{13}	η_{23}
For J=6, need	1	0	0
	0	1	1
	1	2	0
	3	0	0
For J=11, need	1	0	0
	0	1	1
	1	2	0
	3	0	0
	0	3	1
	1	4	0
	2	1	1
	3	2	0
	1	2	2
	5	0	0

As an example* consider the evaluation of $\tilde{I}_2(\eta_{12} = 2, \eta_{13} = 1, \eta_{23} = 1)$ which from Eq. (C.4) is:

$$I_2(2,1,1) = \int_{s_1=0}^{t} \int_{s_2=0}^{t} \int_{\sigma_1=0}^{t} \int_{\pi_1=0}^{t}$$

$$E_x(\sin\theta_t \sin\theta_{s_1} \sin\theta_{s_2} \sin\theta_{\sigma_1} \qquad)$$

$$E_x(\sin\theta_t \sin\theta_{s_1} \sin\theta_{s_2} \qquad \sin\theta_{\pi_1})$$

$$\cdot$$

$$Ex(\qquad \sin\theta_{\sigma_1} \sin\theta_{\pi_1})$$

$$ds_1\, ds_2\, d\sigma_1\, d\pi_1$$

where as before $\theta_s = \omega s + x_s + \phi$. Making use of the trigonometric identity $\sin\alpha \sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$ and of Lemma 43 it is easy to show that

$$\tilde{I}_2(2,1,1) = \frac{1}{512} \int_{s_4=0}^{t} \int_{s_3=0}^{t} \int_{s_2=0}^{t} \int_{s_1=0}^{t} E_x\{\cos(x_{s_3}-x_{s_4})\}\cdot$$

$$\cdot\ E_x\{\cos(x_t-x_{s_1}-x_{s_2}+x_{s_4})\}\cdot E_x\{\cos(x_t-x_{s_1}-x_{s_2}+x_{s_3})\}$$

$$ds_1\quad ds_2\quad ds_3\quad ds_4$$

$$+\ \frac{1}{512} \int_{s_4=0}^{t} \int_{s_3=0}^{t} \int_{s_2=0}^{t} \int_{s_1=0}^{t} E_x\{\cos(x_{s_3}-x_{s_4})\}\cdot$$

$$\cdot\ E_x\{\cos(x_t-x_{s_1}+x_{s_2}-x_{s_4})\}\cdot E_x\{\cos(x_t-x_{s_1}+x_{s_2}-x_{s_3})\}$$

$$ds_1\quad ds_2\quad ds_3\quad ds_4 \qquad \text{(C.6)}$$

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*Among the $\tilde{I}_2(.,.,.)$ needed for J=11, $\tilde{I}_2(2,1,1)$ is third in terms of complexity.

We can write an expression for the integrand in both terms of Eq. (C.6) provided there is a definite order between the s_1, s_2, s_3, s_4. Thus each of the integrals in Eq. (C.6) can be broken up into 4! = 24 integrals since R^4 can be partitioned; up toa set of Lebesgue measure zero, into 4! regions where a definite order exists between the variables. Fortunately most of the resulting 48 integrals are identical and this fact can be conveniently determined beforehand by having a computer go through all 4! orders of the s_i. In this case the first term and second terms of Eq. (C.6) produce respectively 3 (rather than 24) and 6 (rather than 24) different integrals:

$$\tilde{I}_2(2, 1, 1) =$$

$$\frac{1}{512} \int_{s_4=0}^{t} \int_{s_3=0}^{s_4} \int_{s_2=0}^{s_3} \int_{s_1=0}^{s_2} \{4 \exp(s_1+3s_2-s_3-2s_4=t)$$

$$+ 8 \exp(s_1+2s_3-2s_4-t)$$

$$+ 12 \exp(s_1-s_3+s_4-t\} \, ds_1 \, ds_2 \, ds_3 \, ds_4$$

$$+ \frac{1}{512} \int_{s_4=0}^{t} \int_{s_3=0}^{s_4} \int_{s_2=0}^{s_3} \int_{s_1=0}^{s_2} \{4 \exp(s_1-s_2+s_3-t)$$

$$+ 4 \exp (s_1+2s_2-2s_3-t)$$

$$+ 4 \exp (s_1+2s_2+s_3-3s_4-t)$$

$$+ 4 \exp (s_1-t)$$

$$+ 2 \exp (s_1+3s_3-3s_4-t)$$

$$+ 6 \exp (s_1-s_3+s_4-t)\} \, ds_1 \, ds_2 \, ds_3 \, ds_4 \quad (C.7)$$

Clearly the integrals in Eq. (C.7) can be evaluated with the aid of an ordinary table of integrals.

Step 3. Once an expression of the form of Eq.(C.7) has been obtained all that remains is the integration of the resulting integrals. Further we note that as evident from Lemma 43B, evaluation of the $\tilde{I}_2(\eta_{12}, \eta_{13}, \eta_{23})$ will always result in integrals of the form

$$\int_{s_I=0}^{t} \int_{s_{I-1}=0}^{s_i} \cdots \int_{s_1=0}^{s_2} \exp \{ \sum_{i=1}^{I} a_i s_i \} \, ds_1 \cdots ds_{I-1} \, ds_I$$

where the a_i are integers. As was the case with I(.,.,.,.,.) considered in the previous subsection, these integrals can be evaluated either "by hand" or more conveniently using symbolic integration, the specific method used in this study being the MACZYMA System developed at MIT's Project MAC by Prof. J. Moses et,al. As was the case with the integrals of subsection 6.3.1 and Appendix B, the present ones turn out to be linear combinations of $t^m \varepsilon^{-nt}$ for m, n non-negative integers.

BIOGRAPHICAL NOTE

Jorge I. Galdos was born in Habana, Cuba on December 10, 1946 and immigrated to the U.S. in 1960. He received the B.E.E. (magna cum laude) and M.E.E. degrees from Marquette University and the University of Virginia in 1967 and 1969 respectively. While at M.I.T. he was a teaching and research assistant and a C.S. Draper Laboratories Fellow. In 1972 he was commissioned a second lieutenant in the U.S. Air Force and in 1973 served a three month tour of active duty with the Electronics Systems Division (USAF) as a Site Activation Test Director for tactical radar equipment. His interests are in the area of estimation, detection, and stochastic systems theory and application. He is a member of Tau Beta Pi, Eta Kappa Nu, and Sigma Xi.