OPTIMAL FILTERING OF GYROSCOPIC NOISE

by

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ABSTRACT

An infinite-dimensional model is given for the generation of gyroscopic noise, which exhibits power spectral density proportional to (1/f) over a wide frequency range. An optimal filter is derived for separating a statistically described signal from additive gyroscopic noise. This filter is expressed as a discrete-time infinite-dimensional Kalman filter, with an associated Riccati covariance operator equation.

For general estimation problems, sufficient conditions are presented for asymptotic stability of Hilbert space Kalman filters in the time-varying model case, and sufficient conditions are presented in the time-invariant model case for:

(1) Asymptotic stability of the Kalman filter.

(2) Weak convergence of the Riccati operator, from arbitrary positive semidefinite initial conditions, to a unique positive definite (bounded below) operator, which is the unique positive semidefinite solution of the steady-state Riccati operator equation.

(3) Asymptotic stability of the steady-state Kalman filter.

The sufficient conditions are then specified to the gyroscopic noise filtering problem. Finally, finite-dimensional approximate modeling of gyroscopic noise (resulting in a finite-dimensional Kalman filter) is discussed with regard to its effect on filter mean-squared estimation error. It is shown that through the use of a sufficient number of dimensions in the approximate model it is possible to come arbitrarily close to the optimal mean-squared estimation error.

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INTRODUCTION

The gyroscope is an instrument used to detect angular movement. The problem of the removal from the gyro output signal of noise inherent to the gyroscope in a constant gravitational field is one which has received considerable attention in the literature. Sutherland and Gelb[1], for example, discuss an aided inertial guidance system, where periodic telescopic sightings are used along with gyro output to develop gyro error observations. The error observations are used as the input to a Kalman filter, which is used to estimate the gyro error at the observation times. An estimate of the true angular position is then obtained by subtracting the estimated gyro error from the gyro output samples. Mehra and Bryson[2] discuss smoothing of the gyro output to obtain estimates of the input signal.

Gyroscopic noise has often been modeled as either a first-order Gauss-Markov process[3], or as a Gaussian random walk (integral of Gaussian white noise)[4,5]. However, recent studies performed at The Charles Stark Draper Laboratory[6] of the power spectral characteristics of the random noise associated with various gyroscopes indicate that gyro noise is often characterized by a (1/f) behavior in power spectral density over a wide frequency range. An explanation of the source of this noise in the magnetic materials of the gyroscope (e.g. the gyro float rebalance torquer) is proposed by Harris and Koenigsberg[7]. In Section 2 we discuss their findings and add others. We present an infinite-dimensional state space model which generates noise with the power spectral properties of gyroscopic noise. We also discuss the possible relationship between magnetic disaccommodation and gyroscopic noise.

In Section 3, we introduce and solve the filtering problem to be treated in the thesis. Using discrete-time observations, a statistically described input process is optimally separated from additive gyroscopic noise. Because observations are made at discrete times, we first determine a discrete-time infinite-dimensional linear system to generate samples of the gyroscopic noise, as modeled in continuous time in Section 2. We solve the filtering problem as a conditional expectation filter in the case where the input signal is Gaussian (this solution being equivalent to the minimum variance linear estimator for non-Gaussian input signals). The resulting optimal filter is expressed as a discrete-time infinite-dimensional Kalman filter with an associated Riccati covariance operator equation.

In Section 4 we discuss general properties of Hilbert space Kalman filters and Riccati operator equations. For time-varying models we present sufficient conditions for asymptotic stability of the Kalman filter. In the time-invariant model case we present sufficient conditions for:

(1) Asymptotic stability of the Kalman filter.

(2) Weak convergence of the Riccati covariance operator, from arbitrary positive semidefinite initial conditions, to a unique positive definite (bounded below) operator, this operator being the unique positive semidefinite solution of the steady-state Riccati operator equation.

(3) Asymptotic stability of the steady-state Kalman filter.

The sufficient conditions are then specified to the gyro noise filtering problem and are expressed as conditions on the system generating the signal to be recovered.

The optimal filter derived in Section 3 involves integrations over a free time constant parameter. In applications, these integrations must be implemented discretely. This discretization can be achieved by making a finite-dimensional approximation to the infinite-dimensional gyroscopic noise model. The optimal filter becomes an ordinary finite-dimensional discrete-time Kalman filter, with an associated matrix Riccati equation. In Section 5 we show that the mean-squared estimation error incurred in using the Kalman filter of the finite-dimensional approximate model can be made, through the use of a sufficient number of dimensions in the approximation, to approach the mean-squared estimation error associated with optimal filtering of gyroscopic noise.

SECTION 2

AN INFINITE-DIMENSIONAL MODEL FOR GYROSCOPIC NOISE

Gyroscopic noise has often been modeled as either a first-order Gauss-Markov process[3], or as a Gaussian random walk (integral of Gaussian white noise)[4,5]. The Gaussian nature of the noise is inferred from histogram plots of gyro output. A linearized version on log-log scales of the power spectral density of a first-order Gauss-Markov process is shown in Figure 1. The random walk has a variance proportional to time, hence is nonstationary. Thus in a strict sense the power spectral density of a random walk process does not exist.

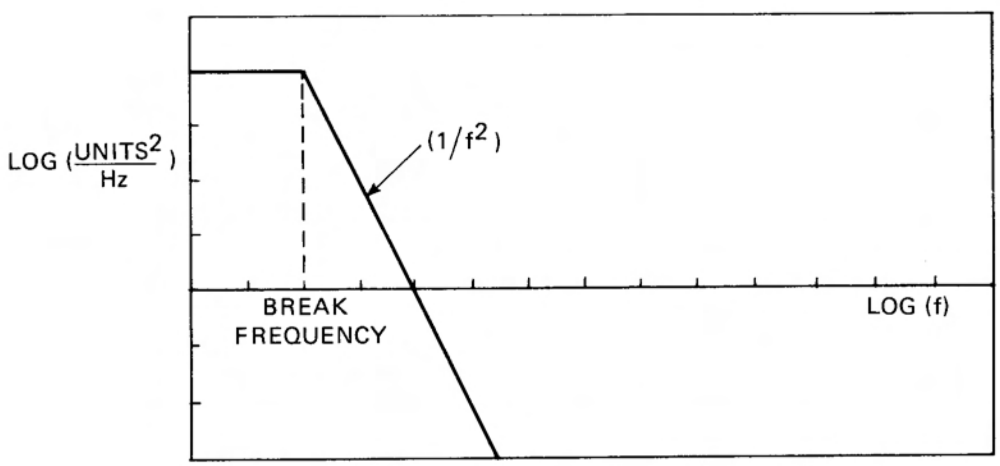

Figure 1. Linearized power spectral density of first-order Markov process.

When discrete samples of bandlimited white noise are generated by computer and summed (to resemble the integration of white noise), the resulting noise is found to be characterized by a $(1/f^2)$ power spectral density over the bandwidth of the original bandlimited white noise. (The power spectral density is found through evaluation of the squared magnitudes of the Fourier coefficients of the output signal.) For the following reasons we intuitively expect this result. The power spectral density, $S_{yy}(f)$, of the output of a time-invariant linear system (transfer function $H(f)$) to an input signal of PSD (power spectral density) $S_{xx}(f)$ is given by:

$$S_{yy}(f) = S_{xx}(f) \cdot |H(f)|^2 \tag{1}$$

The transfer function of an integrator is proportional to (1/s), hence we would have:

$$|H(f)|^2 = \frac{1}{(j2\pi f)(-j2\pi f)} = \frac{1}{4\pi^2 f^2} \tag{2}$$

Bandlimited white noise has a PSD constant with frequency (over its band limits), so we would intuitively expect our approximation to random walk to have behavior proportional to $(1/f^2)$. The PSD resulting from the computer simulation described above is shown in Figure 2. Notice that both random processes discussed here exhibit $(1/f^2)$ behavior in PSD (slopes of (-2) on log-log scales).

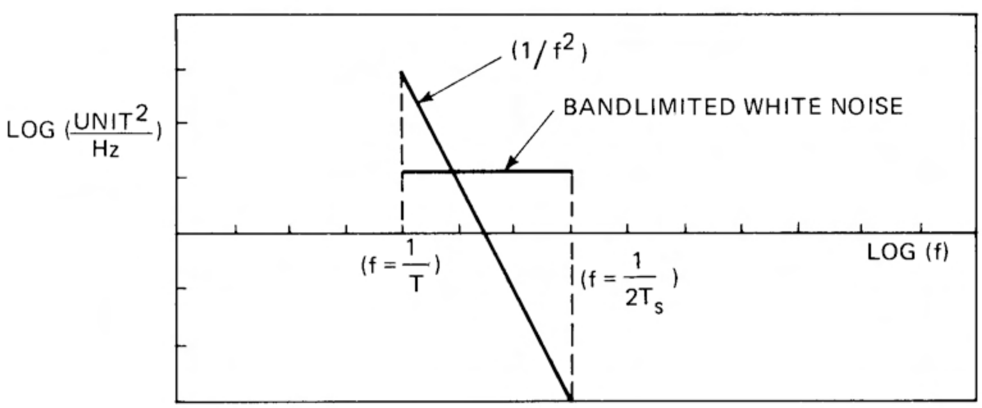

Figure 2. Linearized power spectral density of computer-simulated random walk. The lowest frequency sample (aside from the sample at 0 corresponding to the mean) of the power spectral density is at $(f = 1/T)$, where T is the record length used for analysis. The highest frequency sample is at $(f = 1/2T_s)$, where T_s is the sample time.

Recent studies performed at The Charles Stark Draper Laboratory[6] of the power spectral characteristics of the random noises associated with various gyroscopes indicate that gyro noise is often characterized by a (1/f) behavior in power spectral density. (The gyro is set up as an input rate integrator, with a binary torque rebalance loop. The units of PSD are (input rate)^2/Hz.) A linearized graph of the observed form of gyro power spectral density is given in Figure 3. The (f^2) portion of this graph is primarily attributed to quantization noise due to the binary torque loop. This effect of quantization is currently

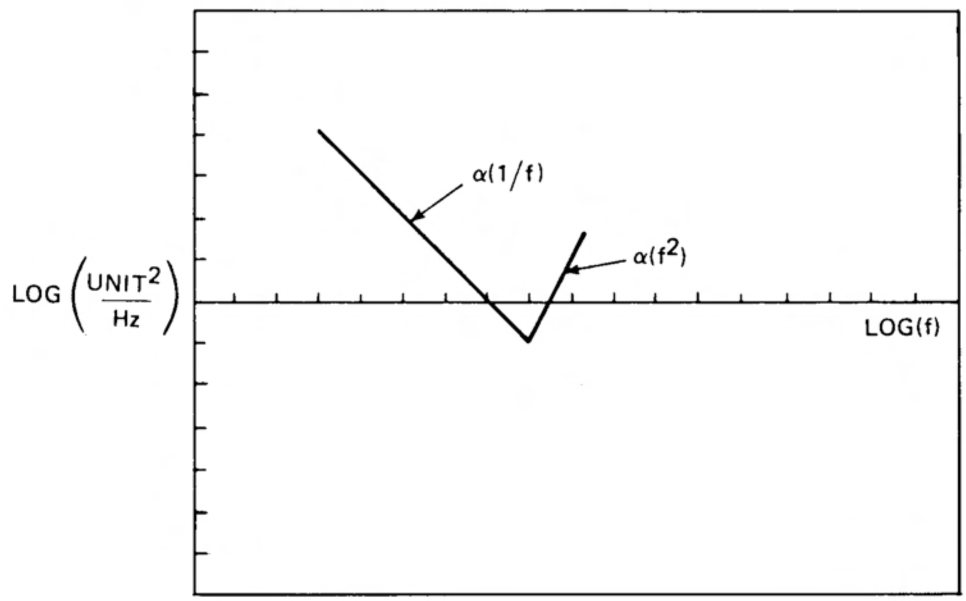

Figure 3. The observed form of gyro noise power spectral density.

under investigation. Power spectral analyses of separate record lengths of gyro noise show the power spectral density to be constant in time, hence we will treat the gyro noise as stationary. An explanation of the source of this noise in the magnetic materials of the gyroscope (e.g. the gyro float rebalance torquer) is proposed by Harris and Koenigsberg[(7)]. In this section we shall discuss their findings and add others. We first discuss a model for magnetic relaxation (disaccommodation). This model is then used to develop an infinite-dimensional state space model for the generation of gyroscopic noise.

Examination of the literature on magnetic relaxation (e.g. Ref. (8)) indicates that the response of iron to transients in applied magnetic field can be characterized as the impulse response of a continuum of first-order linear systems with a uniform volume density distribution of time constants. The term "uniform volume density distribution" is used here to mean a spatial distribution of systems such that each volume element contains many systems, and such that the systems in each volume element have time constants distributed according to the same probability density function. Each individual system is characterized by a transfer function of the form:

$$G_\tau(s) = \frac{\tau}{\tau s + 1} \tag{3}$$

The probability density function of time constants (τ) is given by (see insert in Figure 4):

$$p_d(\tau) = \begin{cases} (1/\ell n(\tau_2/\tau_1))(1/\tau) & ; \quad \tau_1 \le \tau \le \tau_2 \\ \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{4}$$

We shall demonstrate that the above density function is effective in explaining the gyro noise PSD in addition to magnetic relaxation, which is observed when the gyro is operated in the presence of power supply transients. Incidentally, other possible density forms (for anelastic relaxation of strain in crystalline solids, a related phenomenon) are discussed by Nowick and Berry[9]. The impulse response of each linear system (Eq. (3)) is given by:

$$h_\tau(t) = e^{-t/\tau} \tag{5}$$

The magnetic relaxation of the material is then characterized (see Ref. (8) by the weighted integral of the impulse responses of the linear systems, with the time constant density of (Eq. 4):

$$m(t) \triangleq K \int_0^\infty p_d(\tau) h_\tau(t) d\tau \tag{6}$$

Substituting Eq. (4) and Eq. (5) into Eq. (6) and simplifying (See Appendix A), we find that:

$$m(t) = \frac{K}{\ell n(\tau_2/\tau_1)} [E_1(t/\tau_2) - E_1(t/\tau_1)] \tag{7}$$

where $E_1(z)$ is the exponential integral, defined by:

$$E_1(z) = \int_z^\infty \left(\frac{e^{-u}}{u}\right) du \tag{8}$$

We choose K to normalize m(t) to $\psi(t)$, where we require for normalization that:

$$\psi(0) = 1 \; ; \quad \psi(\infty) = 0 \tag{9}$$

We find that:

$$K = 1 \tag{10}$$

Thus, the magnetic disaccommodation (relaxation) is normalized to:

$$\psi(t) = \left(\frac{1}{\ell n(\tau_2/\tau_1)}\right)[E_1(t/\tau_2) - E_1(t/\tau_1)] \tag{11}$$

Graphs of $\psi(t)$, for $\tau_1 = 0.01$, $\tau_2 = 1.0$, on linear-linear, semilog, and log-log scales are found in Figures 4, 5, and 6, respectively. As discussed in Ref. (7), $\psi(t)$, with proper choice of τ_1 and τ_2, often fits the time record of gyro output in the presence of power supply transients. Gyro output is the record of the torques applied by the magnetic gyro torquer in order to keep the gyro float angle close to zero. For (t) between τ_1 and τ_2, $\psi(t)$ is proportional to $(-\ell n(t))$, a familiar result in the study of magnetic relaxation (see Ref. (25)). (Incidentally, τ_1 and τ_2 may be estimated by observing the gyro output and using an analytic approximation[10] for $\psi(t)$, for t between τ_1 and τ_2.) In summary, the time constant density given in Eq. (4) can be used to explain the deterministic gyro response to transients. Note that while we have used the weighted integral (Eq. (6)) of the impulse responses (Eq. (5)) of first-order linear systems (Eq. (3)) in order to account for magnetic relaxation, this phenomenon can also be accounted for by interpreting Eq. (5) as the negative step response (with unit positive bias) of a linear system with transfer function given by:

$$G'_\tau(s) = \frac{1}{\tau s + 1}$$

The time constant density function (Eq. (4)) is the same, and the weighted integral (Eq. (6)) is also the same, with interpretation as the negative step response of magnetic material to the removal of an applied magnetic field. The reader should be aware that we do not have empirical confirmation that the relaxation exhibited by gyro output in the presence of power supply transients is necessarily magnetic in origin. We can only suggest this as a possible source, and note that this mechanism is effective in explaining the observed power spectral characteristics of gyroscopic output noise, which we shall now discuss.

If a linear system (Eq. (3)) with time constant (τ) is fed by an input function $w(\tau, t)$ then its reponse, $x(\tau, t)$, is characterized by:

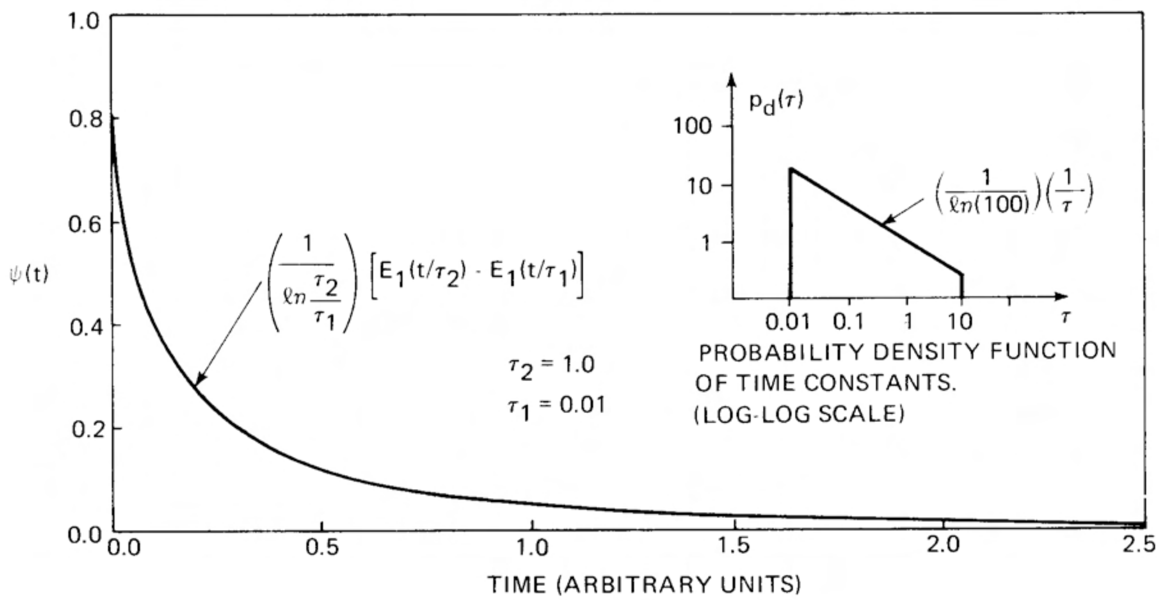

Figure 4. ψ(t) vs. t; linear-linear scale.

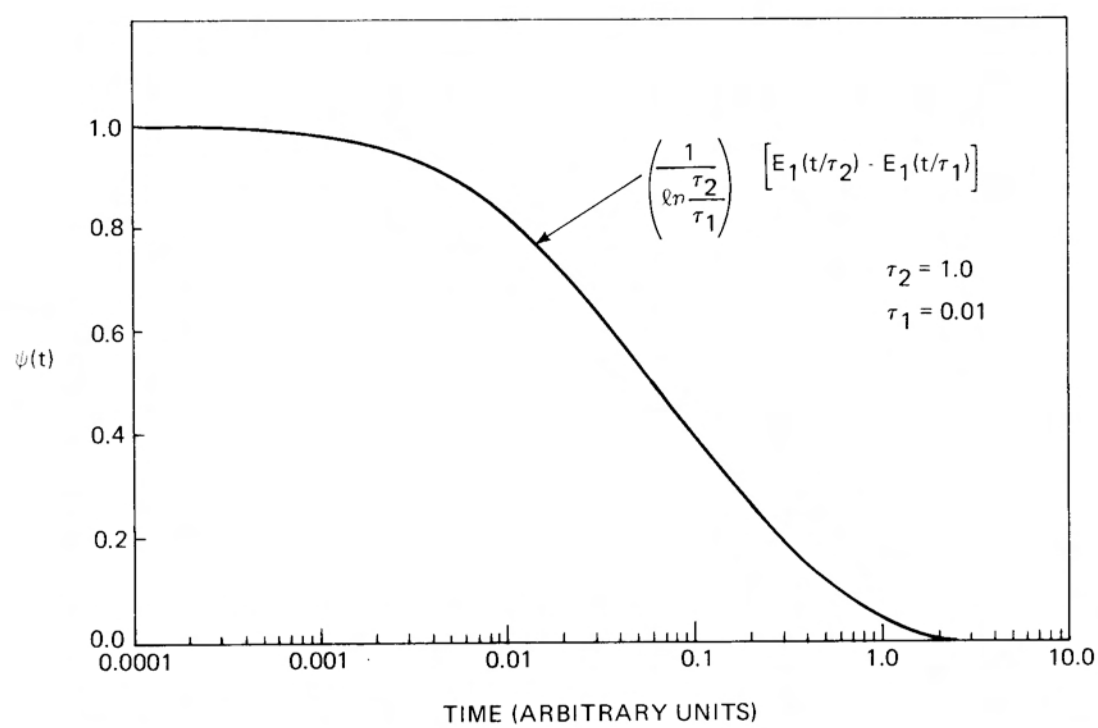

Figure 5. ψ(t) vs. t; semilog scale.

$$\frac{\partial}{\partial t} x(\tau, t) = -(\frac{1}{\tau})x(\tau, t) + w(\tau, t) \tag{12}$$

Let the input function of two variables, $w(\tau, t)$, to the systems be characterized by covariance: ($\delta(\cdot)$ is the Dirac delta function)

$$E\{w(\tau, t)w(\gamma, t - \alpha)\} \triangleq W\delta(\tau - \gamma)\delta(\alpha) \tag{13}$$

$w(\tau, t)$ is formally a "two-dimensional white noise". The inputs to two systems with time constants τ and γ are independent if $\tau \neq \gamma$. The variation of constants formula, applied formally to Eq. (12), yields (in steady state as $(t) \to \infty$) the correlation function:

$$R_{x_\tau x_\gamma}(\alpha) \triangleq E\{x(\tau, t)x(\gamma, t - \alpha)\} = \frac{W\tau}{2} \cdot e^{-|\alpha|/\tau} \cdot \delta(\tau - \gamma) \tag{14}$$

Gyroscopic noise is now modeled as the weighted integral of the outputs of the filters (where $x(\tau, t)$ is the output (at time t) of a filter with time constant τ), and is given by:

$$g(t) = \int_0^\infty x(\tau, t)p_d(\tau)d\tau \tag{15}$$

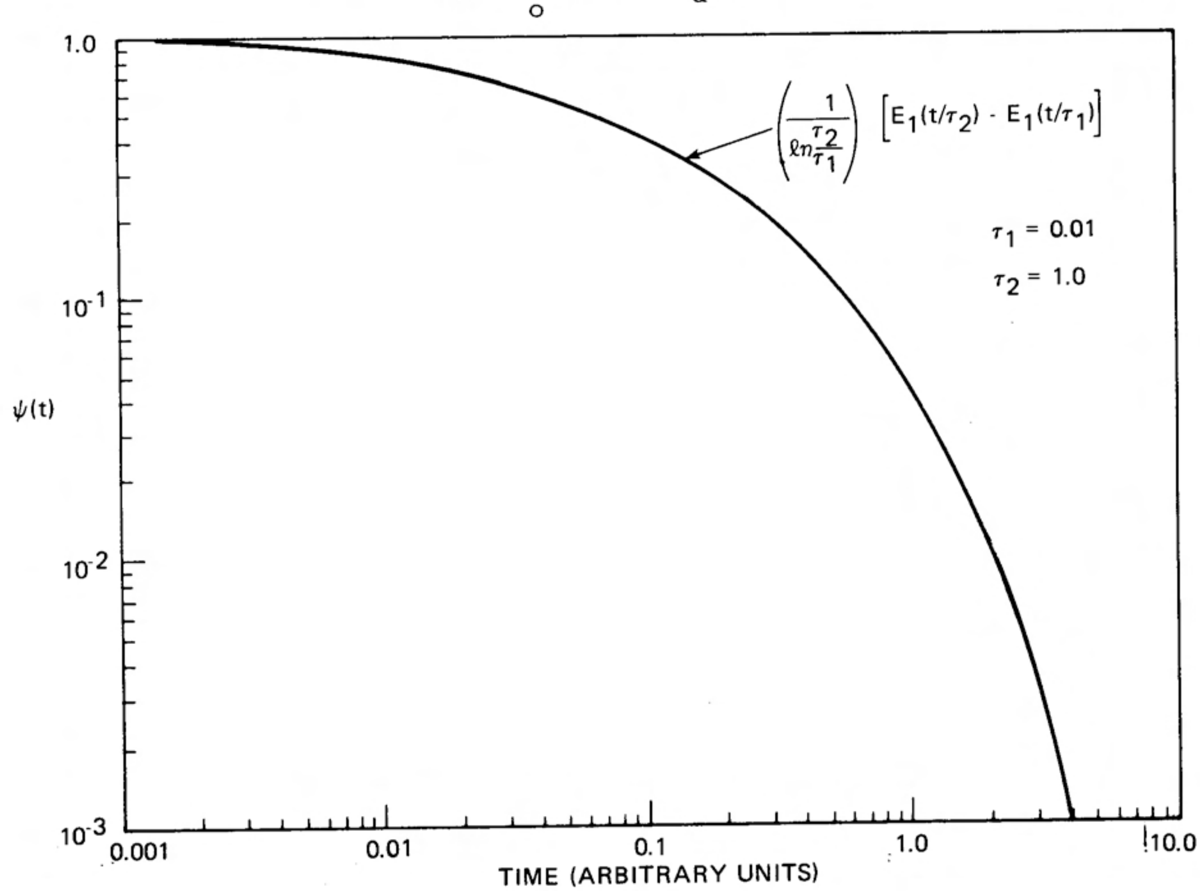

Figure 6. $\psi(t)$ vs. t; log-log scale.

As shown in Appendix A, normalization of g(t) so that the noise has unit variance requires:

$$W = 2 \ln(\tau_2/\tau_1) \tag{16}$$

The power spectral density of the noise is then given by:

$$S_{gg}(f) = \left(\frac{2}{\ln(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\tan^{-1}\left[\frac{2\pi f(\tau_2 - \tau_1)}{1 + 4\pi^2 f^2 \tau_2 \tau_1}\right] \tag{17}$$

A graph of $S_{gg}(f)$ is plotted in Figure 7. Note that the (1/f) characteristic of gyro noise observed experimentally is inherent in the linearized version of this plot. The (f^2) section of Figure 3, the experimentally observed gyro noise, due to quantization dominates over the $(1/f^2)$ line of Figure 7 at high frequencies, masking that portion of the gyro noise. Further, it is felt that the low frequency breakpoint of Figure 7 corresponds to times longer than the record lengths normally employed for observations of gyro output, accounting for its absence from Figure 3 (see caption of Figure 2). Ongoing experiments at The Charles Stark Draper Laboratory with long record lengths of gyro output indicate that the power spectral density is flat at very low frequencies (~1 cycle/month) for some of the gyroscopes being tested.

Henceforth, we shall use the term gyroscopic noise to refer to the stochastic process generated by our state space model, which we now review. The state equation is given by (where state $x(\tau, t)$ is a function of $\tau\varepsilon[\tau_1, \tau_2]$):

$$\frac{\partial}{\partial t} x(\tau, t) = (-\frac{1}{\tau})x(\tau, t) + w(\tau, t) \tag{18}$$

where $w(\tau, t)$ is zero mean, and:

$$E[x(\tau, 0)] = 0 \quad (\forall\tau) \tag{19}$$

$$E[w(\tau, t)w(\beta, s)] = W \cdot \delta(\tau - \beta) \cdot \delta(t - s) \tag{20}$$

The gyroscopic noise, g(t), is given by:

$$g(t) = \int_{\tau_1}^{\tau_2} p_d(\tau)x(\tau, t)d\tau \tag{21}$$

The gyroscopic noise is assumed to have started at $(t = -\infty)$, hence to be stationary at $(t = 0)$. We thus model the linear systems of Eq. (13) as each being in steady state at time zero. The steady-state covariance of two systems with time constants (γ) and (β) is given (from Eq. (14)) by:

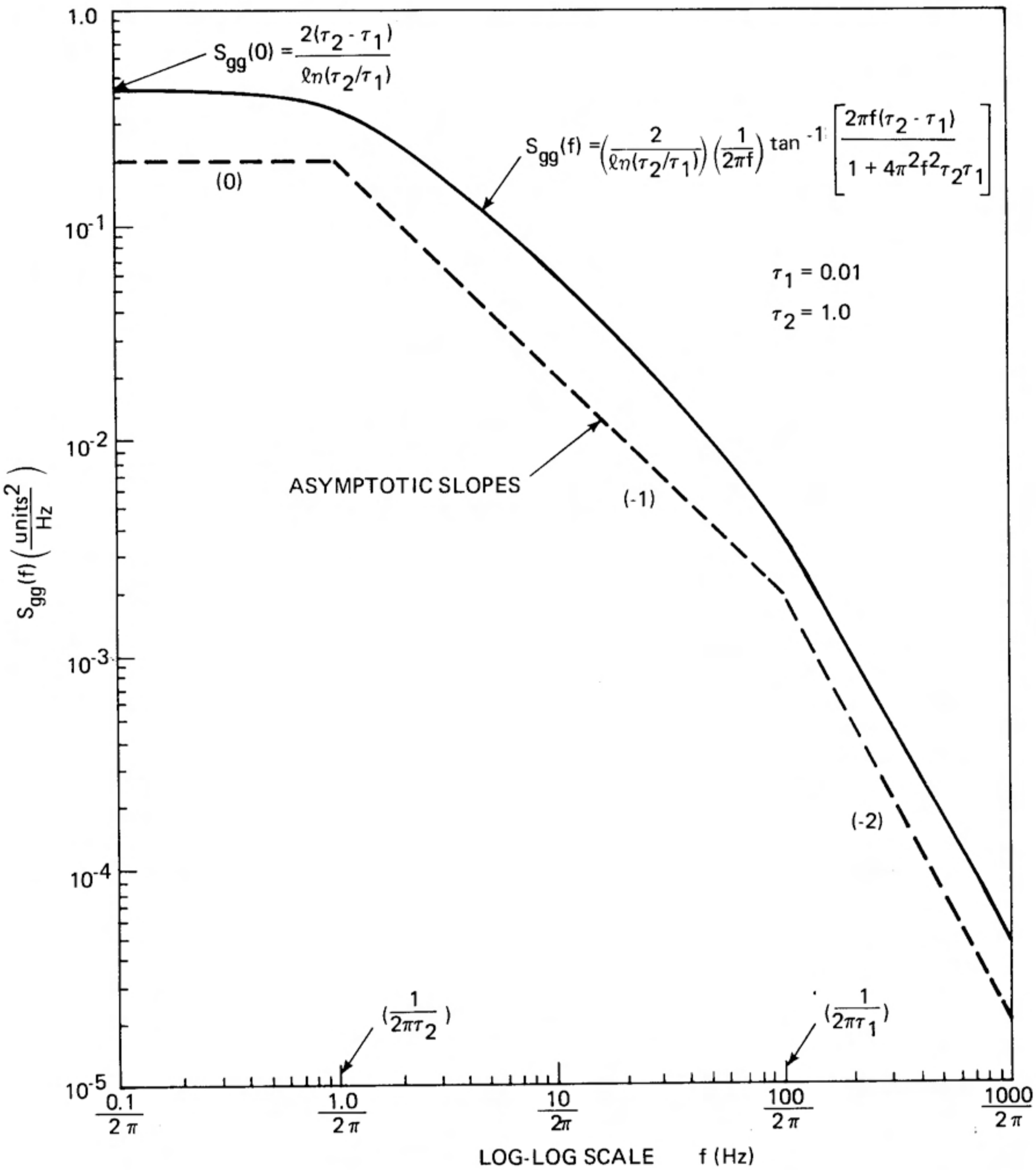

Figure 7. Power spectral density of gyroscopic noise model.

$$E\{x(\gamma, 0)x(\beta, 0)\} = \frac{W\gamma}{2} \cdot \delta(\gamma - \beta) \tag{22}$$

In more rigorous form, Eq. (18), (20), and (21) are shorthand for:

$$g(t) = \int_{\tau_1}^{\tau_2} p_d(\tau)e^{-t/\tau}\, d\mu(\tau, 0) + \int_{\tau_1}^{\tau_2}\int_0^t p_d(\tau)e^{-(t-s)/\tau}\, d\beta(\tau, s) \tag{23}$$

where the first integral, the initial condition propagation, is a Wiener integral and the second is a "two-dimensional Wiener integral", defined in Appendix B. In this appendix we also discuss the two-dimensional Wiener process $\beta(\tau, s)$ whose (formal) mixed double partial derivative is the two-dimensional white noise, $w(\tau, s)$, in Eq. (20). Note that the term "two-dimensional Wiener process" has two different meanings in the literature, to which, for lack of a better name, we are now adding a third. First, by a "two-dimensional Wiener process" it is sometimes meant a map

$$\alpha : R \to R^2 \tag{24}$$

representing Brownian motion in the plane[27,28]. Second, the term can refer[29,30] to a map $\gamma : R^2 \to R$ satisfying:

$$E[\gamma(\underline{x})] = 0 \tag{25}$$

$$E[\gamma(\underline{x})\gamma(\underline{y})] = \frac{1}{2}(||\underline{x}|| + ||\underline{y}|| - ||\underline{x} - \underline{y}||) \tag{26}$$

where:

$$||\underline{x}|| \triangleq [\sum_{i=1}^{2} (x_i)^2]^{1/2} \tag{27}$$

From Eq. (26) we obtain:

$$E[(\gamma(\underline{x}) - \gamma(\underline{y}))^2] = ||\underline{x} - \underline{y}|| \tag{28}$$

In our context by "two-dimensional Wiener process" we mean a map:

$$\beta : R^2 \to R$$

with covariance function satisfying (in contrast with Eq. (26)):

$$E[\beta(\tau, t)\beta(\sigma, s)] = W \times \min(\tau, \sigma) \times \min(t, s) \tag{29}$$

As desired, the formal mixed double partial derivative of $\beta(\tau, t)$ is a "two-dimensional white noise", $w(\tau, t)$, satisfying Eq. (20). (See Appendix B):

$$E[w(\tau, t)w(\sigma, s)] = W \times \delta(\tau - \sigma) \times \delta(t - s) \tag{30}$$

Equation (17) for the power spectral density of the gyroscopic noise is also rederived in Appendix B, using the more rigorous definitions. Incidentally, alternative models, in terms of diffusion mechanisms, for stochastic processes with the power spectral characteristics of gyroscopic noise are discussed in Ref. (23) and (24). Note however that because our gyro noise filter is a linear estimator only the second-order properties of the gyroscopic noise influence the filter mean-squared error sequence. Thus all mathematical models which generate stochastic processes with the same PSD as gyro noise (hence the same second-order properties as gyro noise) will yield the same optimal (minimum variance) filter. In the next section we introduce the estimation problem which we shall consider.

SECTION 3

THE GYRO NOISE FILTERING PROBLEM

Having obtained a model for the generation of gyroscopic noise, we wish to use this model in order to filter away gyroscopic noise from gyro output signals composed of the sum of:

(1) Gyro response to actual angular rate inputs to the gyroscope, or changes in gravitational field.

and

(2) Gyroscopic noise.

(Note that for subsequent clarity we shall call (1) above simply the "input", which then, as outlined here, is corrupted by additive gyroscopic noise to yield the total gyro output.) This problem will be quantitatively defined later in this section.

In order to separate gyro noise from gyro output signals due to actual angular rate inputs, there are at least three approaches which may be taken. First, for stationary inputs we may use knowledge of the input autocorrelation function and $R_{gg}(\alpha)$ to solve the Wiener-Hopf equation for the optimal causal, time-invariant, linear filter for steady-state estimation of the input signal. Actually, we would be solving a discrete summation version of the Wiener-Hopf integral equation, since samples, instead of continuous functions of time, are involved. This approach suffers from two difficulties. First, we are able to deal only with stationary input processes. Second, only the steady-state optimal filter is found.

The second approach, which would permit Kalman-Bucy filtering of the gyro output signal, is to model gyro noise as the output of a linear system driven by (possibly multidimensional) white noise. It is clear that gyro noise cannot be modeled as the output of a finite-dimensional time-invariant linear system to scalar white noise since, by Eq. (1), we would have:

$$S_{gg}(f) = S_{xx}(f)\,|H(f)|^2 = C|H(f)|^2 \qquad (31)$$

$H(f)$ would be rational, and $S_{gg}(f)$ would be asymptotically characterized by power spectral behavior proportional to even powers of frequency. Recall that gyro noise exhibits (1/f) behavior. It can also be demonstrated that the scalar response of a finite-dimensional time-invariant linear system to finite-dimensional stationary white noise is also characterized asymptotically by power spectral behavior proportional to even powers of frequency, ruling out this form of system for a model of gyro noise. Nonetheless, as we shall show in Section 5, it is possible to approximate gyroscopic noise PSD arbitrarily closely with finite-dimensional systems, and consequently, as we shall also show, we can construct a finite-dimensional Kalman filter whose mean-squared error is arbitrarily close to that of the optimal filter which we shall derive in this section.

The third approach, the one we shall pursue, is to retain an infinite-dimensional model with the state a function of continuous parameter (τ), and an output equation using a weighted integral of the state function. A conditional expectation filter is then formulated to optimally separate the actual input from the additive gyro noise. We now discuss this approach. The results are summarized in Eq. (101) to (109).

Because of the fact that gyro output is sampled discretely in time, we must discretize Eq. (18) and (21). From Eq. (18) and the variation of constants formula, we obtain:

$$x(\tau,\ n\tilde{T} + \tilde{T}) = e^{-\tilde{T}/\tau}\ x(\tau,\ n\tilde{T}) + \int_{n\tilde{T}}^{n\tilde{T}+\tilde{T}} e^{-(n\tilde{T}+\tilde{T}-\sigma)/\tau} \cdot w(\tau,\ \sigma)d\sigma \tag{32}$$

(Where $\tilde{T}$ is the sample time increment.) Make the following identifications in Eq. (32):

$$w(\tau,\ n) = \int_{n\tilde{T}}^{(n+1)\tilde{T}} e^{-([n+1]\tilde{T}-\sigma)/\tau} \cdot w(\tau,\ \sigma)d\sigma \tag{33}$$

$$\Phi_\tau = e^{-\tilde{T}/\tau} \tag{34}$$

The discrete-time system is then given by:

$$x(\tau,\ n+1) = \Phi_\tau x(\tau,\ n) + w(\tau,\ n) \tag{35}$$

$$g(n) = \int_{\tau_1}^{\tau_2} p_d(\tau) \cdot x(\tau,\ n)d\tau \tag{36}$$

It is easily verified, using Eq. (16) and (20) that:

$$E\{w(\tau, n)w(\gamma, m)\} = (\frac{W\tau}{2})(1 - e^{-2\tilde{T}/\tau})\delta_{n,m}\delta(\tau - \gamma)$$

$$= (\ell n(\tau_2/\tau_1))(\tau)(1 - e^{-2\tilde{T}/\tau})\delta_{n,m}\delta(\tau - \gamma)$$

$$E\{w(\tau, n)w(\gamma, m)\} \triangleq Q(\tau)\delta_{n,m}\delta(\tau - \gamma) \tag{37}$$

where $\delta_{n,m}$ is the Kronecker delta function, defined by:

$$\delta_{n,m} = \begin{cases} 1 \; ; \; n = m \\ 0 \; ; \; n \neq m \end{cases}$$

The covariance propagation equation is given[12] by:

$$E\{x(\tau, n + 1)x(\gamma, n + 1)\} = \Phi_\tau \; E\{x(\tau, n)x(\gamma, n)\}\Phi_\gamma + E\{w(\tau, n)w(\gamma, n)\}$$

$$E\{x(\tau, n + 1)x(\gamma, n + 1)\} = \Phi_\tau \; E\{x(\tau, n)x(\gamma, n)\}\Phi_\gamma + Q(\tau)\delta(\tau - \gamma) \tag{38}$$

The steady-state covariance is thus:

$$\lim_{n\to\infty} E\{x(\tau,n)x(\gamma,n)\} = \sum_{p=0}^{\infty} (e^{-\tilde{T}/\tau})^p \; (\frac{W\tau}{2})(1-e^{-2\tilde{T}/\tau})(e^{-\tilde{T}/\gamma})^p \; \delta(\tau - \gamma)$$

$$= (\frac{W\tau}{2})\delta(\tau - \gamma) \tag{39}$$

(where the last equality follows because the summation is geometric) The stationary gyroscopic noise is taken to be in steady state at time zero, thus:

$$E\{x(\tau, 0)x(\gamma, 0)\} = \frac{W\tau}{2} \cdot \delta(\tau - \gamma) \; ; \quad E\{x(\tau, 0)\} = 0 \tag{40}$$

which, as expected, is exactly the answer we had in the continuous time case (Eq. 22).

We desire to solve the following conditional expectation filtering problem (diagrammed in Figure 8). Consider a Gaussian input signal $p_1(n)$ (if $p_1(n)$ is not Gaussian, then the estimate which we obtain is the minimum mean-squared error linear estimate, and not necessarily the conditional expectation) generated as the scalar output of a finite-dimensional (state $\underline{a}(n)$) linear system driven by finite-dimensional discrete white noise. Add to the signal the sum of gyro noise $g(n)$ and an additional scalar white noise $v(n)$, which may, for example, represent roundoff noise in digital data transmission. Discrete white noise $v(n)$ may be left out if desired for a particular application, with the possibility of added computational problems, which, incidentally, can be overcome by decreasing the dimensionality of the state in the formulation of the estimation problem. Call this sum of three signals z(n). It is desired to estimate $p_1(n)$, or $\underline{a}(n)$ itself, given the output signal z(n). Thus: (where $p(\tau) \triangleq p_d(\tau)$)

$$z(n) = p_1(n) + g(n) + v(n) = \underline{H}(n)\underline{a}(n) + \int_{\tau_1}^{\tau_2} p(\tau)x(\tau, n)d\tau + v(n) \tag{41}$$

Specifically, the finite-dimensional linear system is given by:

$$\underline{a}(n+1) = \underline{\Phi}(n)\underline{a}(n) + \underline{B}(n)\underline{u}(n) \quad (\underline{a} \text{ is } n\times 1) \tag{42}$$

$$p_1(n) = \underline{H}(n)\underline{a}(n) \tag{43}$$

where:

$$E[\underline{a}(0)] = \underline{0} \tag{44}$$

$$E[\underline{a}(0)\underline{a}'(0)] \triangleq \underline{P} \tag{45}$$

$$E[\underline{u}(n)] = \underline{0} \quad ; \quad (\forall n) \tag{46}$$

$$E[\underline{u}(n)\underline{u}'(m)] \triangleq \underline{Q}_1(n)\delta_{n,m} \quad ; \quad (\forall n,m) \tag{47}$$

and the white noise term v(n) is given by:

$$E[v(n)] = 0 \tag{48}$$

$$E[v(n)v(m)] \triangleq r(n)\delta_{n,m} \tag{49}$$

Further, we assume that $\underline{a}(0)$, $x(\tau, 0)$, $\underline{u}(n)$, $w(\tau, m)$, and $v(q)$ are statistically independent $(\forall n,m,q,\tau)$.

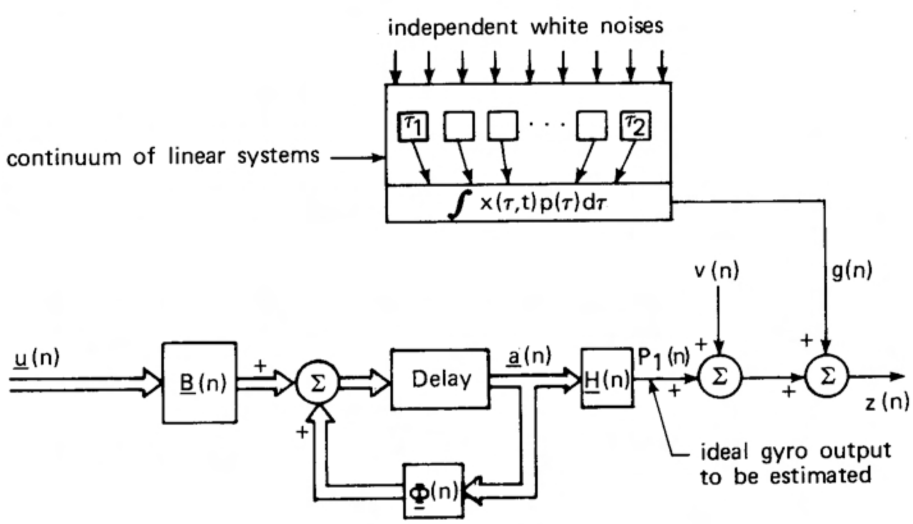

Figure 8. Block diagram displaying noise and signal addition.

The conditional expectation filter estimates the state $\underline{a}(n)$ of the finite-dimensional linear system, and the state $x(\tau, n)$ $(\tau \varepsilon [\tau_1, \tau_2])$ of the infinite-dimensional system generating the gyro noise $g(n)$. We shall give the equations for this filter, which is derived in Appendix C using the ideas of the derivation of the Kalman-Bucy filter given by Schweppe[12] for the finite-dimensional case. A more general derivation of discrete-time infinite-dimensional filtering can be found, for example, in Ref.(13).

Define the following conditional expectation estimates:

$$\hat{x}(\tau, n|m) = E\{x(\tau, n) | (z(1), \ldots, z(m)) \triangleq \underline{z}_m\} \tag{50}$$

$$\underline{\hat{a}}(n|m) = E\{\underline{a}(n) | \underline{z}_m\} \tag{51}$$

To formulate the filter we further define the following quantities:

A scalar function of two variables (γ, β):

$$\Sigma_{\gamma,\beta}(n|n) = E\{[x(\gamma, n) - \hat{x}(\gamma, n|n)][x(\beta, n) - \hat{x}(\beta, n|n)]\} \tag{52}$$

and similarly:

$$\Sigma_{\gamma,\beta}(n+1|n) = E\{[x(\gamma, n+1) - \hat{x}(\gamma, n+1|n)][x(\beta, n+1) - \hat{x}(\beta, n+1|n)]\} \tag{53}$$

A column vector function of one variable (τ):

$$\Sigma_{\underline{a},\tau}(n|n) = E\{[\underline{a}(n) - \underline{\hat{a}}(n|n)][x(\tau, n) - \hat{x}(\tau, n|n)]\} \tag{54}$$

and similarly:

$$\Sigma_{\underline{a},\tau}(n+1|n) = E\{[\underline{a}(n+1) - \underline{\hat{a}}(n+1|n)][x(\tau, n+1) - \hat{x}(\tau, n+1|n)]\} \tag{55}$$

A row vector function of one variable (τ):

$$\Sigma_{\tau,\underline{a}}(n|n) = E\{[x(\tau, n) - \hat{x}(\tau, n|n)][\underline{a}(n) - \underline{\hat{a}}(n|n)]'\} \tag{56}$$

and similarly:

$$\Sigma_{\tau,\underline{a}}(n+1|n) = E\{[x(\tau, n+1) - \hat{x}(\tau, n+1|n)][\underline{a}(n+1) - \underline{\hat{a}}(n+1|n)]'\} \tag{57}$$

Finally, an $(n\times n)$ matrix:

$$\Sigma_{\underline{a},\underline{a}}(n|n) = E\{[\underline{a}(n) - \underline{\hat{a}}(n|n)][\underline{a}(n) - \underline{\hat{a}}(n|n)]'\} \tag{58}$$

and similarly:

$$\Sigma_{\underline{a},\underline{a}}(n+1|n) = E\{[\underline{a}(n+1) - \underline{\hat{a}}(n+1|n)][\underline{a}(n+1) - \underline{\hat{a}}(n+1|n)]'\} \tag{59}$$

It is shown in Appendix C that the conditional expectation filter for the states $\underline{a}(n)$ and $x(\tau, n)$ is given in recursive form by:

$$\hat{x}(\tau,\ n+1|n+1) = \Phi_\tau \hat{x}(\tau,\ n|n) + A_{12,\tau}(n+1)A_{22}^{-1}(n+1)[z(n+1) - \int_{\tau_1}^{\tau_2} p(\tau)\Phi_\tau \hat{x}(\tau,\ n|n)d\tau - \underline{H}(n+1)\underline{\Phi}(n)\underline{\hat{a}}(n|n)] \tag{60}$$

where $\tau \varepsilon [\tau_1,\ \tau_2]$, and:

$$\underline{\hat{a}}(n+1|n+1) = \underline{\Phi}(n)\underline{\hat{a}}(n|n) + A_{12,\underline{a}}(n+1)A_{22}^{-1}(n+1)[z(n+1) - \int_{\tau_1}^{\tau_2} p(\tau)\Phi_\tau \hat{x}(\tau,\ n|n)d\tau - \underline{H}(n+1)\underline{\Phi}(n)\underline{\hat{a}}(n|n)] \tag{61}$$

Where we have defined the following quantities:

First, a scalar:

$$A_{22}(n+1) = \begin{bmatrix} r(n+1) + \underline{H}(n+1)\Sigma_{\underline{a},\underline{a}}(n+1|n)\underline{H}'(n+1) \\ + \int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2} p(\gamma)p(\beta)\Sigma_{\gamma,\beta}(n+1|n)d\gamma d\beta \\ + \int_{\tau_1}^{\tau_2} p(\tau)\Sigma_{\tau,\underline{a}}(n+1|n)\underline{H}'(n+1)d\tau \\ + \int_{\tau_1}^{\tau_2} p(\tau)\underline{H}(n+1)\Sigma_{\underline{a},\tau}(n+1|n)d\tau \end{bmatrix} \tag{62}$$

Second, another scalar, a function of (τ):

$$A_{12,\tau}(n+1) = [\int_{\tau_1}^{\tau_2} p(\gamma)\Sigma_{\tau,\gamma}(n+1|n)d\gamma + \Sigma_{\tau,\underline{a}}(n+1|n)\underline{H}'(n+1)] \tag{63}$$

Finally, a column vector:

$$A_{12,\underline{a}}(n+1) = [\int_{\tau_1}^{\tau_2} p(\tau)\Sigma_{\underline{a},\tau}(n+1|n)d\tau + \Sigma_{\underline{a},\underline{a}}(n+1|n)\underline{H}'(n+1)] \tag{64}$$

Further, it is shown in Appendix C that the covariance equations are:

$$\Sigma_{\gamma,\beta}(n+1|n) = \Phi_\gamma \Sigma_{\gamma,\beta}(n|n)\Phi_\beta + Q(\gamma)\delta(\gamma - \beta) \tag{65}$$

$$\Sigma_{\tau,\underline{a}}(n+1|n) = \Phi_\tau \Sigma_{\tau,\underline{a}}(n|n)\underline{\Phi}'(n) \tag{66}$$

$$\Sigma_{\underline{a},\tau}(n+1|n) = \underline{\Phi}(n)\Sigma_{\underline{a},\tau}(n|n)\Phi_{\tau} \tag{67}$$

$$\Sigma_{\underline{a},\underline{a}}(n+1|n) = \underline{\Phi}(n)\Sigma_{\underline{a},\underline{a}}(n|n)\underline{\Phi}'(n) + \underline{B}(n)\underline{Q}_1(n)\underline{B}'(n) \tag{68}$$

$$\Sigma_{\gamma,\beta}(n+1|n+1) = \Sigma_{\gamma,\beta}(n+1|n) - A_{12,\gamma}(n+1)A_{22}^{-1}(n+1)A_{12,\beta}(n+1) \tag{69}$$

$$\Sigma_{\tau,\underline{a}}(n+1|n+1) = \Sigma_{\tau,\underline{a}}(n+1|n) - A_{12,\tau}(n+1)A_{22}^{-1}(n+1)A_{12,\underline{a}}'(n+1) \tag{70}$$

$$\Sigma_{\underline{a},\tau}(n+1|n+1) = \Sigma_{\underline{a},\tau}(n+1|n) - A_{12,\underline{a}}(n+1)A_{22}^{-1}(n+1)A_{12,\tau}(n+1) \tag{71}$$

$$\Sigma_{\underline{a},\underline{a}}(n+1|n+1) = \Sigma_{\underline{a},\underline{a}}(n+1|n) - A_{12,\underline{a}}(n+1)A_{22}^{-1}(n+1)A_{12,\underline{a}}'(n+1) \tag{72}$$

The initial conditions for Eq. (65)-(72) are obtained from Eq. (40), Eq. (45), and the assumption that the input signal and gyro noise are independent. That is:

$$\Sigma_{\gamma,\beta}(0|0) = \frac{W\gamma}{2}\delta(\gamma - \beta) \tag{73}$$

$$\Sigma_{\tau,\underline{a}}(0|0) = \underline{0} \tag{74}$$

$$\Sigma_{\underline{a},\tau}(0|0) = \underline{0} \tag{75}$$

$$\Sigma_{\underline{a},\underline{a}}(0|0) = \underline{P} \tag{76}$$

Boundary conditions for Eq. (60) and Eq. (61) are obtained from Eq. (40) and Eq. (44):

$$\hat{x}(\tau, 0|0) = 0 \;; \quad (\tau \in [\tau_1, \tau_2]) \tag{77}$$

$$\underline{\hat{a}}(0|0) = \underline{0} \tag{78}$$

Note that in order to propagate the covariances in Eq. (69) to (72), we must compute the integrals in Eq.(62) to Eq.(64). Another integral computation is necessary in the evaluation of the residual (final factors in the second terms) in Eq.(60) and (61). Equations (62) to (72) bear some resemblance to the familiar discrete Riccati equations, and we shall now express them in this form. The result is summarized in Eq.(101) to (109).

In order to carry out our aim, we juxtapose the states of the gyro noise system and the finite-dimensional system generating $p_1(n)$.

$$x(n) \triangleq \begin{bmatrix} x(\tau, n) \\ \underline{a}(n) \end{bmatrix} \tag{79}$$

Here x(n) is a vector whose first element is a function of (τ) over

$[\tau_1, \tau_2]$. Vectors constructed in this manner will be considered as members of a vector space V_s, with inner product defined as follows:

$$<x, y>_{V_s} \triangleq \begin{bmatrix} x(\tau) \\ \underline{a} \end{bmatrix} \circledS \begin{bmatrix} y(\tau) \\ \underline{b} \end{bmatrix} \triangleq \int_{\tau_1}^{\tau_2} x(\tau)y(\tau)d\tau + \underline{a}'\underline{b} \tag{80}$$

Letting $x(\tau)$ be functions in $L^2([\tau_1, \tau_2])$, V_s is a Hilbert space. The norm on V_s induced by the inner product is:

$$||x|| = (<x, x>_{V_s})^{1/2} = \left(\begin{bmatrix} x(\tau) \\ \underline{a} \end{bmatrix} \circledS \begin{bmatrix} x(\tau) \\ \underline{a} \end{bmatrix}\right)^{1/2} = \left[\int_{\tau_1}^{\tau_2} x^2(\tau)d\tau + \underline{a}'\underline{a}\right]^{1/2} \tag{81}$$

We now define some operators. First:

$$H_s(n) : V_s \to R \; ; \quad H_s(n)(x) = \int_{\tau_1}^{\tau_2} p(\tau)(x(\tau))d\tau + \underline{H}(n)\underline{a} \tag{82}$$

(where $\underline{H}(n)$ is defined in Eq. (43))
The adjoint operator of $H_s(n)$ (i.e. $H_s^*(n) : R \to V_s$ such that $<\beta, H_s(n)x>_R = <H_s^*(n)\beta, x>_{V_s}$, where the inner product on R is multiplication ($\beta \in R$, $x \in V_s$)) is shown in Appendix A to be given by:

$$H_s^*(n) : R \to V_s \; ; \quad H_s^*(n)(\beta) = \begin{bmatrix} p(\tau)\beta \\ \underline{H}'(n)\beta \end{bmatrix} \tag{83}$$

We now define a state transition operator $\Phi_s(n) : V_s \to V_s$:

$$\Phi_s(n) : V_s \to V_s \; ; \quad \Phi_s(n)(x) = \begin{bmatrix} \Phi_\tau & 0 \\ 0 & \underline{\Phi}(n) \end{bmatrix} \begin{bmatrix} x(\tau) \\ \underline{a} \end{bmatrix} = \begin{bmatrix} \Phi_\tau x(\tau) \\ \underline{\Phi}(n)\underline{a} \end{bmatrix} \tag{84}$$

The adjoint of $\Phi_s(n)$ also maps V_s into V_s, and is given by ($y \in V_s$):

$$\Phi_s^*(n) : V_s \to V_s \; ; \quad \Phi_s^*(n)(y) = \begin{bmatrix} \Phi_\tau y(\tau) \\ \underline{\Phi}'(n)\underline{b} \end{bmatrix} \tag{85}$$

By Eq.(79, 84) and model equations Eq. (35, 42), we have:

$$x(n+1) = \Phi_s(n)x(n) + \begin{bmatrix} w(\tau, n) \\ \underline{B}(n)\underline{u}(n) \end{bmatrix} \tag{86}$$

Also, by Eq.(79) and (82) and model equations Eq. (36), (41), and (43) we have:

$$z(n) = H_s(n)x(n) + v(n) \tag{87}$$

Equations (63) and (64) define transformations:

$$A_{12,\tau}(n+1) : R \to \text{function of } (\tau)$$

$$A_{12,\underline{a}}(n+1) : R \to \text{column vector } (n \times 1)$$

If we juxtapose these, we obtain a transformation:

$$A_{12}(n+1) : R \to V_s \; ; \quad (\alpha \in R)$$

$$A_{12}(n+1)(\alpha) \triangleq \left[\begin{array}{c} \left(\int_{\tau_1}^{\tau_2} p(\gamma)\Sigma_{\tau,\gamma}(n+1|n)d\gamma\right)(\alpha) + \Sigma_{\tau,\underline{a}}(n+1|n)\underline{H}'(n+1)(\alpha) \\ \hline \left(\int_{\tau_1}^{\tau_2} p(\tau)\Sigma_{\underline{a},\tau}(n+1|n)d\tau\right)(\alpha) + \Sigma_{\underline{a},\underline{a}}(n+1|n)\underline{H}'(n+1)(\alpha) \end{array}\right] \begin{array}{l} \}\ 1 \\ \\ \}\ n \end{array} \tag{88}$$

We now define an operator $\Sigma_s(n+1|n) : V_s \to V_s$:

$$\Sigma_s(n+1|n)(y) = \underbrace{\left[\begin{array}{c|c} \int_{\tau_1}^{\tau_2} \Sigma_{\tau,\gamma}(n+1|n)(\cdot)d\gamma & \Sigma_{\tau,\underline{a}}(n+1|n) \\ \hline \int_{\tau_1}^{\tau_2} \Sigma_{\underline{a},\gamma}(n+1|n)(\cdot)d\gamma & \Sigma_{\underline{a},\underline{a}}(n+1|n) \end{array}\right]}_{1 \qquad\qquad n} \begin{array}{l} \}\ 1 \\ \\ \}\ n \end{array} \begin{bmatrix} y(\gamma) \\ \\ \underline{b} \end{bmatrix} \tag{89}$$

So that:

$$\Sigma_s(n+1|n)(y) = \begin{bmatrix} \int_{\tau_1}^{\tau_2} \Sigma_{\tau,\gamma}(n+1|n)y(\gamma)d\gamma + \Sigma_{\tau,\underline{a}}(n+1|n)\underline{b} \\ \\ \int_{\tau_1}^{\tau_2} \Sigma_{\underline{a},\gamma}(n+1|n)y(\gamma)d\gamma + \Sigma_{\underline{a},\underline{a}}(n+1|n)\underline{b} \end{bmatrix} \in V_s \tag{90}$$

Define $\Sigma_s(n|n)$ similarly. Now compare Eq. (88) and Eq. (90) to get:

$$A_{12}(n+1)(\alpha) = \Sigma_s(n+1|n)H_s^*(n+1)(\alpha) \tag{91}$$

Thus:

$$A_{12}^*(n+1) \; : \; V_s \to R = H_s(n+1)\Sigma_s^*(n+1|n) \tag{92}$$

Straightforward algebraic manipulation, similar to that in Appendix A, shows that:

$$\Sigma_s^*(n+1|n) = \Sigma_s(n+1|n) \tag{93}$$

so that $\Sigma_s(n+1|n)$ (and similarly $\Sigma_s(n|n)$) is self-adjoint. Continued manipulation shows that:

$$A_{22}(n+1) = [r(n+1) + H_s(n+1)\Sigma_s(n+1|n)H_s^*(n+1)] \tag{94}$$

and thus that Eq. (69) to (72) become:

$$\Sigma_s(n+1|n+1) = \Sigma_s(n+1|n) - \Sigma_s(n+1|n)H_s^*(n+1)[r(n+1) + H_s(n+1)\Sigma_s(n+1|n)H_s^*(n+1)]^{-1} H_s(n+1)\Sigma_s(n+1|n) \tag{95}$$

Group Eq. (65) to (68) and manipulate them (see Appendix A) to obtain:

$$\Sigma_s(n+1|n) = \Phi_s(n)\Sigma_s(n|n)\Phi_s^*(n) + D(n) \tag{96}$$

where we define $D(n) \; : \; V_s \to V_s$ by:

$$D(n)\left(\begin{bmatrix} f(\tau) \\ \underline{d} \end{bmatrix}\right) = \begin{bmatrix} Q(\tau) \int_{\tau_1}^{\tau_2} \delta(\gamma - \tau) f(\gamma) d\gamma \\ \underline{B}(n)\underline{Q}_1(n)\underline{B}'(n)\underline{d} \end{bmatrix} = \begin{bmatrix} Q(\tau) f(\tau) \\ \underline{B}(n)\underline{Q}_1(n)\underline{B}'(n)\underline{d} \end{bmatrix} \tag{97}$$

Note that the operator $Q(\tau)$ above is used as a multiplication operator. The initial condition for Eq. (96) is obtained using Eq. (73) to (76) and Eq. (89) as: (Note again that $(\frac{W\tau}{2})$ is a multiplication operator.)

$$\Sigma_s(0|0) = \begin{bmatrix} \frac{W\tau}{2} & \underline{0} \\ \underline{0} & \underline{P} \end{bmatrix} \tag{98}$$

Further, observing the above definitions, we see that the filtering equations (Eq. (60) and (61)) become (by Eq. (91) and (94)):

$$\hat{x}(n+1|n+1) = \Phi_s(n)\hat{x}(n|n) + \Sigma_s(n+1|n)H_s^*(n+1)[r(n+1) + H_s(n+1)\Sigma_s(n+1|n)H_s^*(n+1)]^{-1} \times [z(n+1) - H_s(n+1)\Phi_s(n)\hat{x}(n|n)] \tag{99}$$

with initial condition, from Eq. (77) and (78):

$$\hat{x}(0|0) = \begin{bmatrix} 0 \\ \underline{0} \end{bmatrix} \tag{100}$$

In summary, we have defined the augmented state:

$$x(n) = \begin{bmatrix} x(\tau, n) \\ \underline{a}(n) \end{bmatrix} \tag{101}$$

as a member of a vector space V_s, with inner product:

$$\langle x, y\rangle_{V_s} = \begin{bmatrix} x(\tau) \\ \underline{a} \end{bmatrix} \circledS \begin{bmatrix} y(\tau) \\ \underline{b} \end{bmatrix} = \int_{\tau_1}^{\tau_2} x(\tau)y(\tau)d\tau + \underline{a}'\underline{b} \tag{102}$$

Our state and output equations (Eq. (35), (36), (41), (42), and (43)) become:

$$x(n+1) = \Phi_s(n)x(n) + \begin{bmatrix} w(\tau, n) \\ \underline{B}(n)\underline{u}(n) \end{bmatrix} \tag{103}$$

and:

$$z(n) = H_s(n)x(n) + v(n) \tag{104}$$

where $\Phi_s(n)$ and $H_s(n)$ are linear transformations defined in Eq. (84) and Eq. (82), respectively. The recursive conditional expectation filtering equation for the augmented state x(n) is given by:

$$\begin{aligned} \hat{x}(n+1|n+1) &= \Phi_s(n)\hat{x}(n|n) \\ &+ \Sigma_s(n+1|n)H_s^*(n+1)[r(n+1) + H_s(n+1)\Sigma_s(n+1|n)H_s^*(n+1)]^{-1} \\ &\times [z(n+1) - H_s(n+1)\Phi_s(n)\hat{x}(n|n)] \end{aligned} \tag{105}$$

which corresponds to the form of the discrete-time Kalman filter for finite-dimensional linear systems. The initial conditions for Eq. (105) are given by:

$$\hat{x}(0|0) = \begin{bmatrix} 0 \\ \underline{0} \end{bmatrix} \tag{106}$$

The following operator equations are used to determine the "gains":

$$\Sigma_s(n+1|n) = \Phi_s(n)\Sigma_s(n|n)\Phi_s^*(n) + \begin{bmatrix} Q(\tau), & \underline{0} \\ \underline{0}, & \underline{B}(n)\underline{Q}_1(n)\underline{B}'(n) \end{bmatrix} \tag{107}$$

$$\Sigma_s(n+1|n+1) = \Sigma_s(n+1|n) - \Sigma_s(n+1|n)H_s^*(n+1)[r(n+1) + H_s(n+1)\Sigma_s(n+1|n)H_s^*(n+1)]^{-1} H_s(n+1)\Sigma_s(n+1|n) \quad (108)$$

Equations (107) and (108) correspond to the discrete-time matrix Riccati equations for finite-dimensional Kalman filtering. The initial conditions for Eq. (107) and (108) are given by:

$$\Sigma_s(0|0) = \begin{bmatrix} \frac{W\tau}{2} & \underline{0} \\ \underline{0} & \underline{P} \end{bmatrix} \quad (109)$$

We have thus seen that the conditional expectation filtering equations may be expressed in a Kalman-Bucy filter form, with an operator Riccati equation. In the next section we discuss properties of infinite-dimensional Kalman filters and apply these to the gyro noise filter which we have derived.

SECTION 4

PROPERTIES OF DISCRETE-TIME HILBERT SPACE KALMAN FILTERS

In Section 3 we derived the equations defining the conditional expectation filter for the gyro noise filtering problem. The equations were shown to take the form of the discrete-time Kalman filter, with an associated operator Riccati equation. The purpose of this section is twofold. First, we shall discuss conditions to guarantee the asymptotic stability of Hilbert space Kalman filters (note that Deyst and Price(15) discuss conditions for asymptotic stability which unfortunately are not satisfied in the case of gyro noise filtering, due to the scalar observations $z(n)$ and the infinite dimensional state $x(n)$, necessitating a different approach here). Second, we shall restrict attention to the time-invariant case, and discuss conditions for the convergence of the Riccati operator to a unique steady-state value from arbitrary positive semidefinite initial conditions, positive definiteness of this steady-state operator, uniqueness of this operator as a positive semidefinite solution of the steady-state Riccati equation, and asymptotic stability of the resulting steady-state Kalman filter. Results similar to these are obtained by Caines and Mayne(26) for time-invariant models, where the Hilbert space, however, is restricted to being finite dimensional. Finally, we shall specify the sufficient conditions found in Subsections 4.1 and 4.2 to the gyro noise filtering problem, and express them as conditions on the system generating the signal to be recovered. For the interested reader, topics in the theory of Hilbert spaces and linear operators are discussed by Berberian(17), while Bachman and Narici(16) discuss types of operator convergence.

4.1 Behavior of the Kalman Filter for Time-Varying Models

In Section 3 we saw that the gyro noise filtering problem can be solved by estimation (from observations $\{z(i)\}$) of the state of the following dynamical system:

$$x(n+1) = \Phi_s(n)x(n) + w(n) \tag{110}$$

$$z(n) = H_s(n)x(n) + v(n) \tag{111}$$

where state $x(n)$, a vector in Hilbert space V_s, is composed of a function of $\tau \,\varepsilon [\tau_1, \tau_2]$ augmented by an n-dimensional vector, and $z(n)$ is a scalar. $w(n)$ and $v(n)$ are white noises, in the appropriate vector spaces. The dot product in V_s is defined by:

$$\begin{bmatrix} f(\tau) \\ \underline{a} \end{bmatrix} \circledS \begin{bmatrix} g(\tau) \\ \underline{b} \end{bmatrix} = \int_{\tau_1}^{\tau_2} f(\tau)g(\tau)d\tau + \underline{a}'\underline{b} < \infty \tag{112}$$

The optimal estimator was shown to be given by the following:

$$\begin{aligned}\hat{x}(n+1|n+1) &= \Phi_s(n)\hat{x}(n|n) \\ &\quad + \Sigma_s(n+1|n)H_s^*(n+1)[r(n+1) + H_s(n+1)\Sigma_s(n+1|n)H_s^*(n+1)]^{-1} \\ &\quad \times [z(n+1) - H_s(n+1)\Phi_s(n)\hat{x}(n|n)]\end{aligned} \tag{113}$$

where:

$$E(v^2(n)) = r(n) \tag{114}$$

and $\Sigma_s(n+1|n)$ is found through the discrete-time Riccati equation:

$$\begin{aligned}\Sigma_s(n+1|n) &= \Phi_s(n)(\Sigma_s(n|n-1) \\ &\quad - \Sigma_s(n|n-1)H_s^*(n)[r(n) + H_s(n)\Sigma_s(n|n-1)H_s^*(n)]^{-1} \\ &\quad \times H_s(n)\Sigma_s(n|n-1))\Phi_s^*(n) + D(n)\end{aligned} \tag{115}$$

where $D(n)$ is a positive semidefinite operator, defined in the previous section, associated with the gyro and input noise models. The initial conditions are (note that by Eq. (115) $\Sigma_s(n+1|n)$ is self-adjoint as long as $\Sigma_s(0|-1)$ is self-adjoint):

$$\Sigma_s(0|-1) = \Sigma_0 \geq 0 \tag{116}$$

$$\hat{x}(0|0) = 0 \tag{117}$$

(There is no loss in generality in starting with $\Sigma_s(0|-1)$ instead of $\Sigma_s(0|0)$, since we may set $H_s(0) = 0$.) To preserve the generality of the treatment we will not explicitly use the fact that the measurements, $\{z(n)\}$, are scalar, making $r(n)$ a scalar. Henceforth in this section we will let the observation space V_0 and the state space V_s be arbitrary Hilbert spaces. We make the following assumptions:

$$\|D(n)\| < d \; ; \quad (\forall n), \; (\text{V}d \in R^+) \text{ (i.e. For some } d > 0)$$

$$\|H_s(n)\| < h \; ; \quad (\forall n), \; (\text{V}h \in R^+)$$

$$\|\Sigma_0\| < \sigma_0 \; ; \quad (\text{V}\sigma_0 \in R^+)$$

$$\|\Phi_s(n)\| < \phi \; ; \quad (\forall n), \; (\text{V}\phi \in R^+) \tag{118}$$

Self-adjoint operator $r(n) : V_0 \to V_0$ satisfies:

$$r_s\|z\|^2 \le \langle r(n)z,z\rangle_{V_0} \le r_b\|z\|^2 \; ; \quad (\forall n), \; (\text{V}r_s, r_b \in R^+), \; (\forall z \in V_0) \tag{118}$$

The following is the major result of this section:

Theorem 1: The Kalman filter is asymptotically stable (defined below) if the following conditions hold:

(1) The sequence $\{(H_s(n), \Phi_s(n))\}$ is uniformly detectable, i.e. there exist operators $\{L(n)\}$ such that:

$$\|L(n)\| < \ell \; ; \quad (\text{V}\ell \in R^+)$$

and:

$$\left\| \prod_{n=i}^{i+M-1} (\Phi_s(n) - L(n)H_s(n)) \right\| \le q \; ; \quad (\forall i)$$

for some $0 \le q < 1$, for some $M \in Z^+$ (i.e. some M a positive integer). (Here $\prod$ [L] means to apply succeeding operators on the left.)

(2) For some $N \in \{0, 1, \ldots\}$, all $T \in \{N, N+1, \ldots\}$, and all $p \in \{0, 1, \cdots, T-N\}$, we have that:

$$G\|x\|^2 \le x'\left(\sum_{i=p}^{p+N-1} [\Phi_s(T-p-1)\cdots\Phi_s(T-i)]D(T-i-1)[\Phi_s^*(T-i)\cdots\Phi_s^*(T-p-1)]\right)x \; ;$$

$$(\forall x \in V_s)$$

for some $G \in R^+$. Where $x'Fx$ denotes $\langle Fx, x\rangle_{V_s}$. The above may be written more simply as:

$$G\|x\|^2 \le x'\left(\sum_{k=s}^{s+N-1} [\Phi_s(s+N-1)\cdots\Phi_s(k+1)]D(k)[\Phi_s^*(k+1)\cdots\Phi_s^*(s+N-1)]\right)x \; ;$$

$$(\forall s \in \{0, 1, \ldots, T-N\})$$

(The summand here is $(D(s+N-1))$ for $(k = s+N-1)$. Summands throughout this discussion should be interpreted similarly.) By asymptotic stability we mean that for any given initial condition, $\hat{x}(0|0)$, the homogeneous propagation ($z(n) = 0$; $(\forall n \geq 0)$) of Eq. (113) will generate estimates satisfying

$$\lim_{n\to\infty}\left[\frac{\|\hat{x}(n|n)\|}{\|\hat{x}(0|0)\|}\right] = 0$$

In order to reach the above result, we first introduce the dual control problem:

$$y(n+1) = A(n)y(n) + B(n)u(n) \tag{119}$$

where we desire to choose the sequence of vectors (in the gyro noise case, scalars) $\{u(n)\}$, $n \in \{0, 1, \ldots, T-1\}$, such that the penalty function: (starting from a given initial state $y(0)$)

$$J(\{u(n)\}, (\Sigma_0, T), 0) = \frac{1}{2}\{\sum_{i=0}^{T-1} (y'(i)Q(i)y(i) + u'(i)e(i)u(i)) + y'(T)\Sigma_0 y(T)\} \tag{120}$$

is minimized. To make this the dual of our estimation problem, we choose:

$$\begin{aligned} B(n) &= H_s^*(T-1-n) \\ A(n) &= \Phi_s^*(T-1-n) \\ Q(n) &= D(T-1-n) \\ e(n) &= r(T-1-n) \end{aligned} \tag{121}$$

The solution of the dual control problem can be found in Ref. (14), or through dynamic programming, which uses the same argument as Ref. (32), except with completing-the-square instead of differentiation for the optimization of $u(n)$, because $u(n)$, a member of the arbitrary Hilbert space V_0 in the present discussion, is restricted to being finite dimensional in Ref. (32). The solution is given by:

$$u(i) = -[e(i) + B^*(i)K(i+1)B(i)]^{-1} B^*(i)K(i+1)A(i)y(i) \tag{122}$$

or, by Eq. (121):

$$u(i) = -[r(T-i-1) + H_s(T-i-1)K(i+1)H_s^*(T-i-1)]^{-1} H_s(T-i-1)K(i+1)\Phi_s^*(T-i-1)y(i) \tag{122a}$$

where we find $K(i)$ by backward-in-time recursion:

$$K(i) = A^*(i)(K(i+1) - K(i+1)B(i)[e(i) + B^*(i)K(i+1)B(i)]^{-1} B^*(i)K(i+1))A(i) + Q(i) \tag{123}$$

or by Eq. (121):

$$K(i) = \Phi_s(T-i-1)(K(i+1) - K(i+1)H_s^*(T-i-1)[r(T-i-1) + H_s(T-i-1)K(i+1)H_s^*(T-i-1)]^{-1} \times$$

$$\times H_s(T-i-1)K(i+1))\Phi_s^*(T-i-1) + D(T-i-1) \tag{123a}$$

where:

$$K(T) = \Sigma_0 \tag{124}$$

Comparison of Eq. (115), (116), (123a), and (124) shows that:

$$K(i, (\Sigma_0, T)) = \Sigma_s(T-i|T-i-1, (\Sigma_0, 0)) \tag{125}$$

(The left-hand-side notation means that K(i) which is generated by Eq. (123) and (124). The RHS notation means the $\Sigma_s(T-i|T-i-1)$ generated by Eq. (115) and (116).) The minimum cost-to-go of the dual control problem, from a given state y(i) at time (i), is given by:

$$J_{inf}(\{u(n)\}, (\Sigma_0, T), i) = \frac{1}{2}y'(i)K(i, (\Sigma_0, T))y(i) \ ; \quad (i = 0, \ldots, T) \tag{126}$$

We now prove the following lemma:

<u>Lemma 1</u>: $K(j, (0, T))$ is uniformly bounded for all T, $j \le T$, if the sequence $\{(H_s(n), \Phi_s(n))\}$ is uniformly detectable.

<u>Proof</u>: We first prove the assertion for the case where M = 1, that is when we have that $||\Phi_s(n) - L(n)H_s(n)|| \le q < 1$, $(\forall n)$, where $||L(n)|| < \ell$. Observing Eq. (119) and (121), we see that application, in the dual control problem, of control sequence:

$$\{u(n)\}_1 = L^*(T-n-1)y(n) \tag{127}$$

will lead to state y(n) being characterized by:

$$y(n) = [(\Phi_s^*(T-n) - H_s^*(T-n)L^*(T-n))\cdots(\Phi_s^*(T-1) - H_s^*(T-1)L^*(T-1))]y(0) \tag{128}$$

So that:

$$||y(n)|| \le q^n||y(0)|| \tag{129}$$

The cost-to-go with these controls (with no terminal penalty) is (from Eq. (120) and (121)):

$$J(\{u(n)\}_1, (0,T), 0) = \frac{1}{2} \sum_{n=0}^{T-1} (y'(n)D(T-n-1)y(n) + u'(n)r(T-n-1)u(n)) \tag{130}$$

But by the above estimates, and Eq. (118), we have that:

$$\sum_{n=0}^{T-1} [y'(n)D(T-n-1)y(n) + u'(n)r(T-n-1)u(n)]$$

$$\leq \sum_{n=0}^{T-1} [(\|y(0)\|q^n)^2 d + \|y(n)\|^2 \ell^2 \; r_b] \tag{131}$$

$$\leq \|y(0)\|^2 (d+\ell^2 \; r_b) \sum_{n=0}^{T-1} q^{2n}$$

$$\leq \|y(0)\|^2 (d+\ell^2 \; r_b)\left(\frac{1}{1-q^2}\right); \quad (\forall T) \tag{132}$$

However, by optimality:

$$\frac{1}{2}y'(0)K(0, (0,T))y(0) = J_{inf}(\{u(n)\}, (0,T), 0)$$

$$\leq J(\{u(n)\}_1, (0,T), 0)$$

$$\leq \frac{1}{2}\|y(0)\|^2 (d+\ell^2 \; r_b)\left(\frac{1}{1-q^2}\right) ; \quad (\forall T) \tag{133}$$

Equation (133) establishes an upper bound, k_0, on the gain of $K(0, (0,T))$, $(\forall T)$:

$$\|K(0, (0, T))\| \leq k_0 ; \quad (\forall T) \tag{134}$$

However, due to the structure of the dual control problem, with the same sequence of operators starting backward from whatever time is selected as T, we have that:

$$K(j, (0,T)) = K(0, (0,T-j)) ; \quad (0 \leq j \leq T)$$

hence:

$$\|K(j, (0,T))\| \leq k_0 ; \quad (0 \leq j \leq T) \tag{134a}$$

In the case where $(M \neq 1)$, we note first that with the application of the sequence of controls Eq. (127) we have:

$$\|y(n)\| \leq (\max_{0 \leq i \leq M-1} \{(\phi+\ell h)^i\}) \|y(0)\| \triangleq C_M \|y(0)\| ; (0 \leq n \leq M-1)$$

We may apply the argument of the above proof separately to each of the sequences $\{y(0), y(M), y(2M), \ldots\}$, $\{y(1), y(M+1), y(2M+1), \ldots\}$, ..., and finally $\{y(M-1), y(2M-1), \ldots\}$. We thus obtain as in Eq. (132) that (where $g(x)$ denotes the greatest integer in x):

$$\sum_{n=0}^{T-1} (y'(n)D(T-n-1)y(n) + u'(n)r(T-n-1)u(n))$$

$$= \sum_{i=0}^{M-1} \sum_{n=0}^{g(\frac{T-1}{M})} (y'(Mn+i)D(T - (Mn+i) - 1)y(Mn+i)$$

$$+ u'(Mn+i)r(T - (Mn+i) - 1)u(Mn+i))$$

$$\leq M[(C_M^2 \|y(0)\|^2)(d+\ell^2 r_b)\left(\frac{1}{1-q^2}\right)]$$

where for the first equality above we have set $D(j) = 0$, $r(j) = 0$; $(\forall j<0)$ in the double summation. Thus again, for some $k_0 > 0$, we have Eq. (134a).

QED

We now generalize Lemma 1:

<u>Lemma 2</u>: $K(j, (\Sigma_0, T))$ is uniformly bounded for all T, and $j \leq T$, if the sequence $\{(H_s(n), \Phi_s(n))\}$ is uniformly detectable.

<u>Proof</u>: It is clear from the construction of the controls in the proof of Lemma 1 that the following condition results, for any terminal time T:

$$\|y'(T)\Sigma_0 y(T)\| \leq C_M^2 \|y(0)\|^2 \|\Sigma_0\| \tag{135}$$

Then we have, from Eq. (120), (133), and (135):

$$\tfrac{1}{2}y'(0)K(0, (\Sigma_0,T))y(0) \leq \tfrac{1}{2}\|y(0)\|^2(d+\ell^2 r_b)M C_M^2\left(\frac{1}{1-q^2}\right)$$

$$+ \tfrac{1}{2}\|y(0)\|^2\|\Sigma_0\|C_M^2 \tag{136}$$

Thus, as in Eq. (134), we have that for some $k_{\Sigma_0} > 0$:

$$\|K(0, (\Sigma_0,T))\| \leq k_{\Sigma_0} \;; \quad (\forall T) \tag{137}$$

and, as in Eq. (134a):

$$\|K(j, (\Sigma_0,T))\| \leq k_{\Sigma_0} \;; \quad (\forall T,\ 0\leq j\leq T) \tag{137a}$$

QED

We now examine the behavior of the magnitude of the terminal state of the dual control problem as we let the terminal time become large:

Lemma 3: The optimal control for the dual control problem (with arbitrary terminal cost) yields $((\|y(T)\|)/(\|y(0)\|)) \to 0$ and $((\|y(T-1)\|)/(\|y(0)\|)) \to 0$ as $T \to \infty$, if the following condition is satisfied, in addition to that of Lemma 2:

$\exists N$ such that $\forall p \in \{0, 1, \ldots, T-N\}$ we have that:

$$y'\left(\sum_{i=p}^{p+N-1} (A^*(p)\cdots A^*(i-1)Q(i)A(i-1)\cdots A(p))\right)y \geq G\|y\|^2 \qquad (138)$$

Proof: We have from Lemma 2 and Eq. (126) that:

$$J_{inf}(\{u(n)\}, (\Sigma_0, T), 0) = \frac{1}{2}\left(\sum_{i=0}^{T-1} (y'(i)Q(i)y(i) + u'(i)e(i)u(i))\right.$$

$$\left. + y'(T)\Sigma_0 y(T)\right) \leq \frac{1}{2}k_{\Sigma_0}\|y(0)\|^2 ;$$

$$(\forall T), \quad (\forall y(0) \in V_s) \qquad (139)$$

Pick any $\varepsilon \in R^+$. Then pick $T > N/\varepsilon$. Because some N-term portion of the summation in Eq. (139) must be at most as large as the average of such portions, we have, for some $L<T-N+1$:

$$\sum_{i=L}^{L+N-1} (y'(i)Q(i)y(i) + u'(i)e(i)u(i))$$

$$\leq \left(\frac{k_{\Sigma_0}\|y(0)\|^2}{T/N}\right) < k_{\Sigma_0}\, \varepsilon\, \|y(0)\|^2 \qquad (140)$$

From Eq. (140) we see that:

$$r_s\|u(i)\|^2 \leq u'(i)e(i)u(i) \leq k_{\Sigma_0}\, \varepsilon\, \|y(0)\|^2 ; \quad (L\leq i\leq L+N-1) \qquad (141)$$

where the first inequality follows by Eq. (118) and (121). Now, we have that (from Eq. (119)):

$$y(L+k) = A(L+k-1)\cdots A(L)y(L) + \sum_{i=0}^{k-1} A(L+k-1)\cdots A(L+i+1)B(L+i)u(L+i) \qquad (142)$$

From Eq. (141):

$$\|u(i)\|^2 \leq \left(\frac{k_{\Sigma_0}}{r_s}\right) \varepsilon\, \|y(0)\|^2 ; \quad (L\leq i\leq L+N-1) \qquad (143)$$

From Eq. (118) and (142), we see that:

$$y(L+k) = A(L+k-1)\cdots A(L)y(L) + \rho_{L+k} \tag{144}$$

where:

$$\|\rho_{L+k}\| = \|\sum_{i=0}^{k-1} A(L+k-1)\cdots A(L+i+1)B(L+i)u(L+i)\|$$

$$\leq \sum_{i=0}^{k-1} \|A(L+k-1)\cdots A(L+i+1)B(L+i)u(L+i)\|$$

$$\leq \sum_{i=0}^{k-1} \|A(L+k-1)\cdots A(L+i+1)\| \, \|B(L+i)u(L+i)\|$$

$$\leq \sum_{i=0}^{k-1} s^{k-i-1}\|B(L+i)\| \, \|u(L+i)\| \text{ (s defined in Eq. (148))} \tag{145}$$

$$\leq \sum_{i=0}^{k-1} s^{N}h\left(\sqrt{\left(\frac{k_{\Sigma_0}}{r_s}\right)\varepsilon}\right)\|y(0)\| \leq Ns^{N}h\left(\sqrt{\left(\frac{k_{\Sigma_0}}{r_s}\right)\varepsilon}\right)\|y(0)\|;\ (0\leq k\leq N-1)$$

(146) (147)

where:

$$s = \max[1, \phi] \tag{148}$$

Equation (146) is gotten from Eq. (118) and (121), and from Eq. (143). Equation (140) implies that:

$$\|\sum_{i=0}^{N-1} y'(L+i)Q(L+i)y(L+i)\| \leq k_{\Sigma_0} \varepsilon \|y(0)\|^2 \tag{149}$$

Using Eq. (144), this becomes:

$$\|\sum_{i=0}^{N-1} (A(L+i-1)\cdots A(L)y(L) + \rho_{L+i})'Q(L+i)(A(L+i-1)\cdots A(L)y(L) + \rho_{L+i})\| \leq k_{\Sigma_0} \varepsilon \|y(0)\|^2 \tag{150}$$

Multiply out the terms in the summand:

$$\|\sum_{i=0}^{N-1} y'(L)A^{*}(L)\cdots A^{*}(L+i-1)Q(L+i)A(L+i-1)\cdots A(L)y(L)$$

$$+ \sum_{i=0}^{N-1} \rho'_{L+i}Q(L+i)\rho_{L+i} + \sum_{i=0}^{N-1} \rho'_{L+i}Q(L+i)A(L+i-1)\cdots A(L)y(L)$$

$$+ \sum_{i=0}^{N-1} y'(L)A^{*}(L)\cdots A^{*}(L+i-1)Q(L+i)\rho_{L+i}\| \leq k_{\Sigma_0} \varepsilon \|y(0)\|^2 \tag{151}$$

However, we have by Eq. (147) that:

$$\left\| \sum_{i=0}^{N-1} \rho'_{L+i} Q(L+i) \rho_{L+i} \right\| \le Nd \left(\max_{i \in \{0,\ldots,N-1\}} \left(\|\rho_{L+i}\|^2 \right) \right)$$

$$\le NdN^2 s^{2N} h^2 \left(\frac{k_{\Sigma_0}}{r_s} \right) \varepsilon \ \|y(0)\|^2 \tag{152}$$

Similarly:

$$\left\| \sum_{i=0}^{N-1} y'(L) A^*(L) \cdots A^*(L+i-1) Q(L+i) \rho_{L+i} \right\| = \left\| \sum_{i=0}^{N-1} \rho'_{L+i} Q(L+i) A(L+i-1) \ldots A(L) y(L) \right\|$$

$$\le N \left(N s^N h \left(\sqrt{\frac{k_{\Sigma_0}}{r_s} \varepsilon} \right) \|y(0)\| \right) d s^N \|y(L)\| \tag{153}$$

and, from Eq. (138) and (151):

$$G\|y(L)\|^2 \le \left\| \sum_{i=0}^{N-1} y'(L) A^*(L) \cdots A^*(L+i-1) Q(L+i) A(L+i-1) \cdots A(L) y(L) \right\|$$

$$\le \quad k_{\Sigma_0} \ \varepsilon \ \|y(0)\|^2 + \left\| \sum_{i=0}^{N-1} \rho'_{L+i} Q(L+i) A(L+i-1) \cdots A(L) y(L) \right.$$

$$+ \sum_{i=0}^{N-1} \rho'_{L+i} Q(L+i) \rho(L+i)$$

$$\left. + \sum_{i=0}^{N-1} y'(L) A^*(L) \cdots A^*(L+i-1) Q(L+i) \rho_{L+i} \right\| \tag{154}$$

So that, from Eq. (152) and (153):

$$G\|y(L)\|^2 \le k_{\Sigma_0} \ \varepsilon \ \|y(0)\|^2 + \left(N^3 d s^{2N} h^2 \left(\frac{k_{\Sigma_0}}{r_s} \right) \right) \varepsilon \ \|y(0)\|^2$$

$$+ 2 \left(N^2 s^{2N} d h \left(\sqrt{\frac{k_{\Sigma_0}}{r_s} \varepsilon} \right) \right) \|y(0)\| \ \|y(L)\| \tag{155}$$

Rewriting, with obvious definitions for α and β, ($\alpha > 0$, $\beta > 0$):

$$G\|y(L)\|^2 \le \alpha \ \varepsilon \ \|y(0)\|^2 + \beta\sqrt{\varepsilon}\|y(0)\| \ \|y(L)\| \tag{156}$$

$$\|y(L)\|^2 - \frac{\beta\sqrt{\varepsilon}}{G}\|y(0)\| \ \|y(L)\| - \frac{\alpha\varepsilon\|y(0)\|^2}{G} \le 0 \tag{157}$$

$$\underbrace{\left(\|y(L)\| - \frac{\beta\sqrt{\varepsilon}}{2G}\|y(0)\| + \|y(0)\| \frac{\sqrt{\frac{\beta^2\varepsilon}{G^2} + \frac{4\alpha\varepsilon}{G}}}{2} \right)}_{T_1} \times$$

$$\times \underbrace{\left(\|y(L)\| - \frac{\beta\sqrt{\varepsilon}}{2G}\|y(0)\| - \frac{\sqrt{\frac{\beta^2\varepsilon}{G^2} + \frac{4\alpha\varepsilon}{G}}}{2}\|y(0)\| \right)}_{T_2} \leq 0 \qquad (158)$$

Either $T_1 \geq 0$, and $T_2 \leq 0$, or $T_1 \leq 0$, and $T_2 \geq 0$. Clearly $T_1 \geq T_2$, so if $T_2 \geq 0$, then $T_1 \geq 0$ also, eliminating the second possibility. Thus $T_1 \geq 0$, and $T_2 \leq 0$. Rewriting this we have:

$$\left(\frac{\beta}{2G} - \frac{\sqrt{\frac{\beta^2}{G^2} + \frac{4\alpha}{G}}}{2} \right)\|y(0)\|\sqrt{\varepsilon} \leq \|y(L)\| \leq \left(\frac{\beta}{2G} + \frac{\sqrt{\frac{\beta^2}{G^2} + \frac{4\alpha}{G}}}{2} \right)\|y(0)\|\sqrt{\varepsilon}$$

(159) (160)

The RHS (Eq. (160)) is the part of importance here. Now observe Eq. (137a). This equation says that:

$$\|K(j, (\Sigma_0,T))\| \leq k_{\Sigma_0} \;; \quad (\forall T,\ 0 \leq j \leq T) \qquad (161)$$

The optimal cost-to-go from time L and state y(L) is:

$$\tfrac{1}{2}y'(L)K(L, (\Sigma_0,T))y(L) \leq \tfrac{1}{2}k_{\Sigma_0}\|y(L)\|^2$$

$$\leq \tfrac{1}{2}k_{\Sigma_C}\left(\frac{\beta}{2G} + \frac{\sqrt{\frac{\beta^2}{G^2} + \frac{4\alpha}{G}}}{2} \right)^2 \|y(0)\|^2\,\varepsilon \qquad (162)$$

where the last inequality is from Eq. (160).

Next, we consider the position of L with respect to T. We consider two cases, first $L > T-N-1$, second $L \leq T-N-1$. If $L > T-N-1$, then consider the application of the optimal controls starting at time L and continuing until time T. From Eq. (122) we see that:

$$u(i) = -[e(i) + B^*(i)K(i+1)B(i)]^{-1}\, B^*(i)K(i+1)A(i)y(i) \qquad (163)$$

so that by Eq. (119):

$$y(i+1) = \Big(A(i) - B(i)[e(i) + B^*(i)K(i+1)B(i)]^{-1} B^*(i)K(i+1)A(i)\Big)y(i) \tag{164}$$

Thus:

$$\begin{aligned} y(T) = {} & [A(T-1) - B(T-1)[e(T-1) + B^*(T-1)K(T)B(T-1)]^{-1} B^*(T-1)K(T)A(T-1)]\cdots \\ & \cdots[A(L) - B(L)[e(L) + B^*(L)K(L+1)B(L)]^{-1} B^*(L)K(L+1)A(L)]y(L) \end{aligned} \tag{165}$$

so that:

$$\begin{aligned} \|y(T)\| \le {} & \|A(T-1) - B(T-1)[e(T-1) \\ & + B^*(T-1)K(T)B(T-1)]^{-1} B^*(T-1)K(T)A(T-1)\|\cdots\|(\cdot)\| \; \|y(L)\| \end{aligned} \tag{166}$$

Because all the operators in Eq. (166) are bounded independent of T (see Eq. (118), (121), and (161)), we see that for some appropriate F_1, independent of L and T (where $T-N-1 < L \le T$):

$$\|y(T)\| \le F_1\|y(L)\| \le F_1\left(\frac{\beta}{2G} + \frac{\sqrt{\frac{\beta^2}{G^2} + \frac{4\alpha}{G}}}{2}\right)\|y(0)\|\sqrt{\varepsilon} \; ; \quad (\forall y(0) \; \varepsilon \; V_s) \tag{167}$$

where the second inequality follows from Eq. (160). Now say $L \le T-N-1$. Then the expression for the optimal cost-to-go from y(L) at time L is (from Eq. (162)):

$$\begin{aligned} \tfrac{1}{2}y'(L)K(L, (\Sigma_0,T))y(L) = {} & \tfrac{1}{2}\Big(\sum_{i=L}^{T-1}(y'(i)Q(i)y(i) \\ & + u'(i)e(i)u(i)) + y'(T)\Sigma_0 y(T)\Big) \\ \le {} & \tfrac{1}{2}k_{\Sigma_0}\left(\frac{\beta}{2G} + \frac{\sqrt{\frac{\beta^2}{G^2} + \frac{4\alpha}{G}}}{2}\right)^2\|y(0)\|^2\varepsilon \end{aligned} \tag{168}$$

In particular, from Eq. (168):

$$\sum_{i=T-N}^{T-1}(y'(i)Q(i)y(i) + u'(i)e(i)u(i)) \le k_{\Sigma_0}\left(\frac{\beta}{2G} + \frac{\sqrt{\frac{\beta^2}{G^2} + \frac{4\alpha}{G}}}{2}\right)^2\|y(0)\|^2\varepsilon \tag{169}$$

However, by the same argument as Eq. (140) through (160), we see that for some $F_2 \; \varepsilon \; R^+$, independent of T, we have that:

$$||y(T-N)|| \leq F_2\sqrt{\varepsilon}||y(0)|| \tag{170}$$

Then apply the argument of Eq. (163) through (167), to see that for some $F_3 \ \varepsilon \ R^+$, independent of T:

$$||y(T)|| \leq F_3F_2\sqrt{\varepsilon}||y(0)|| \quad ; \quad (\forall y(0) \ \varepsilon \ V_s) \tag{171}$$

Equations (167) and (171), together with the fact that ε can be chosen arbitrarily small if T is large enough, imply that:

$$\left(\frac{||y(T)||}{||y(0)||}\right) \to 0 \ ; \tag{172}$$

as the terminal time $T \to \infty$, for the dual optimal control problem. It is also clear from the proof that:

$$\left(\frac{||y(T-1)||}{||y(0)||}\right) \to 0 \qquad \text{QED}$$

as terminal time $T \to \infty$.

We are now ready for the major theorem:

Theorem 1: Under the conditions of Lemmas 2 and 3, the Kalman filter is asymptotically stable. (Note that the conditions of Lemmas 2 and 3 can be equated to those in the first statement of Theorem 1, found earlier in this section, through the use of Eq. (121).)

Proof: We know, for the dual control problem, from Lemma 3:

$$||y(T-1)|| < \delta||y(0)|| \quad ; \quad (\forall y(0) \ \varepsilon \ V_s), \ (\forall T > T(\delta)) \tag{173}$$

However, we have that (from Eq. (122)):

$$y(T-1) = \left[\prod_{i=0}^{T-2}(A(i) - B(i)[e(i) + B^*(i)K(i+1)B(i)]^{-1} B^*(i)K(i+1)A(i))\right]y(0) \tag{174}$$

Thus, from Eq. (173) we see that:

$$||\prod_{i=0}^{T-2}(A(i) - B(i)[e(i) + B^*(i)K(i+1)B(i)]^{-1} B^*(i)K(i+1)A(i))|| < \delta \tag{175}$$

Now substitute Eq. (121) and (125) into Eq. (175), to obtain:

$$||\prod_{i=0}^{T-2}(\Phi^*(T-i-1) - H_s^*(T-i-1)[r(T-i-1) + H_s(T-i-1)\Sigma_s(T-i-1|T-i-2)H_s^*(T-i-1)]^{-1} \times \times H_s(T-i-1)\Sigma_s(T-i-1|T-i-2)\Phi_s^*(T-i-1))|| < \delta \tag{176}$$

In order to test the asymptotic stability of the Kalman filter, we need to rewrite the homogeneous portion of Eq. (113) ($z(n) = 0$; $(\forall n \geq 0)$):

$$\hat{x}_h(n+1|n+1) = \Big(\Phi_s(n) - \Sigma_s(n+1|n)H_s^*(n+1)[r(n+1) + H_s(n+1)\Sigma_s(n+1|n)H_s^*(n+1)]^{-1} \times H_s(n+1)\Phi_s(n)\Big)\hat{x}_h(n|n) \quad (177)$$

(homogeneous propagation, indicated by the arrow to the subscript h)

Thus, we see that, for the homogeneous part of the Kalman filter:

$$\hat{x}_h(T|T) = \left[\prod_{i=0}^{T-1}{}^{L}(I - \Sigma_s(i+1|i)H_s^*(i+1)[r(i+1) + H_s(i+1)\Sigma_s(i+1|i)H_s^*(i+1)]^{-1}\ H_s(i+1))(\Phi_s(i))\right]\hat{x}_h(0|0) \quad (178)$$

Now take the adjoint of Eq. (176):

$$\left\|\prod_{i=0}^{T-2}{}^{R}(\Phi_s(T-i-1))(I - \Sigma_s(T-i-1|T-i-2)H_s^*(T-i-1)[r(T-i-1) + H_s(T-i-1)\Sigma_s(T-i-1|T-i-2)H_s^*(T-i-1)]^{-1}\ H_s(T-i-1))\right\| < \delta \quad (179)$$

Rewrite this as:

$$\left\|\prod_{i=1}^{T-1}{}^{L}(\Phi_s(i))(I - \Sigma_s(i|i-1)H_s^*(i)[r(i)+H_s(i)\Sigma_s(i|i-1)H_s^*(i)]^{-1}\ H_s(i))\right\| < \delta \quad (180)$$

We wish to regroup terms in Eq. (178). Rewrite a sequence from the interior of the operator product. Regroup as with the dotted brackets:

$$[(I - \Sigma_s(13|12)H_s^*(13)[r(13)+H_s(13)\Sigma_s(13|12)H_s^*(13)]^{-1}\ H_s(13))\,\vdots\,\vdots\,(\Phi_s(12))] \times$$
$$\times [(I - \Sigma_s(12|11)H_s^*(12)[r(12)+H_s(12)\Sigma_s(12|11)H_s^*(12)]^{-1}\ H_s(12))\,\vdots \times$$
$$\times \vdots\,(\Phi_s(11))][(I - \Sigma_s(11|10)H_s^*(11)\ \times$$
$$\times\ [r(11)+H_s(11)\Sigma_s(11|10)H_s^*(11)]^{-1}\ H_s(11))\,\vdots\,\vdots\,(\Phi_s(10))] \quad (181)$$

Then the operator in Eq. (178) is:

$$\prod_{i=0}^{T-1}(I - \Sigma_s(i+1|i)H_s^*(i+1)[r(i+1)+H_s(i+1)\Sigma_s(i+1|i)H_s^*(i+1)]^{-1} H_s(i+1))(\Phi_s(i))$$

$$= [I - \Sigma_s(T|T-1)H_s^*(T)[r(T)+H_s(T)\Sigma_s(T|T-1)H_s^*(T)]^{-1} H_s(T)] \times$$

$$\times [\prod_{i=0}^{T-1}(\Phi_s(i))(I - \Sigma_s(i|i-1)H_s^*(i)[r(i)+H_s(i)\Sigma_s(i|i-1)H_s^*(i)]^{-1} H_s(i))][\Phi_s(0)] \tag{182}$$

Since the first and third operators are bounded, by Eq. (118), Lemma 2, and Eq. (125), we use Eq. (180) to obtain:

$$\left\|\prod_{i=0}^{T-1}(I - \Sigma_s(i+1|i)H_s^*(i+1)[r(i+1)+H_s(i+1)\Sigma_s(i+1|i)H_s^*(i+1)]^{-1} H_s(i+1))(\Phi_s(i))\right\|$$

$$\leq F_4\delta\|\Phi_s(0)\| \leq F_4\delta\phi \tag{183}$$

Thus, from Eq. (178), we see that:

$$\|\hat{x}(T|T)\|_h \leq F_4\delta\phi\|\hat{x}(0|0)\|_h \tag{184}$$

where δ may be chosen arbitrarily small for T large enough, so that the Kalman filter is indeed asymptotically stable. QED

In summary, we have proven the major result of this part of Section 4:

<u>Theorem 1</u>: The Hilbert space Kalman filter is asymptotically stable if the following conditions hold:

(1) The sequence $\{(H_s(n), \Phi_s(n))\}$ is uniformly detectable.

(2) For some $N\varepsilon\{0, 1, \ldots\}$, some $G\varepsilon R^+$, and all $s\varepsilon\{0, 1, \ldots, T-N\}$:

$$G\|x\|^2 \leq <(\sum_{k=s}^{s+N-1}[\Phi_s(s+N-1)\cdots\Phi_s(k+1)]D(k)[\Phi_s^*(k+1)\cdots\Phi_s^*(s+N-1)])x,x>_{V_s} ;$$

$$(\forall x \ \varepsilon \ V_s)$$

In the next part of Section 4 we will discuss the behavior of the Kalman filter for time-invariant models.

4.2 <u>Behavior of the Kalman Filter for Time-Invariant Models</u>

In this part of Section 4 we discuss estimation of the state of the time-invariant model:

$$x(n+1) = \Phi_s x(n) + w(n) \;;\quad x(n) \in V_s$$
$$z(n) = H_s x(n) + v(n) \;;\quad z(n) \in V_0 \tag{185}$$

where if bounded linear operator $F : V_0 \to R$, then:

$$E[(F(v(n)))^2] = FrF^* \;;\quad r \text{ is self-adjoint} \tag{185a}$$

where $x(n)$ and $v(n)$ are members of Hilbert spaces V_s and V_0, respectively, which are arbitrary Hilbert spaces in the present section. The Kalman filter is given by (see Eq. (113)):

$$\hat{x}(n+1|n+1) = \Phi_s \hat{x}(n|n) + \Sigma_s(n+1|n)H_s^*[r+H_s\Sigma_s(n+1|n)H_s^*]^{-1}\,[z(n+1)-H_s\Phi_s\hat{x}(n|n)] \tag{186}$$

where $\Sigma_s(n+1|n)$ is found through the Riccati equation:

$$\Sigma_s(n+1|n) = \Phi_s\Big(\Sigma_s(n|n-1) - \Sigma_s(n|n-1)H_s^*[r+H_s\Sigma_s(n|n-1)H_s^*]^{-1}\,H_s\Sigma_s(n|n-1)\Big)\Phi_s^* + D \tag{187}$$

where D is a positive semidefinite operator (defined in Section 3 for the gyro noise filtering problem). The initial conditions are:

$$\Sigma_s(0|-1) = \Sigma_0 \tag{188}$$

$$\hat{x}(0|0) = 0 \tag{189}$$

We assume the same bounds as in Eq. (118). Theorem 1, the main result of Section 4.1, becomes:

<u>Theorem 2</u>: The Kalman filter in the time-invariant case is asymptotically stable if the following conditions are satisfied:

(1) (H_s, Φ_s) is detectable, i.e. there exists an operator L such that $\|L\| < \ell \in R^+$, and $\|(\Phi_s - LH_s)^M\| \le q$, for some $0 \le q < 1$, and some $M \in Z^+$.

(2) For some $N \in \{0, 1, \ldots\}$, and some $G \in R^+$:

$$G\|x\|^2 \le x'\Big(\sum_{i=0}^{N-1} \Phi_s^i D(\Phi_s^*)^i\Big)x \;;\quad (\forall x \in V_s) \tag{190}$$

However, many other interesting results are available to us in the time-invariant case. The main result of the present discussion is the following:

<u>Theorem 3</u>: If (H_s, Φ_s) is detectable, and for some $G \in R^+$, and some N a positive integer:

$$G\|x\|^2 \le x'\left(\sum_{i=0}^{N-1} \Phi_s^i D(\Phi_s^*)^i\right)x \ ; \quad (\forall x \in V_s)$$

then the Riccati operator equation converges weakly, from arbitrary positive semidefinite initial conditions, to a positive definite (bounded below) operator $\hat{K}$, which is the unique positive semidefinite solution of the steady-state operator Riccati equation:

$$\hat{K} = \Phi_s(\hat{K} - \hat{K}H_s^*[r + H_s\hat{K}H_s^*]^{-1} H_s\hat{K})\Phi_s^* + D$$

Furthermore, the steady-state Kalman filter is asymptotically stable.

In order to prove Theorem 3, we introduce the dual control problem, as we did in Section 4.1. The dual control problem is given by:

$$y(n+1) = Ay(n) + Bu(n) \tag{191}$$

where we wish to choose the sequence of vectors (in the gyro noise case scalars) $\{u(n)\}$, $n \in \{0, 1, \ldots, T-1\}$, such that:

$$J(\{u(n)\}, (\Sigma_0,T), 0) = \frac{1}{2}\left(\sum_{i=0}^{T-1} (y'(i)Qy(i) + u'(i)eu(i)) + y'(T)\Sigma_0 y(T)\right) \tag{192}$$

is minimized. For the dual control problem, the operators above are given by:

$$\begin{aligned} B &= H_s^* \\ A &= \Phi_s^* \\ Q &= D \\ e &= r \end{aligned} \tag{193}$$

The solution of the dual control problem is:

$$u(i) = -[e + B^*K(i+1)B]^{-1} B^*K(i+1)Ay(i) \tag{194}$$

or, by Eq. (193):

$$u(i) = -[r + H_sK(i+1)H_s^*]^{-1} H_sK(i+1)\Phi_s^*y(i) \tag{194a}$$

where we find K(i) by backward-in-time recursion:

$$K(i) = A^*(K(i+1) - K(i+1)B[e + B^*K(i+1)B]^{-1} B^*K(i+1))A + Q \tag{195}$$

Again, by Eq. (193):

$$K(i) = \Phi_s(K(i+1) - K(i+1)H_s^*[r + H_s K(i+1)H_s^*]^{-1} H_s K(i+1))\Phi_s^* + D \tag{195a}$$

and

$$K(T) = \Sigma_0 \tag{196}$$

Comparing Eq. (195a) and (196) with Eq. (187) and (188), we find as before (Eq. (125)) that:

$$K(i, (\Sigma_0,T)) = \Sigma(T-i|T-i-1, (\Sigma_0,0)) \tag{197}$$

The minimal cost-to-go of the dual control problem, from state y(i) at time (i) is given by:

$$J_{inf}(\{u(n)\}, (\Sigma_0,T), i) \triangleq \inf[\frac{1}{2}(\sum_{j=i}^{T-1} (y'(j)Qy(j) + u'(j)ru(j)) + y'(T)\Sigma_0 y(T))] = \frac{1}{2}y'(i)K(i, (\Sigma_0,T))y(i) \tag{198}$$

The specialization of Lemma 1 to the time-invariant case is:

<u>Lemma 4</u>: $K(j, (0,T))$ is uniformly bounded for all T, $j \le T$, if (H_s, Φ_s) is detectable.

It is easy to see, as in Section 4.1, that:

$$K(j, (0,T)) = K(0, (0,T-j)) \tag{199}$$

Observing the summation form of the cost-to-go with no terminal cost, we see that:

$$x'K(j, (0,T))x \ge x'K(j+1, (0,T))x \ ; \quad (\forall x \in V_s) \tag{200}$$

Using Eq. (199), we obtain from Eq. (200) (let j = 0):

$$x'K(0, (0,T))x \ge x'K(0, (0,T-1))x \ ; \quad (\forall T), \ (\forall x \in V_s) \tag{201}$$

Thus by Lemma 4, if (H_s, Φ_s) is detectable, then:

$$K(0, (0,T)) \overset{w}{\to} \hat{K} \text{ monotonically, as } T \to \infty \tag{202}$$

where $\hat{K}$ is a bounded self-adjoint operator, and $(\overset{w}{\to})$ denotes weak operator convergence, i.e. for the sequence of operators:

$$K(0, (0,T)) : V_s \to V_s \quad (T = 0, 1, \ldots) \tag{203}$$

where V_s is a Hilbert space, we have that:

$$\lim_{T\to\infty} \langle K(0, (0,T))x, y\rangle_{V_s} = \langle \hat{K}x, y\rangle_{V_s} \; ; \quad (\forall x, y \in V_s) \tag{204}$$

The convergence in Eq. (202) is monotonic, in that:

$$x'K(0, (0,T+1))x \geq x'K(0, (0,T))x \; ; \quad (\forall T), (\forall x \in V_s)$$

By Eq. (197) we see that correspondingly:

$$\Sigma(T|T-1, (0,0)) \xrightarrow{w} \hat{K} \text{ monotonically} \tag{205}$$

We may now state Lemma 5:

Lemma 5: $\Sigma(T|T-1, (0,0)) \xrightarrow{w} \hat{K}$, monotonically upwards from initial condition $\Sigma(0|-1) = 0$, provided that (H_s, Φ_s) is detectable. Also, under the same condition $K(0, (0,T)) \xrightarrow{w} \hat{K}$, monotonically upwards from initial condition $K(T) = 0$. Also, $\hat{K}$ is a bounded self-adjoint operator.

We now discuss properties of $\hat{K}$, the operator to which the zero-initial-condition Riccati equation converges.

Lemma 6: If (H_s, Φ_s) is detectable, and:

$$G||x||^2 \leq x'(\sum_{i=0}^{N-1} \Phi_s^i D(\Phi_s^*)^i)x \; ; \quad (\forall x \in V_s)$$

then $\hat{K}$ is positive definite (bounded below). Furthermore, $\hat{K}$ is a solution of the steady-state Riccati equation:

$$\hat{K} = \Phi_s(\hat{K} - \hat{K}H_s^*[r + H_s\hat{K}H_s^*]^{-1} H_s\hat{K})\Phi_s^* + D \tag{206}$$

Proof: Because, from Eq. (200):

$$\hat{K} \geq K(j, (0,T)) \geq K(j+1, (0,T)) \; ; \quad (0 \leq j \leq T)$$

we see that:

$$\hat{K} \geq K(T-N, (0,T)) \tag{207}$$

So if we can show that $K(T-N, (0,T))$ is positive definite, then we'll know that $\hat{K}$ is positive definite. Suppose $K(T-N, (0,T))$ is not positive definite. Then for any $\varepsilon \in R^+$ we could find $y(T-N)$ such that:

$$\tfrac{1}{2}y'(T-N)K(T-N, (0,T))y(T-N) < \varepsilon \; ||y(T-N)||^2 \triangleq \gamma \tag{208}$$

However, the expression on the left of Eq. (208) is the cost-to-go from time T-N, using optimal controls:

$$\frac{1}{2}y'(T-N)K(T-N,\ (0,T))y(T-N) = \frac{1}{2}\sum_{i=T-N}^{T-1}(y'(i)Qy(i) + u'(i)eu(i)) < \varepsilon\|y(T-N)\|^2 \triangleq \gamma \tag{209}$$

Using the argument of Eq. (140) through (160), we find that:

$$\|y(T-N)\| \leq F_5\sqrt{\gamma} \tag{210}$$

for some F_5 independent of T. Thus:

$$\|y(T-N)\| \leq F_5\sqrt{\varepsilon}\ \|y(T-N)\| \tag{211}$$

and consequently:

$$\varepsilon \geq \frac{1}{(F_5)^2} \tag{212}$$

proving K(T-N, (0,T)) to be bounded below, hence positive definite. As shown by Eq. (207), K(T-N, (0,T)) positive definite implies $\hat{K}$ positive definite.

The fact that $\hat{K}$ satisfies the steady-state Riccati equation is proven in Appendix D. QED

We shall now show that $\hat{K}$ is in fact the unique positive semidefinite operator which satisfies the steady-state Riccati equation, and that the recursive Riccati equation converges to $\hat{K}$, no matter what the positive semidefinite initial conditions. (We established the convergence so far only for K(T) = 0.)

Lemma 7: If (H_s, Φ_s) is detectable, and for some $G \in R^+$, and N some positive integer:

$$G\|x\|^2 \leq x'(\sum_{i=0}^{N-1}\Phi_s^i D(\Phi_s^*)^i)x\ ; \quad (\forall x \in V_s)$$

then the Riccati equation converges weakly, from arbitrary positive semidefinite initial conditions, to positive definite operator $\hat{K}$, the unique positive semidefinite solution of the steady-state Riccati operator equation.

Proof: Because the penalty summation for the dual control problem with no terminal cost is:

$$J(\{u(n)\},\ (0,T),\ 0) = \frac{1}{2}\sum_{i=0}^{T-1}(y'(i)Qy(i) + u'(i)eu(i)) \tag{213}$$

and the penalty with terminal cost is (same as Eq. (213) plus an extra term):

$$J(\{u(n)\}, (\Sigma_0,T), 0) = \frac{1}{2}\left(\sum_{i=0}^{T-1} (y'(i)Qy(i) + u'(i)eu(i)) + y'(T)\Sigma_0 y(T)\right) \tag{214}$$

we see that Eq. (198) implies that for all y(0), all T, and all $\Sigma_0 \geq 0$:

$$\frac{1}{2}y'(0)K(0, (\Sigma_0,T))y(0) \geq \frac{1}{2}y'(0)K(0, (0,T))y(0) \tag{215}$$

The cost represented by the LHS of Eq. (215) is the least possible given the form of the penalty function Eq. (214). Thus if we were to generate controls for this problem using the control law of the 0-terminal cost problem, we would necessarily incur greater cost (LHS of Eq. (216)):

$$\frac{1}{2}(y'(T)\Sigma_0 y(T) + y'(0)K(0, (0,T))y(0)) \geq \frac{1}{2}y'(0)K(0, (\Sigma_0,T))y(0) \tag{216}$$

By Lemma 3 of Section 4.2, specialized to the time-invariant case here, $\|y(T)\| \to 0$ as T, the terminal time of the dual control problem, becomes large. Putting Eq. (215) and (216) together, we find that: (Credit this idea to William W. Hager, Department of Mathematics, MIT):

$$\begin{aligned} y'(T)\Sigma_0 y(T) + y'(0)K(0, (0,T))y(0) &\geq y'(0)K(0, (\Sigma_0,T))y(0) \\ &\geq y'(0)K(0, (0,T))y(0) \end{aligned} \tag{217}$$

Taking the limit as $T \to \infty$, $K(0, (0,T))$ converges weakly to $\hat{K}$, hence:

$$y'(0)\hat{K}y(0) \geq \lim_{T\to\infty}[y'(0)K(0, (\Sigma_0,T))y(0)] \geq y'(0)\hat{K}y(0) \tag{218}$$

Thus:

$$\lim_{T\to\infty}[y'(0)K(0, (\Sigma_0,T))y(0)] = y'(0)\hat{K}y(0) \;; \quad (\forall y(0) \in V_s) \tag{219}$$

and we see that:

$$K(0, (\Sigma_0,T)) \xrightarrow{w} \hat{K}, \text{ as } T \to \infty \tag{220}$$

Thus the Riccati equation converges weakly to $\hat{K}$ no matter what the positive semidefinite initial conditions, where $\hat{K}$, as seen in Eq. (206), is a positive definite (Lemma 6) solution of the steady-state Riccati operator equation:

$$\hat{K} = \Phi_s(\hat{K} - \hat{K}H_s^*[r + H_s\hat{K}H_s^*]^{-1} H_s\hat{K})\Phi_s^* + D \tag{221}$$

Finally, to show that $\hat{K}$ is the unique positive semidefinite solution of the steady-state Riccati equation, if the steady-state Riccati equation had any other positive semidefinite solution, say $K_1 \neq \hat{K}$, then the use of K_1 as an initial condition in the recursive Riccati equation would lead to convergence, in fact stagnation, of $K(0, (K_1,T))$ at K_1, since then we would have:

$$K(0, (K_1,T)) = K_1 \ ; \quad (\forall T)$$

This would be a contradiction of the result already proven in this lemma. Thus $\hat{K}$, a positive definite solution of the steady-state Riccati equation, is the unique positive semidefinite solution of the steady-state Riccati equation. QED

We now show that the steady-state Kalman filter is asymptotically stable. We first show that the steady-state control transition operator (using $\hat{K}$) for the dual control problem yields an asymptotically stable system. First observe the steady-state Riccati equation:

$$\hat{K} = \Phi_s(\hat{K} - \hat{K}H_s^*[r + H_s\hat{K}H_s^*]^{-1} H_s\hat{K})\Phi_s^* + D \tag{222}$$

and the steady-state control, from Eq. (194a):

$$u_s(i) = -[r + H_s\hat{K}H_s^*]^{-1} H_s\hat{K}\Phi_s^* y(i) \tag{223}$$

Thus, from Eq. (191), (193), and (223):

$$y(i+1) = (\Phi_s^* - H_s^*[r + H_s\hat{K}H_s^*]^{-1} H_s\hat{K}\Phi_s^*)y(i) \tag{224}$$

Multiply Eq. (223) on the left by $[r + H_s\hat{K}H_s^*]$:

$$ru_s(i) + H_s\hat{K}H_s^*u_s(i) = -H_s\hat{K}\Phi_s^*y(i)$$

Multiply now on the left by $u_s'(i)$, so that:

$$u_s'(i)H_s\hat{K}H_s^*u_s(i) + u_s'(i)H_s\hat{K}\Phi_s^*y(i) = -u_s'(i)ru_s(i) \tag{225}$$

Use Eq. (224) to write:

$$y'(i+1)\hat{K}y(i+1) = y'(i)\Phi_s\hat{K}\Phi_s^*y(i) - y'(i)\Phi_s\hat{K}H_s^*[r+H_s\hat{K}H_s^*]^{-1} H_s\hat{K}\Phi_s^*y(i) +$$

$$- y'(i)\Phi_s\hat{K}H_s^*[r+H_s\hat{K}H_s^*]^{-1} H_s\hat{K}\Phi_s^*y(i) +$$

$$+ y'(i)\Phi_s\hat{K}H_s^*[r+H_s\hat{K}H_s^*]^{-1} H_s\hat{K}H_s^*[r+H_s\hat{K}H_s^*]^{-1} H_s\hat{K}\Phi_s^*y(i) \tag{226}$$

Using Eq. (222) on the first two terms of the RHS of Eq. (226), and Eq. (225) on the last two terms, we find that:

$$y'(i+1)\hat{K}y(i+1) = y'(i)\hat{K}y(i) - y'(i)Dy(i) - u_s'(i)ru_s(i) \qquad (227)$$

From Eq. (227) we see that (with steady-state controls $\{u_s(i)\}$):

$$y'(i)\hat{K}y(i) = \sum_{j=i}^{M} (y'(j)Dy(j) + u_s'(j)ru_s(j)) + y'(M+1)\hat{K}y(M+1) \; ; \quad (\forall M) \qquad (228)$$

Because $\hat{K}$ is positive definite and bounded (Lemmas 6 and 5) we see that for some $\hat{k} \in R^+$ (let (i=0) in Eq. (228)):

$$\sum_{j=0}^{M} (y'(j)Dy(j) + u_s'(j)ru_s(j)) < \hat{k}||y(0)||^2 \; ; \quad (\forall M), \; (\forall y(0) \in V_s) \qquad (229)$$

along the steady-state control trajectory. This is precisely the condition existing in Eq. (139) of Lemma 3. Thus the steady-state control yields, as did the optimal control in Lemma 3, from Eq. (167) and (171), for M_1 large enough (depending on the ε we pick):

$$||y(M_1)|| < F_6||y(0)||\sqrt{\varepsilon} \; ; \quad \forall M_1 \geq M(\varepsilon), \; (\forall y(0) \in V_s) \qquad (230)$$

for some $F_6 \in R^+$, independent of M_1 or ε. Hence:

$$\left(\frac{||y(M_1)||}{||y(0)||}\right) \to 0 \text{ as } M_1 \to \infty; \text{ for the steady-state control.} \qquad (231)$$

As a sidelight, we show in Appendix E that the steady-state controls $\{u_s(n)\}$ solve the dual control problem with penalty function:

$$J(\{u(n)\}, \infty, i) = \frac{1}{2}\left(\sum_{j=i}^{\infty} (y'(j)Dy(j) + u'(j)ru(j))\right) \qquad (232)$$

and that the minimal cost-to-go for this problem (call it the infinite-interval dual control problem) is given by:

$$J_{inf}(\{u(n)\}, \infty, i) = \left(\frac{1}{2}\right)y'(i)\hat{K}y(i) \qquad (233)$$

Returning to our discussion, Eq. (231) implies that the steady-state control yields an asymptotically stable solution to the infinite-interval dual control problem.

We now relate Eq. (230) to the Kalman filter. By Eq. (224):

$$y(M_1) = (\Phi_s^* - H_s^*[r + H_s\hat{K}H_s^*]^{-1} H_s\hat{K}\Phi_s^*)^{M_1} y(0) \tag{234}$$

Thus by Eq. (230):

$$\|[(I - H_s^*[r + H_s\hat{K}H_s^*]^{-1} H_s\hat{K})(\Phi_s^*)]^{M_1}\| < F_6\sqrt{\varepsilon} \ ; \quad (\forall M_1 \geq M(\varepsilon)) \tag{235}$$

Take the adjoint in the LHS of Eq. (235):

$$\|[\Phi_s(I - \hat{K}H_s^*[r + H_s\hat{K}H_s^*]^{-1} H_s]^{M_1}\| < F_6\sqrt{\varepsilon} \ ; \quad \forall M_1 \geq M(\varepsilon) \tag{236}$$

We now observe the homogeneous steady-state Kalman filter. In steady-state, the homogeneous part of Eq. (186) is:

$$\hat{x}_h(n+1|n+1) = (I - \hat{K}H_s^*[r + H_s\hat{K}H_s^*]^{-1} H_s)\Phi_s \hat{x}_h(n|n) \tag{237}$$

Thus:

$$\hat{x}_h(M_1+1|M_1+1) = ([(I - \hat{K}H_s^*[r + H_s\hat{K}H_s^*]^{-1} H_s)\Phi_s]^{M_1+1})\hat{x}_h(0|0) \tag{238}$$

$$\hat{x}_h(M_1+1|M_1+1) = [I - \hat{K}H_s^*[r + H_s\hat{K}H_s^*]^{-1} H_s][\Phi_s(I - \hat{K}H_s^*[r + H_s\hat{K}H_s^*]^{-1} H_s)]^{M_1} \Phi_s \hat{x}_h(0|0) \tag{239}$$

so that, using Eq. (236):

$$\|\hat{x}_h(M_1+1|M_1+1)\| \leq (\|I - \hat{K}H_s^*[r + H_s\hat{K}H_s^*]^{-1} H_s\|)\sqrt{\varepsilon}\, F_6\|\Phi_s\| \, \|\hat{x}_h(0|0)\| \ ; \quad \forall M_1 \geq M(\varepsilon) \tag{240}$$

Since the operators on the RHS of Eq. (240) are bounded, we see that the steady-state Kalman filter is asymptotically stable, proving the following lemma:

<u>Lemma 8</u>: If (H_s, Φ_s) is detectable, and for some $G \in R^+$, and some N a positive integer:

$$G\|x\|^2 \leq x'(\sum_{i=0}^{N-1} \Phi_s^i D(\Phi_s^*)^i)x \ ; \quad (\forall x \in V_s)$$

then the steady-state Kalman filter is asymptotically stable. Finally, we draw together Lemmas 6, 7, and 8 for the main result of Section 4.2:

<u>Theorem 3</u>: If (H_s, Φ_s) is detectable, and for some $G \in R^+$, and some N a positive integer:

$$G\|x\|^2 \leq x'(\sum_{i=0}^{N-1} \Phi_s^i D(\Phi_s^*)^i)x \; ; \quad (\forall x \in V_s)$$

then the Riccati operator equation converges weakly, from arbitrary positive semidefinite initial conditions, to a positive definite operator $\hat{K}$, which is the unique positive semidefinite solution of the steady-state Riccati equation:

$$\hat{K} = \Phi_s(\hat{K} - \hat{K}H_s^*[r + H_s\hat{K}H_s^*]^{-1} H_s\hat{K})\Phi_s^* + D$$

Furthermore, the steady-state Kalman filter is asymptotically stable.

In summary, Theorem 3 above is the main result of Section 4.2. In Section 4.3 we shall specify the sufficient conditions for the various Kalman filter properties which we have discussed to the gyro noise filtering problem. We note first, however, that all the arguments of Sections 4.1 and 4.2 apply to finite-dimensional, as well as infinite-dimensional, models because the arguments we used assumed only that the vector to be estimated was an element of a Hilbert space V_s, where in Sections 4.1 and 4.2 we let V_s be an arbitrary Hilbert space. If V_s were finite dimensional, operators $\{\Sigma_s(n|n-1)\}$ would be matrices, and weak convergence of $\{\Sigma_s(n|n-1)\}$ would be equivalent to uniform convergence for matrices. In the finite-dimensional time-invariant model case results similar to ours are obtained by Caines and Mayne[26].

4.3 Properties of the Kalman Filter for Gyroscopic Noise

We now discuss sufficient properties of the signal to be filtered from gyroscopic noise in order that the theorems of Sections 4.1 and 4.2 apply to the gyro noise filtering problem. The gyro noise model is time-invariant, however the consideration of time-varying signals necessitates the use of the results of Section 4.1. We saw in Section 4.1 that the following two conditions are sufficient for asymptotic stability of the Kalman filter in the time-varying case:

(1) The sequence $\{(H_s(n), \Phi_s(n))\}$ is uniformly detectable.

(2) $$G\|x\|^2 \leq x'(\sum_{k=s}^{s+N-1} [\Phi_s(s+N-1)\cdots\Phi_s(k+1)]D(k)[\Phi_s^*(k+1)\cdots\Phi_s^*(s+N-1)])x$$

for some $G \in R^+$, some N a positive integer, $(\forall 0 \leq s \leq T-N)$, $(\forall x \in V_s)$.

Recall, from Eq. (34), (37), (84), and (97):

$$\Phi_s(n) = \begin{bmatrix} \Phi_\tau & 0 \\ 0 & \underline{\Phi}(n) \end{bmatrix} = \begin{bmatrix} e^{-\tilde{T}/\tau} & 0 \\ 0 & \underline{\Phi}(n) \end{bmatrix} \tag{241}$$

$$D(k) = \begin{bmatrix} \frac{W\tau}{2}(1-e^{-2\tilde{T}/\tau}) & \underline{0} \\ \underline{0} & \underline{B}(n)\underline{Q}_1(n)\underline{B}'(n) \end{bmatrix} \tag{242}$$

As long as the limits of the time constant probability density function for the generation of gyroscopic noise satisfy:

$$0 < \tau_1 < \tau_2 < \infty \tag{243}$$

we have that:

$$e^{-\tilde{T}/\tau} \leq q < 1 \ ; \quad \frac{W\tau}{2}(1 - e^{-2\tilde{T}/\tau}) \geq s > 0 \ ; \quad \tau_1 \leq \tau \leq \tau_2 \tag{244}$$

Thus, due to the complete decoupling of the discrete-time systems which generate the gyroscopic noise and the signal, conditions (1) and (2) will be satisfied by the gyro noise filtering problem as long as:

(1') The sequence $\{(\underline{H}(n), \underline{\Phi}(n))\}$ is uniformly detectable.

(2') $$G_1 \|\underline{y}\|^2 \leq \underline{y}'\left(\sum_{k=s}^{s+N_1-1} [\underline{\Phi}(s+N_1-1)\cdots \underline{\Phi}(k+1)]\underline{B}(n)\underline{Q}_1(n)\underline{B}'(n)[\underline{\Phi}'(k+1)\cdots\underline{\Phi}'(s+N_1-1)]\right)\underline{y}$$

for some $G_1 \in R^+$, some N_1 a positive integer, $(\forall 0 \leq s \leq T-N_1)$, $(\forall \underline{y} \in R^n)$.

In the time-invariant signal case (signal generated by a time-invariant system (Eq. (42) and (43)) we obtain the Kalman filter properties of Theorem 3 if:

(1") $(\underline{H}, \underline{\Phi})$ is detectable, i.e. there exists a matrix $\underline{L}$ such that $\|(\underline{\Phi} - \underline{LH})^{M_1}\| \leq q_1$; for some $0 \leq q_1 < 1$, and some $M_1 \in Z^+$ (i.e. M_1 a positive integer).

(2") $$G_1\|y\|^2 \leq \underline{y}'\left(\sum_{i=0}^{N_1-1} \underline{\Phi}^i \underline{BQ}_1\underline{B}'(\underline{\Phi}')^i\right)\underline{y};$$ for some $G_1 \in R^+$, some $N_1 \in Z^+$, $(\forall \underline{y} \in R^n)$.

Because $\underline{H}$ and $\underline{\Phi}$ are matrices, condition (1") is equivalent to the following condition (as is proven in Appendix F):

(1''') spectral radius $(\underline{\Phi} - \underline{L}\underline{H}) \triangleq \max\{|\lambda_i| \,|\, (\underline{\Phi} - \underline{L}\underline{H})\underline{x} = \lambda_i \underline{x};\ (\forall \underline{x} \in R^n)\} = \rho < 1$

We thus have seen sufficient conditions for the application of the theorems of Sections 4.1 and 4.2 to the gyro noise filtering problem. In summary, in Section 4 we have first given sufficient conditions for general estimation problems to guarantee various desired properties for Hilbert space Kalman filters, and we have then specified these conditions to the gyro noise filtering problem.

SECTION 5

FINITE-DIMENSIONAL APPROXIMATE MODELS FOR GYROSCOPIC NOISE

In Section 4 we discussed the properties of the optimal filter for separating additive gyroscopic noise from a statistically described signal. This filter involves integrations over a parameter (i.e. a time constant) both in its estimation equation and in its Riccati covariance operator propagation equations (Section 3). In applications, these integrations must be implemented discretely. A practical method of achieving discretization is to make a finite-dimensional approximation to the gyroscopic noise, which has been modeled infinite dimensionally as the integral of the outputs from independent white noise excitations of a continuum of first-order linear systems distributed in time constant parameters. The optimal filter becomes an ordinary finite-dimensional discrete Kalman filter. In making such an approximation, we must show that the error incurred in using the Kalman filter of the finite-dimensional model can be made to approach the error associated with optimal filtering of the gyroscopic noise, through the use of a sufficient number of dimensions in the approximation. This section is devoted primarily to this issue. In addition, we discuss the filter performance effects of errors in the bounds on the system time constant density function used to construct the filter.

We first discuss the effects of finite-dimensional approximation of gyroscopic noise. Using the sequence $\{\overline{e^2(k)}\}$ of mean-squared estimation errors of the filter as our criterion, we shall show that the problem of finite-dimensional approximation to gyroscopic noise is equivalent to the problem of finite-dimensional approximation of the gyroscopic noise power spectral density. The Kalman filter solution is equivalent to the optimal causal time-varying impulse response, $h_k(i)$, at time k, to scalar observation $z(k - i)$, at time $(k - i)$. We formulate, as do Laning and Battin[19] in the continuous time steady-state case, a digital frequency domain expression for the mean-squared estimation error at time (k) in terms of the signal PSD (power spectral density), the gyro noise PSD, the additive discrete white noise variance, and the time-varying impulse response of the filter. The gyro noise filtering model, where an impulse response $h_k(i)$ is used for the filter, has the form shown in Figure 9. Note that we take $\{p_1(n)\}$ to be stationary in this discussion. $h_k(i)$ is causal, hence is zero for $(i < 0)$. In addition, because observation sequence $\{z(i)\}$ begins at time zero, $h_k(i)$ also is zero for $(i > k)$. The error $(e(k))$ in estimation is given by:

$$e(k) \triangleq p_1(k) - \hat{p}_1(k)$$

$$e(k) = \sum_{i=0}^{\infty} \{-[h_k(i) - \delta_{i,0}]p_1(k-i)\} - \sum_{i=0}^{\infty} h_k(i)v_1(k-i) \quad (245)$$

where δ is the Kronecker delta function, defined by:

$$\delta_{n,m} = \begin{Bmatrix} 1 ; & n = m \\ 0 ; & n \neq m \end{Bmatrix}$$

For a fixed k, we define the autocorrelation function, $R_{e_k e_k}(\ell)$, of the sequence of errors $\{e_k(n)\}$ $(n = 0, 1, \ldots)$ which would result if $h_k(n)$ $(n = 0, 1, \ldots, k)$ were used as a time invariant impulse response, $h(n)$, acting on the entire observation sequence $\{z(i)\}$ $(i = 0, 1, \ldots)$. Note that $(\forall n \geq k)$ the sequence $\{e_k(n)\}$ is stationary, permitting such a definition:

$$R_{e_k e_k}(\ell) \triangleq E[e_k(m+\ell)e_k(m)] \quad (246)$$

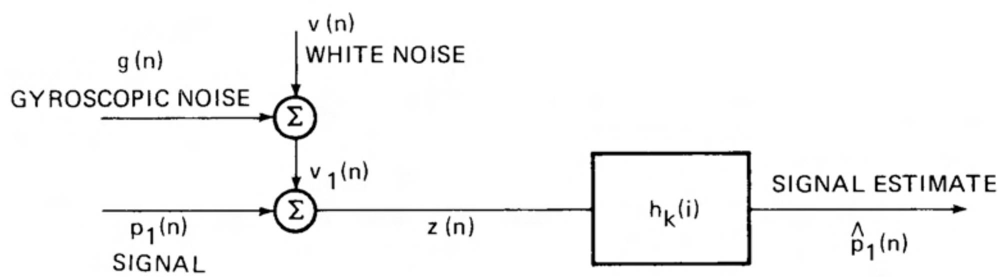

Figure 9. Gyro noise filtering model.

Then from Eq. (245) we obtain:

$$R_{e_k e_k}(\ell) = E[\{\sum_{i=0}^{\infty} \{-[h_k(i)-\delta_{i,0}]p_1(m+\ell-i) - h_k(i)v_1(m+\ell-i)\}\}\{-\sum_{n=0}^{\infty} [h_k(n)-\delta_{n,0}]p_1(m-n)+h_k(n)v_1(m-n)\}] \quad (247)$$

As $p_1(n)$ and $v_1(n)$ are uncorrelated processes, we obtain from Eq. (247):

$$R_{e_k e_k}(\ell) = \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} \{[h_k(i)-\delta_{i,0}][h_k(n)-\delta_{n,0}]R_{p_1 p_1}(\ell-i+n)+h_k(i)h_k(n)R_{v_1 v_1}(\ell-i+n)\} \tag{248}$$

We now define the Fourier transform of a discrete sequence, and the inverse transform (which is easily verified):

$$F(\omega_d) = \sum_{n=-\infty}^{\infty} f(n)e^{-j\omega_d n} \tag{249a}$$

$$f(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\omega_d)e^{j\omega_d n}\, d\omega_d \tag{249b}$$

Note that $F(\omega_d)$ is periodic, of period 2π. Such transforms can be defined for $R_{e_k e_k}(\ell)$, $R_{p_1 p_1}(\ell)$, and $R_{v_1 v_1}(\ell)$, yielding:

$$R_{e_k e_k}(\ell),\ R_{p_1 p_1}(\ell),\ R_{v_1 v_1}(\ell) \iff S_{e_k e_k}(\omega_d),\ S_{p_1 p_1}(\omega_d),\ S_{v_1 v_1}(\omega_d)$$

The above power spectral densities are nonnegative even functions of digital frequency ω_d. We now note that the relation between the error sequence $\{e_k(n)\}$ and the error at time (k), e(k), resulting from using the time-varying impulse response $h_k(n)$ resulting from the Kalman filter solution is:

$$\overline{e^2(k)} = \overline{e_k^2(n)} \quad ; \quad (\forall n \geq k) \tag{250}$$

Thus the mean-squared error at time (k), $\overline{e^2(k)}$, is given by:

$$\overline{e^2(k)} = R_{e_k e_k}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{e_k e_k}(\omega_d)e^{j\omega_d \cdot 0}\, d\omega_d = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{e_k e_k}(\omega_d)d\omega_d \tag{251}$$

We now transform Eq. (248):

$$S_{e_k e_k}(\omega_d) = \sum_{\ell=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \{[h_k(i)-\delta_{i,o}][h_k(n)-\delta_{n,o}]R_{p_1 p_1}(\ell-i+n)e^{-j\omega_d \ell}$$
$$+ h_k(i)h_k(n)R_{v_1 v_1}(\ell-i+n)e^{-j\omega_d \ell}\} \tag{252}$$

Interchanging orders of summation in Eq. (252), we obtain:

$$S_{e_k e_k}(\omega_d) = \sum_{i=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \{[h_k(i)-\delta_{i,0}][h_k(n)-\delta_{n,0}]e^{-j\omega_d i} e^{j\omega_d n} \sum_{\ell=-\infty}^{\infty} R_{p_1 p_1}(\ell-i+n)e^{-j\omega_d(\ell-i+n)}\}$$

$$+ \sum_{i=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \{h_k(i)h_k(n)\, e^{-j\omega_d i} e^{j\omega_d n} \sum_{\ell=-\infty}^{\infty} R_{v_1 v_1}(\ell-i+n)e^{-j\omega_d(\ell-i+n)}\}$$

$$S_{e_k e_k}(\omega_d) = \sum_{i=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \{[h_k(i)-\delta_{i,o}][h_k(n)-\delta_{n,o}]e^{-j\omega_d i} e^{j\omega_d n} S_{p_1 p_1}(\omega_d)$$

$$+ h_k(i)h_k(n)e^{-j\omega_d i} e^{j\omega_d n} S_{v_1 v_1}(\omega_d)\}$$

Finally, using Eq. (249a) we obtain (where $H_k(\omega_d) \Longleftrightarrow h_k(n)$; remember, k is fixed):

$$S_{e_k e_k}(\omega_d) = |H_k(\omega_d) - 1|^2 S_{p_1 p_1}(\omega_d) + |H_k(\omega_d)|^2 S_{v_1 v_1}(\omega_d) \qquad (253)$$

Using Eq. (251), we have that:

$$\overline{e^2(k)} = R_{e_k e_k}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \{|H_k(\omega_d) - 1|^2 S_{p_1 p_1}(\omega_d) + |H_k(\omega_d)|^2 S_{v_1 v_1}(\omega_d)\}d\omega_d \qquad (254)$$

Equation (254) expresses the mean-squared error, at time (k), in estimation of $p_1(n)$ which will result when using the Kalman filter solution. Because the white noise v(n) and the gyroscopic noise g(n) are independent, we have:

$$S_{v_1 v_1}(\omega_d) = S_{vv}(\omega_d) + S_{gg}(\omega_d)$$
$$S_{v_1 v_1}(\omega_d) = r + S_{gg}(\omega_d) \qquad (255)$$

where (r) is the variance of the white noise, v(n). When a finite-dimensional approximation to g(n) is made, $S_{p_1 p_1}(\omega_d)$ and $S_{vv}(\omega_d)$ remain the same. $S_{gg}(\omega_d)$ is approximated by $S_{ggapx}(\omega_d)$. Then by Eq. (254), if $S_{ggapx}(\omega_d)$ were indeed the gyro noise spectral density, and $h_k(n)$ (k=0, 1, ...; n = 0, 1, ..., k) were chosen optimally for this spectral density ($h_k(n) = h_{S_{ggapx}}(n)_{,opt,k}$), the mean-squared error at time (k) would be:

$$\overline{e^2(k)}_{S_{ggapx},opt\ S_{ggapx}\ \text{filter}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \{S_{p_1 p_1}(\omega_d)\,|H_{S_{ggapx}}(\omega_d)_{,opt,k} - 1|^2$$
$$+ (S_{vv}(\omega_d) + S_{ggapx}(\omega_d))|H_{S_{ggapx}}(\omega_d)_{,opt,k}|^2\}d\omega_d \qquad (256)$$

Pick an arbitrary constant ($\beta > 0$). If it were the case that we could choose a finite-dimensional linear system which when excited with finite-dimensional white noise would generate a noise with power spectral density $S_{ggapx}(\omega_d)$ satisfying:

$$S_{gg}(\omega_d) \leq S_{ggapx}(\omega_d) \leq (1 + \beta)S_{gg}(\omega_d) \; ; \quad \omega_d \in [-\pi, \pi] \tag{257}$$

then we could argue as follows. First we write down the expression for the error which would result if we filtered $\{z(n)\}$ (see Figure 9) with actual gyro noise using the impulse response $h_{S_{ggapx},opt,k}(n)$:

$$\overline{e^2(k)}_{S_{gg},opt\ S_{ggapx}\ filter} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \{S_{p_1p_1}(\omega_d)|H_{S_{ggapx},opt,k}(\omega_d) - 1|^2 + (S_{vv}(\omega_d) + S_{gg}(\omega_d))|H_{S_{ggapx},opt,k}(\omega_d)|^2\}d\omega_d \tag{258}$$

Then by the first inequality in Eq. (257) we see that:

$$\overline{e^2(k)}_{S_{gg},opt\ S_{ggapx}\ filter} \leq \overline{e^2(k)}_{S_{ggapx},opt\ S_{ggapx}\ filter} \; ; \; (\forall k) \tag{259}$$

By optimality, we have that:

$$\overline{e^2(k)}_{S_{gg},opt\ S_{gg}\ filter} \leq \overline{e^2(k)}_{S_{gg},opt\ S_{ggapx}\ filter} \; ; \quad (\forall k) \tag{260}$$

Also, exactly as in Eq. (259) and (260) we see that:

$$\overline{e^2(k)}_{(1+\beta)S_{gg},opt(1+\beta)S_{gg}\ filter} \geq \overline{e^2(k)}_{S_{ggapx},opt\ S_{ggapx}\ filter} \; ; \quad (\forall k) \tag{261}$$

However, by optimality:

$$\overline{e^2(k)}_{(1+\beta)S_{gg},opt(1+\beta)S_{gg}\ filter} \leq \overline{e^2(k)}_{(1+\beta)S_{gg},opt\ S_{gg}\ filter} \; ; \quad (\forall k) \tag{262}$$

Writing out the right-hand side of Eq. (262), we have:

$$\overline{e^2(k)}_{(1+\beta)S_{gg},\text{opt } S_{gg} \text{ filter}} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\{S_{p_1p_1}(\omega_d)\,|H_{S_{gg},\text{opt},k}(\omega_d) - 1|^2$$

$$+ (S_{vv}(\omega_d) + (1+\beta)S_{gg}(\omega_d))\,|H_{S_{gg},\text{opt},k}(\omega_d)|^2\}d\omega_d \tag{263}$$

$$\leq \frac{1}{2\pi}(1+\beta)\int_{-\pi}^{\pi}\{S_{p_1p_1}(\omega_d)\,|H_{S_{gg},\text{opt},k}(\omega_d) - 1|^2$$

$$+ (S_{vv}(\omega_d) + S_{gg}(\omega_d))\,|H_{S_{gg},\text{opt},k}(\omega_d)|^2\}d\omega_d \tag{264}$$

$$\overline{e^2(k)}_{(1+\beta)S_{gg},\text{opt } S_{gg} \text{ filter}} \leq (1+\beta)\overline{e^2(k)}_{S_{gg},\text{opt } S_{gg} \text{ filter}} \quad ; \quad (\forall k) \tag{265}$$

Bringing together inequalities Eq. (261), (262), and (265), we see that:

$$\overline{e^2(k)}_{S_{ggapx},\text{opt } S_{ggapx} \text{ filter}} \leq (1+\beta)\overline{e^2(k)}_{S_{gg},\text{opt } S_{gg} \text{ filter}} \quad ; \quad (\forall k) \tag{266}$$

Finally, bringing together inequalities Eq. (259), (260), and (266), we have:

$$\left(\frac{1}{1+\beta}\right)\overline{e^2(k)}_{S_{ggapx},\text{opt } S_{ggapx} \text{ filter}} \leq \overline{e^2(k)}_{S_{gg},\text{opt } S_{gg} \text{ filter}}$$

$$\leq \overline{e^2(k)}_{S_{gg},\text{opt } S_{ggapx} \text{ filter}}$$

$$\leq \overline{e^2(k)}_{S_{ggapx},\text{opt } S_{ggapx} \text{ filter}} \quad ; \quad (\forall k) \tag{267}$$

We may let $k \to \infty$ in Eq. (267) to obtain a relation between the steady-state estimation errors:

$$\left(\frac{1}{1+\beta}\right)\overline{e^2}_{S_{ggapx},\text{opt } S_{ggapx} \text{ filter}} \leq \overline{e^2}_{S_{gg},\text{opt } S_{gg} \text{ filter}}$$

$$\leq \overline{e^2}_{S_{gg},\text{opt } S_{ggapx} \text{ filter}}$$

$$\leq \overline{e^2}_{S_{ggapx},\text{opt } S_{ggapx} \text{ filter}} \tag{267a}$$

To be more rigorous this relation may be rederived using the same arguments as in Eq. (245)-(267) and the fact that in steady state the Kalman filter solution is equivalent to the Wiener-Hopf optimal impulse response, h(n). Thus, as the finite-dimensional approximation is selected to make β small, the estimation error of filtering gyroscopic noise using the finite-dimensional Kalman filter, and the estimation error of filtering the finite-dimensional noise using the finite-dimensional filter approach the estimation error of filtering gyroscopic noise using the optimal gyro noise filter. Thus we see that the finite-dimensional gyro noise approximation problem, with the mean-squared estimation error sequence $\{\overline{e^2(k)}\}$ as a criterion, reduces to the gyro noise PSD approximation problem, where given $\beta>0$ we wish to find a finite-dimensional linear system with output power spectrum related to $S_{gg}(\omega_d)$ as in Eq. (257). Note also that as the estimation error of filtering the finite-dimensional noise using the finite-dimensional filter is more easily determined numerically than the other two quantities in Eq. (267), we may also use this result to bound those quantities.

We now discuss methods of using finite-dimensional systems to approximate the gyro noise PSD as in Eq. (257):

$$S_{gg}(\omega_d) \le S_{ggapx}(\omega_d) \le (1+\beta)S_{gg}(\omega_d) \; ; \quad \omega_d \in [-\pi, \pi] \tag{268}$$

We recall (Eq. (17)) that the gyroscopic noise PSD, normalized to (variance = 1) is given by (continuous time version):

$$S_{gg}(f) = \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\tan^{-1}\left[\frac{2\pi f(\tau_2-\tau_1)}{1+4\pi^2 f^2 \tau_2\tau_1}\right] \; ; \quad S_{gg}(0) = \frac{2(\tau_2-\tau_1)}{\ell n(\tau_2/\tau_1)} \tag{269}$$

Note that the discrete time version, $S_{gg}(\omega_d)$, would be an aliased version of Eq. (269), depending of the sampling time $(\tilde{T})$:

$$S_{gg}(\omega_d) = \left(\frac{1}{\tilde{T}}\right) \sum_{k=-\infty}^{\infty} S_{gg}\left(\frac{\omega_d}{2\pi\tilde{T}} + \frac{k}{\tilde{T}}\right) \tag{269a}$$

In Eq. (269), τ_1 and τ_2 are the limits of the time constant probability density function of the linear systems excited by independent white noises. The density function is given by Eq. (4):

$$p_d(\tau) = \begin{cases} (1/\ell n(\tau_2/\tau_1))(1/\tau) \; ; & \tau_1 \le \tau \le \tau_2 \\ 0 \; ; & \text{otherwise} \end{cases} \tag{270}$$

Recall that each linear system is characterized by the transfer function (Eq. (3)):

$$\left(\frac{\tau}{1 + \tau s}\right) \tag{271}$$

and has output $x(\tau, t)$ from input of white noise with autocorrelation function (Eq. (13) and (16)):

$$E[w(\tau, t)w(\beta, s)] = (2\ln(\tau_2/\tau_1))\delta(\tau - \beta)\delta(t - s) \tag{271a}$$

and that gyroscopic noise was modeled as (Eq. (15)):

$$g(t) = \int_{\tau_1}^{\tau_2} x(\tau, t)p_d(\tau)d\tau \tag{271b}$$

We shall discuss three methods of approximating $S_{gg}(\omega_d)$ as in Eq. (268). The first method is to approximate $S_{gg}(\omega_d)$ by a series of the form:

$$S_{ggapx}(\omega_d) = \sum_{i=0}^{N} \alpha_i \cos(i\omega_d) \tag{272}$$

where $\{\alpha_i\}$ is a sequence of real constants. Due to the continuity and evenness of $S_{gg}(\omega_d)$, such an approximation can be made to arbitrary accuracy (as the number N is increased) with respect to the uniform norm:

$$||S_{ggapx}(\omega_d) - S_{gg}(\omega_d)||_\infty = \sup|S_{ggapx}(\omega_d) - S_{gg}(\omega_d)| \;; \quad \omega_d \in [-\pi, \pi] \tag{273}$$

This is a result of Jackson's Theorem[20], which states that if $q(\omega_d)$ is continuous on $[-\pi, \pi]$, is even, and has period of 2π, then:

$$||q(\omega_d) - q_N^*(\omega_d)||_\infty \leq 6\lambda(1/N) \tag{274}$$

where $q_N^*(\omega_d)$ is the best uniform approximation (which exists and is even) among trigonometric polynomials of degree at most N, and $\lambda(\cdot)$ is the modulus of continuity of $q(\omega_d)$, defined as:

$$\lambda(\gamma) = \sup|q(\omega_1) - q(\omega_2)| \;;$$

$$\omega_1, \omega_2 \in [-\pi, \pi]$$

$$|\omega_1 - \omega_2| \leq \gamma \tag{275}$$

Once such an approximation is made, we could write it in the form:

$$S_{ggapx}(\omega_d) = \alpha_0 + \sum_{i=1}^{N} \frac{\alpha_i}{2}(e^{ji\omega_d} + e^{-ji\omega_d}) \tag{276}$$

(where α_0 may have to be increased somewhat in order that Eq. (268) be satisfied). Then by the symmetry of terms in $e^{ji\omega_d}$ and $e^{-ji\omega_d}$ we could factor Eq. (276) into the form:

$$S_{ggapx}(\omega_d) = \prod_{i=1}^{N} (\beta_i - \gamma_i e^{j\omega_d})(\beta_i - \gamma_i e^{-j\omega_d}) \tag{277}$$

Digressing for a moment, the z-transform relations for a discrete sequence f(n) are given by (z is a complex number):

$$F(z) = \sum_{n=-\infty}^{\infty} f(n) z^{-n}$$

$$f(n) = \frac{1}{2\pi j} \oint F(z) z^{n-1}\, dz$$

In the z-transform domain (replace $e^{j\omega_d}$ by z), we see that Eq. (277) becomes (The PSD in the z-domain is defined as the z-transform of the autocorrelation function.):

$$S_{ggapx}(z) = \prod_{i=1}^{N} (\beta_i - \gamma_i z)(\beta_i - \gamma_i z^{-1}) \tag{278}$$

A noise with this power spectral density could be obtained as the output of the following z-transform response (realized nonrecursively) to a white noise input of unit variance:

$$F(z) = \prod_{i=1}^{N} (\beta_i - \gamma_i z) \tag{279}$$

The above is due to the fact that if a digital system of z-transform response F(z) is excited by a white noise of unit variance, the PSD of the output is given by:

$$S(z) = F(z)F(z^{-1}) \tag{280}$$

The system in Eq. (279) could in turn be represented by the output of an N-state difference equation. Thus we would have a model to generate our gyro noise approximation. The shortcoming of this approach is that due to the rapid falloff with frequency of the gyro noise PSD (Eq. (269)), the modulus of continuity of $S_{gg}(\omega_d)$ is large, forcing a very high order digital system to generate the approximate noise. Because the ease of implementation of the Kalman filter which would result from our finite-dimensional gyroscopic noise model is highly dependent on the dimensionality of the noise model, we use a different approximation method (soon to be discussed), which requires a number of states generally orders of magnitude less than does this method, for a given degree of approximation per Eq. (268).

The second method of approximation is to make an optimal (per the criterion of Eq.(268)) N^{th} order rational approximation to $S_{gg}(\omega_d)$ of the following form:

$$S_{ggapx}(\omega_d) = \frac{\alpha_0 + \sum_{i=1}^{N} \alpha_i (e^{ji\omega_d} + e^{-ji\omega_d})}{\beta_0 + \sum_{i=1}^{N} \beta_i (e^{ji\omega_d} + e^{-ji\omega_d})} \qquad (281)$$

Given τ_1, τ_2, and N, the above would be carried out by a digitial computer. The primary shortcomings of this method are first that the answer would yield little insight relative to our original infinite-dimensional model, and second that we would not be able to change the parameters (i.e. τ_1 and τ_2) of the gyro noise without complete recomputation of the finite-dimensional model.

The third method is one which makes use of the exact form of the gyroscopic noise power spectral density (Eq. (269)). We shall use finite-dimensional continuous-time linear systems to approximate the continuous time PSD, and shall show that the digital aliased version of $S_{ggapx}(f)$ approximates Eq. (269a) in the same manner (per Eq. (268)). Qualitatively, the approximation is made as follows: we split the time constant probability density function (Eq. (4)) of linear systems into segments, where the outputs of the systems with time constants in each segment are weighted and integrated to yield a noise due to that segment. The noises from each of the segments, when added, would yield gyroscopic noise. Using the Taylor series for the $\tan^{-1}[\cdot]$ function, we approximate the noise from each segment as the output of a first-order linear system fed by white noise, and add the resulting approximate noises together to achieve a finite-dimensional approximation to gyroscopic noise.

The first step in our approximation is to write the gyro noise as a sum of N independent noises as follows (see Eq. (271b)) ($\tau^1 = \tau_1$, $\tau^{N+1} = \tau_2$):

$$g(t) = \int_o^\infty x(\tau, t) \sum_{i=1}^{N} p_{\tau_i}(\tau)d\tau = \sum_{i=1}^{N} \int_o^\infty x(\tau, t)p_{\tau_i}(\tau)d\tau \qquad (282)$$

where $p_{\tau_i}(\tau)$ is a density function given by, for a selected constant ($\delta > 0$):

$$p_{\tau_i}(\tau) = \begin{cases} (1/\ln(\tau_2/\tau_1))(1/\tau) \; ; & \left\{ \begin{array}{r} \tau^i \le \tau \le \tau^{i+1} = (1+\delta)\tau^i; \; i=1,\ldots,(N-1) \\ \text{and } \tau^N \le \tau \le \tau^{N+1} = \tau_2 \; ; \; i=N \end{array} \right\} \\ & (\text{note } \tau_2 \le (1+\delta)\tau^N) \\ 0 \; ; & \text{otherwise} \end{cases} \tag{283}$$

The power spectral density of the i^{th} noise in Eq. (282) is given by:

$$S^i_{gg}(f) = \left(\frac{2}{\ln(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\tan^{-1}\left[\frac{2\pi f(\tau^{i+1} - \tau^i)}{1 + 4\pi^2 f^2 \tau^{i+1}\tau^i}\right] \tag{284}$$

Using Eq. (283) in Eq. (284), we obtain: (redefine δ for (i=N))

$$S^i_{gg}(f) = \left(\frac{2}{\ln(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\tan^{-1}\left[\frac{2\pi f\delta\tau^i}{1 + 4\pi^2 f^2(1+\delta)(\tau^i)^2}\right] \; ; \quad i=1,\ldots,N \tag{285}$$

Note that $S_{gg}(f)$ can be recovered by adding these, using the trigonometric identity:

$$\tan^{-1}[A] - \tan^{-1}[B] = \tan^{-1}\left[\frac{A - B}{1 + AB}\right] \tag{286}$$

That is (where the first equality is from Eq. (17)):

$$S_{gg}(f) = \left(\frac{2}{\ln(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\tan^{-1}\left[\frac{2\pi f(\tau_2 - \tau_1)}{1 + 4\pi^2 f^2 \tau_2\tau_1}\right] = \sum_{i=1}^{N} S^i_{gg}(f)$$

$$= \sum_{i=1}^{N} \left(\frac{2}{\ln(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\tan^{-1}\left[\frac{2\pi f\delta\tau^i}{1 + 4\pi^2 f^2(1+\delta)(\tau^i)^2}\right] \tag{287}$$

Our goal is to make a rational approximation for each of the noises (Eq. (285)), and then to add the rational approximations together to approximate $S_{gg}(f)$. The first step is to note that the Taylor series (about $x = 0$) for $\tan^{-1}(x)$, to first order, with remainder term, is given by:

$$\tan^{-1}(x) = x - \frac{\xi x^2}{(\xi^2 + 1)^2} \tag{288}$$

where:

$$\xi\varepsilon(0, x) \text{ if } (x > 0), \text{ and } \xi\varepsilon(x, 0) \text{ if } (x < 0) \tag{289}$$

We note next the following, from Eq. (288):

$$|\tan^{-1}[x]| \le |x| \tag{290}$$

and:

$$|x - \tan^{-1}[x]| = \left|\frac{\xi x^2}{(\xi^2 + 1)^2}\right| \leq |x^3| \tag{291}$$

Note that the above quantity is small for small x. We now make the following approximation in Eq. (285):

$$\tan^{-1}\left[\frac{2\pi f\delta\tau^i}{1 + 4\pi^2 f^2(1+\delta)(\tau^i)^2}\right] \simeq \left[\frac{2\pi f\delta\tau^i}{1 + 4\pi^2 f^2(1+\delta)(\tau^i)^2}\right]$$

Hence the rational approximation to $S^i_{gg}(f)$ in Eq. (285) is given by:

$$S^i_{ggapx}(f) = \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\left(\frac{2\pi f\delta\tau^i}{1 + 4\pi^2 f^2(1+\delta)(\tau^i)^2}\right) \tag{292}$$

As we noted in Eq. (287) the PSD of gyro noise can be expressed as:

$$S_{gg}(f) = \sum_{i=1}^{N} S^i_{gg}(f) = \sum_{i=1}^{N} \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\tan^{-1}\left(\frac{2\pi f\delta\tau^i}{1 + 4\pi^2 f^2(1+\delta)(\tau^i)^2}\right) \tag{293}$$

Thus the approximation to $S_{gg}(f)$ is given by:

$$S_{ggapx}(f) = \sum_{i=1}^{N} S^i_{ggapx}(f) = \sum_{i=1}^{N} \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\left(\frac{2\pi f\delta\tau^i}{1 + 4\pi^2 f^2(1+\delta)(\tau^i)^2}\right) \tag{294}$$

In the following discussion we show that as we decrease the multiplicative width $(1 + \delta)$ of the intervals into which the time constant probability density function is separated (hence increasing N, the number of dimensions of our approximation to gyroscopic noise), the PSD of the finite-dimensional approximation to gyro noise will more closely (i.e. with smaller β) approximate the PSD of the gyro noise per Eq. (257). A conservative version (Eq. (335)) of the final result (Eq. (334)) of the discussion then states that:

$$\begin{aligned}\left(\frac{1}{1+10\delta^2}\right)\overline{e^2(k)}_{S_{ggapx},\text{opt } S_{ggapx} \text{ filter}} &\leq \overline{e^2(k)}_{S_{gg},\text{opt } S_{gg} \text{ filter}} \\ &\leq \overline{e^2(k)}_{S_{gg},\text{opt } S_{ggapx} \text{ filter}} \quad ; \quad (\forall k),\ (\forall \delta \leq 2),\ (\forall 5 \leq (\tau_2/\tau_1) \leq 10^7) \\ &\leq \overline{e^2(k)}_{S_{ggapx},\text{opt } S_{ggapx} \text{ filter}}\end{aligned}$$

which relates the mean-squared error sequences of interest to the quantity δ. Note that as $\delta \to 0$ the leftmost quantity approaches the rightmost, hence closely bounding the quantities of interest in the interior of the inequality.

Digressing for a moment, the maximum value of the quantity:

$$\left[\frac{2\pi f \delta \tau^i}{1 + 4\pi^2 f^2 (1+\delta)(\tau^i)^2}\right]$$

may be found by equating the first derivative to zero:

$$\frac{\partial}{\partial f}\left[\frac{2\pi f \delta \tau^i}{1 + 4\pi^2 f^2 (1+\delta)(\tau^i)^2}\right] = 0$$

We find that:

$$f_{max} = \frac{1}{2\pi\tau^i \sqrt{1 + \delta}}$$

and that:

$$\left[\frac{2\pi f \delta \tau^i}{1 + 4\pi^2 f^2 (1+\delta)(\tau^i)^2}\right]\Bigg|_{f_{max}} = \frac{\delta}{2\sqrt{1 + \delta}} \qquad (295)$$

Note that the last multiplicative width in the time constant density function is actually less than $(1 + \delta)$, since we will not reach τ_2 exactly with products of the form $(1 + \delta)^n \tau_1$. All bounds to be derived below are only tightened if this fact is explicitly taken into account, so we will ignore it. By Eq. (290), (293), and (294) we see that:

$$S_{ggapx}(f) \geq S_{gg}(f) \qquad (296)$$

and that (both $S_{gg}(f)$ and $S_{ggapx}(f)$ are even functions of (f), so we only consider $(f > 0)$):

$$|S_{ggapx}(f) - S_{gg}(f)| = \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\left|\sum_{i=1}^{N}\left\{\frac{2\pi f \delta \tau^i}{1 + 4\pi^2 f^2 (1+\delta)(\tau^i)^2} - \tan^{-1}\left[\frac{2\pi f \delta \tau^i}{1 + 4\pi^2 f^2 (1+\delta)(\tau^i)^2}\right]\right\}\right|$$

$$\leq \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\sum_{i=1}^{N}\left|\frac{2\pi f \delta \tau^i}{1 + 4\pi^2 f^2 (1+\delta)(\tau^i)^2} - \tan^{-1}\left[\frac{2\pi f \delta \tau^i}{1 + 4\pi^2 f^2 (1+\delta)(\tau^i)^2}\right]\right|$$

Hence by Eq. (291):

$$|S_{ggapx}(f) - S_{gg}(f)| \le \left(\frac{2}{\ln(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\sum_{i=1}^{N}\left(\frac{2\pi f\delta\tau^i}{1 + 4\pi^2 f^2(1+\delta)(\tau^i)^2}\right)^3 \tag{297}$$

We shall translate Eq. (297) into three separate bounds. First, by Eq. (295) and Eq. (297) we see that:

$$|S_{ggapx}(f) - S_{gg}(f)| \le \left(\frac{2}{\ln(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\sum_{i=1}^{N}\left(\frac{\delta}{2\sqrt{1+\delta}}\right)^3$$

$$\le \left(\frac{2}{\ln(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)(\log_{(1+\delta)}(\tau_2/\tau_1) + 1)\left(\frac{\delta}{2\sqrt{1+\delta}}\right)^3$$

$$\le \left(\frac{2}{\ln(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\left(\frac{\ln(\tau_2/\tau_1) + \ln(1+\delta)}{\ln(1+\delta)}\right)\left(\frac{\delta^3}{8}\right)$$

We evaluate this in two ranges. First we restrict δ to be less than or equal to 2. Then we tighten to $\delta \le 0.5$. All of our arguments will be done separately for these two levels of restriction in δ. We thus obtain for the first range:

$$|S_{ggapx}(f) - S_{gg}(f)| \le \left(\frac{2}{\ln(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\left(\frac{\ln(3\tau_2/\tau_1)}{\delta/2}\right)\left(\frac{\delta^3}{8}\right)$$

$$|S_{ggapx}(f) - S_{gg}(f)| \le \left(\frac{\ln(3\tau_2/\tau_1)}{\ln(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\left(\frac{1}{2}\right)\delta^2 \; ; \quad (\forall\delta \le 2) \tag{298}$$

The second range gives the bound a bit tighter:

$$|S_{ggapx}(f) - S_{gg}(f)| \le \left(\frac{2}{\ln(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\left(\frac{\ln(1.5\tau_2/\tau_1)}{4\delta/5}\right)\left(\frac{\delta^3}{8}\right)$$

$$|S_{ggapx}(f) - S_{gg}(f)| \le \left(\frac{\ln(1.5\tau_2/\tau_1)}{\ln(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\left(\frac{5}{16}\right)\delta^2 \; ; \quad (\forall\delta \le 0.5) \tag{299}$$

Equations (298) and (299) comprise the first bound. The second bound is obtained from Eq. (297) as follows:

$$|S_{ggapx}(f)-S_{gg}(f)| \le \left(\frac{2}{\ln(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\sum_{i=1}^{N}\left(\frac{2\pi f\delta\tau^i}{1+4\pi^2 f^2(1+\delta)(\tau^i)^2}\right)^3 \le \left(\frac{2}{\ln(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\sum_{i=1}^{N}(2\pi f\delta\tau^i)^3$$

$$= \left(\frac{2}{\ln(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)(2\pi f\delta)^3\sum_{i=1}^{N}[(1+\delta)\tau_1]^3$$

Using a geometric sum argument (ratio = $(1+\delta)^{-3}$), we have:

$$|S_{ggapx}(f) - S_{gg}(f)| \leq \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)(2\pi f\delta)^3\left(\frac{\tau_2^3}{1 - \left(\frac{1}{1+\delta}\right)^3}\right) \quad (300)$$

Because the bounds in Eq. (298) and Eq. (299) are in terms of δ^2, we wish to find a constant γ such that:

$$1 - \left(\frac{1}{1+\delta}\right)^3 > \gamma\delta$$

That is:

$$\frac{1}{\delta} - \frac{1}{\delta(1+\delta)^3} > \gamma \quad (301)$$

But the derivative of the left hand side of Eq. (301) can easily be seen to be negative everywhere ($\delta > 0$). Thus we merely evaluate the left hand side for the two range limits of δ of concern. We obtain:

$$1 - \left(\frac{1}{1+\delta}\right)^3 \geq 0.481\delta \ ; \ \delta \leq 2$$

or better: (302)

$$1 - \left(\frac{1}{1+\delta}\right)^3 \geq 1.40\delta \ ; \ \delta \leq 0.5$$

Thus Eq. (300) becomes:

$$|S_{ggapx}(f) - S_{gg}(f)| \leq \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)(2\pi f\delta)^3\left(\frac{\tau_2^3}{0.481\delta}\right); \quad (\delta \leq 2)$$

$$= \left(\frac{1}{\ell n(\tau_2/\tau_1)}\right)(4.16\tau_2^3)(2\pi f)^2\delta^2 \quad ; \quad (\delta \leq 2) \quad (303)$$

or better:

$$|S_{ggapx}(f) - S_{gg}(f)| \leq \left(\frac{1}{\ell n(\tau_2/\tau_1)}\right)(1.43\tau_2^3)(2\pi f)^2\delta^2 \ ; \quad (\delta \leq 0.5) \quad (304)$$

Equations (303) and (304) comprise the second bound. We now formulate the third bound. From Eq. (297) once more, we have:

$$|S_{ggapx}(f) - S_{gg}(f)| \leq \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\sum_{i=1}^{N}\left(\frac{2\pi f\delta\tau^i}{1 + 4\pi^2 f^2(1+\delta)(\tau^i)^2}\right)^3$$

$$\leq \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\sum_{i=1}^{N}\left(\frac{2\pi f\delta\tau^i}{4\pi^2 f^2(1+\delta)(\tau^i)^2}\right)^3$$

$$\leq \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\left(\frac{\delta}{1+\delta}\right)^3\left(\frac{1}{2\pi f}\right)^3\sum_{i=1}^{N}\left(\frac{1}{\tau^i}\right)^3$$

$$\leq \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\left(\frac{\delta}{1+\delta}\right)^3\left(\frac{1}{2\pi f}\right)^3\left(\frac{(1/\tau_1)^3}{1 - \left(\frac{1}{1+\delta}\right)^3}\right)$$

Using Eq. (302) again, we have that:

$$|S_{ggapx}(f) - S_{gg}(f)| \le \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)^4 \left(\frac{\delta}{1+\delta}\right)^3 \left(\frac{(1/\tau_1)^3}{0.481\delta}\right); \quad (\delta \le 2)$$

$$= \left(\frac{1}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)^4 \left(\frac{4.16}{\tau_1^3}\right)\delta^2 \; ; \quad (\delta \le 2) \quad (305)$$

or better:

$$|S_{ggapx}(f) - S_{gg}(f)| \le \left(\frac{1}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)^4 \left(\frac{1.43}{\tau_1^3}\right)\delta^2 \; ; \quad (\delta \le 0.5) \quad (306)$$

The third bound is made up of Eq. (305) and (306).

We now have three bounds for $|S_{ggapx}(f) - S_{gg}(f)|$. Each of these bounds will possess a region in frequency over which it is smallest. We shall determine these regions by finding the frequencies where the bounds intersect. First we shall deal with the case where we specify that $\delta \le 2$. We can find the intersection of the first two bounds by taking:

$$\left(\frac{\ell n(3\tau_2/\tau_1)}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\left(\frac{1}{2}\right)\delta^2 = \left(\frac{1}{\ell n(\tau_2/\tau_1)}\right)(4.16\tau_2^3)(2\pi f)^2\delta^2$$

Then:

$$f_{\delta\le 2}(1 - 2) = \left(\frac{(\ell n(3\tau_2/\tau_1))^{1/3}}{2.03}\right)\left(\frac{1}{2\pi\tau_2}\right) \quad (307)$$

The value of the bounds at this frequency is:

$$B_{\delta\le 2}(1 - 2) = \left(\frac{(\ell n(3\tau_2/\tau_1))^{2/3}}{\ell n(\tau_2/\tau_1)}\right)(1.02)(\tau_2)\delta^2 \quad (308)$$

We next find the intersection of the second two bounds:

$$\left(\frac{1}{\ell n(\tau_2/\tau_1)}\right)(4.16\tau_2^3)(2\pi f)^2\delta^2 = \left(\frac{1}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)^4 \left(\frac{4.16}{\tau_1^3}\right)\delta^2$$

so that:

$$f_{\delta\le 2}(2 - 3) = \frac{1}{2\pi\sqrt{\tau_1\tau_2}} \quad (309)$$

The value of the bounds at this frequency is:

$$B_{\delta\le 2}(2 - 3) = \left(\frac{1}{\ell n(\tau_2/\tau_1)}\right)(\tau_2/\tau_1)(\tau_2)(4.16)\delta^2 \quad (310)$$

Finally, we find the intersection of the first and third bounds:

$$\left(\frac{\ln(3\tau_2/\tau_1)}{\ln(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\left(\frac{1}{2}\right)\delta^2 = \left(\frac{1}{\ln(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)^4\left(\frac{4.16}{\tau_1^3}\right)\delta^2$$

Then:

$$f_{\delta\le 2}(1-3) = \left(\frac{2.03}{(\ln(3\tau_2/\tau_1))^{1/3}}\right)\left(\frac{1}{2\pi\tau_1}\right) \quad (311)$$

The value of the bounds at this frequency is:

$$B_{\delta\le 2}(1-3) = \frac{(\ln(3\tau_2/\tau_1))^{4/3}}{\ln(\tau_2/\tau_1)}\left(\frac{\tau_1}{4.06}\right)\delta^2 \quad (312)$$

We plot the three bounds and their intersections on log-log scales in Figure 10. Note that the (1/f) bound line falls below the intersection of the other bound lines because $f_{\delta\le 2}(1-2) < f_{\delta\le 2}(1-3)$, a fact easily verified. Similarly, for $\delta \le 0.5$, we obtain the following:

$$f_{\delta\le 0.5}(1-2) = (0.602(\ln(1.5\tau_2/\tau_1))^{1/3})\left(\frac{1}{2\pi\tau_2}\right) \quad (313)$$

$$B_{\delta\le 0.5}(1-2) = \left(\frac{(\ln(1.5\tau_2/\tau_1))^{2/3}}{\ln(\tau_2/\tau_1)}\right)(0.519)(\tau_2)\delta^2 \quad (314)$$

$$f_{\delta\le 0.5}(2-3) = \frac{1}{2\pi\sqrt{\tau_2\tau_1}} \quad (315)$$

$$B_{\delta\le 0.5}(2-3) = \left(\frac{1}{\ln(\tau_2/\tau_1)}\right)(1.43)(\tau_2/\tau_1)(\tau_2)\delta^2 \quad (316)$$

$$f_{\delta\le 0.5}(1-3) = \left(\frac{1}{0.602(\ln(1.5\tau_2/\tau_1))^{1/3}}\right)\left(\frac{1}{2\pi\tau_1}\right) \quad (317)$$

$$B_{\delta\le 0.5}(1-3) = \left(\frac{(\ln(1.5\tau_2/\tau_1))^{4/3}}{\ln(\tau_2/\tau_1)}\right)(0.188)(\tau_1)\delta^2 \quad (318)$$

We graph these tighter bounds and intersections on a log-log scale in Figure 11.

We have thus obtained bounds on $|S_{ggapx}(f) - S_{gg}(f)|$ for the case where $\delta \le 2$, and in the case where we further tighten to $\delta \le 0.5$ (note, incidentally, that the coefficients of δ^2 in the bounds do not change substantially if we further restrict δ). Henceforth, we define:

$$e(f) \triangleq S_{ggapx}(f) - S_{gg}(f) \ ; \quad e'(f) \triangleq \text{our bound on } [S_{ggapx}(f) - S_{gg}(f)] \quad (319)$$

We desire to fulfill the criterion of Eq. (268). Thus we are interested not in $e(f)$, but in $(e(f)/S_{gg}(f))$. In order to bound this last quantity

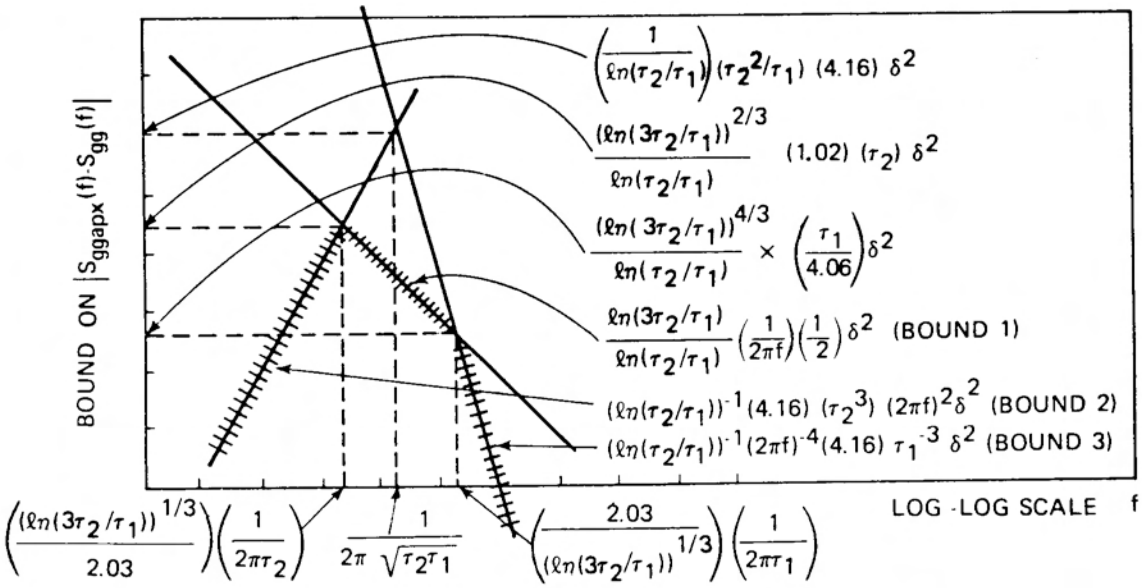

Figure 10. Bounds on $|S_{ggapx}(f) - S_{gg}(f)|$ for $\delta \le 2$. The hatchmarked line becomes the actual bound.

we again use three regions. We shall first concern ourselves with time constant spacings for our linear systems restricted only to $\delta \le 2$. The low frequency range is:

$$f \in \left[0, \left(\frac{(\ell n(3\tau_2/\tau_1))^{1/3}}{2.03}\right)\left(\frac{1}{2\pi\tau_2}\right)\right]$$

As $S_{gg}(f)$ is monotonically decreasing everywhere $(f \ge 0)$, and $e'_{\delta\le 2}(f)$ is monotonically increasing in this frequency range, we see that throughout this range we have that:

$$\frac{e_{\delta\le 2}(f)}{S_{gg}(f)} \le \left[\frac{e'_{\delta\le 2}\left[\left(\frac{(\ell n(3\tau_2/\tau_1))^{1/3}}{2.03}\right)\left(\frac{1}{2\pi\tau_2}\right)\right]}{S_{gg}\left[\left(\frac{(\ell n(3\tau_2/\tau_1))^{1/3}}{2.03}\right)\left(\frac{1}{2\pi\tau_2}\right)\right]}\right] = \left[\frac{(\ell n(3\tau_2/\tau_1))^{2/3}\ (1.02)(\tau_2)\delta^2}{S_{gg}\left[\left(\frac{(\ell n(3\tau_2/\tau_1))^{1/3}}{2.03}\right)\left(\frac{1}{2\pi\tau_2}\right)\right]}\right];$$

$$\forall f \in \left[0, \left(\frac{(\ell n(3\tau_2/\tau_1))^{1/3}}{2.03}\right)\left(\frac{1}{2\pi\tau_2}\right)\right] \quad (320)$$

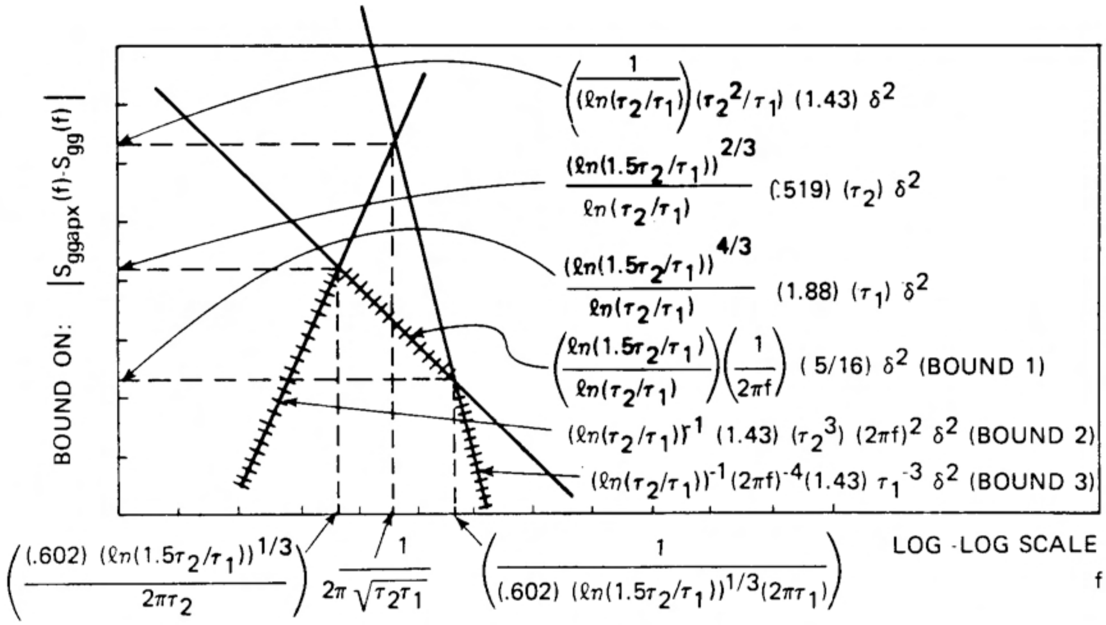

Figure 11. Bounds on $|S_{ggapx}(f) - S_{gg}(f)|$ for $\delta \leq (0.5)$. The hatchmarked line becomes the actual bound.

We next work on the high frequency range. We wish first to find a rational function of frequency less than $S_{gg}(f)$ for high frequencies. The intent is then to divide our bound on $e_{\delta\leq 2}(f)$ by this function, in order to bound $(e_{\delta\leq 2}(f)/S_{gg}(f))$ for high frequencies. To this end, we desire to apply the Taylor series to the $\tan^{-1}(\cdot)$ component of $S_{gg}(f)$:

$$S_{gg}(f) = \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\tan^{-1}\left[\frac{2\pi f(\tau_2 - \tau_1)}{1 + 4\pi^2 f^2 \tau_2 \tau_1}\right] \tag{321}$$

From the Taylor series, we see that:

$$\tan^{-1}[x] = x - \frac{\xi x^2}{(\xi^2 + 1)^2} \geq x - x^3 > 0 \ ; \quad (\forall 0 < x < 1) \tag{322}$$

where we require $(x < 1)$ in order to guarantee the last inequality. Thus we want f large enough so that:

$$\frac{2\pi f(\tau_2 - \tau_1)}{1 + 4\pi^2 f^2 \tau_2 \tau_1} < 1$$

But

$$\frac{2\pi f(\tau_2 - \tau_1)}{1 + 4\pi^2 f^2 \tau_2 \tau_1} \leq \frac{2\pi f(\tau_2 - \tau_1)}{4\pi^2 f^2 \tau_2 \tau_1} = \frac{(\tau_2 - \tau_1)}{2\pi f \tau_2 \tau_1}$$

So pick:

$$f > \frac{(\tau_2 - \tau_1)}{2\pi\tau_2\tau_1} \tag{323}$$

Thus, by Eq. (321) and Eq. (322):

$$S_{gg}(f) \geq \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\left[\frac{2\pi f(\tau_2-\tau_1)}{1 + 4\pi^2 f^2 \tau_2\tau_1} - \left(\frac{2\pi f(\tau_2-\tau_1)}{1 + 4\pi^2 f^2 \tau_2\tau_1}\right)^3\right] \geq 0 \tag{324}$$

and we obtain, for

$$f \geq \max\left[2\left(\frac{(\tau_2 - \tau_1)}{2\pi\tau_2\tau_1}\right),\ \left(\frac{2.02}{(\ell n(3\tau_2/\tau_1))^{1/3}}\right)\left(\frac{1}{2\pi\tau_1}\right)\right]$$

(we doubled the first frequency for reasons to be seen below):

$$\frac{e_{\delta\leq 2}(f)}{S_{gg}(f)} \leq \frac{\left(\frac{1}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)^4 \left(\frac{4.16}{\tau_1^3}\right)\delta^2}{\left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\left(\frac{2\pi f(\tau_2-\tau_1)}{1+4\pi^2 f^2\tau_2\tau_1}\right)\left(1 - \left(\frac{2\pi f(\tau_2-\tau_1)}{1+4\pi^2 f^2\tau_2\tau_1}\right)^2\right)}$$

$$\leq \left(\frac{2.08}{\tau_1^3}\right)\left(\frac{1}{2\pi f}\right)^3 \left[\frac{1+4\pi^2 f^2\tau_2\tau_1}{2\pi f(\tau_2-\tau_1)}\right]\left[1 - \left(\frac{\tau_2-\tau_1}{2\pi f\tau_2\tau_1}\right)^2\right]^{-1} \delta^2$$

$$\leq \left(\frac{2.08}{\tau_1^3}\right)\left(\frac{1}{2\pi f}\right)^2 \left(\frac{\tau_1\tau_2}{\tau_2-\tau_1}\right)\left(\frac{1+4\pi^2 f^2\tau_2\tau_1}{4\pi^2 f^2\tau_2\tau_1}\right)\left[1 - \left(\frac{\tau_2 - \tau_1}{2\pi f\tau_2\tau_1}\right)^2\right]^{-1} \delta^2$$

$$\leq \left(\frac{2.08}{\tau_1^3}\right)\left(\frac{1}{2\pi f}\right)^2 \left(\frac{\tau_1\tau_2}{\tau_2-\tau_1}\right)\left(1 + \frac{1}{4\pi^2 f^2\tau_2\tau_1}\right)\left[1 - \left(\frac{\tau_2-\tau_1}{2\pi f\tau_2\tau_1}\right)^2\right]^{-1} \delta^2$$

By the definition of our frequency range, we see that:

$$\frac{e_{\delta\leq 2}(f)}{S_{gg}(f)} \leq \left(\frac{2.08}{\tau_1^3}\right)\left(\frac{1}{2\pi f}\right)^2 \left(\frac{\tau_1\tau_2}{\tau_2 - \tau_1}\right)\left(1 + \frac{\tau_2\tau_1}{4(\tau_2 - \tau_1)^2}\right)\left(\frac{1}{1 - (1/4)}\right)\delta^2$$

$$\frac{e_{\delta\leq 2}(f)}{S_{gg}(f)} \leq (0.693)\left(\frac{1}{1 - (\tau_1/\tau_2)}\right)^3\left(1 + \frac{1}{4((\tau_2/\tau_1) - 1)(1 - (\tau_1/\tau_2))}\right)\delta^2 \tag{325}$$

Equation (325) is our high frequency ratio bound.

We next work on the mid-range frequencies:

$$f \;\varepsilon\; \left[\left(\frac{(\ln(3\tau_2/\tau_1))^{1/3}}{2.03}\right)\left(\frac{1}{2\pi\tau_2}\right),\; \max\left\{2\left(\frac{\tau_2-\tau_1}{2\pi\tau_2\tau_1}\right),\; \left(\frac{2.03}{(\ln(3\tau_2/\tau_1))^{1/3}}\right)\left(\frac{1}{2\pi\tau_1}\right)\right\}\right]$$

The error $e_{\delta\leq 2}(f)$ in this range can be bounded by Eq. (298), a function proportional to (1/f). Thus if we can find another function proportional to (1/f) which is less than $S_{gg}(f)$ in this range, we can divide the error by it to bound $(e_{\delta\leq 2}(f)/S_{gg}(f))$. Because $S_{gg}(f)$ is concave on log-log scales (see sample graph given in Figure 13), it will cross a line proportional to (1/f) at most twice. Thus if we find the lower of the (1/f) lines passing through the points:

$$\left(\left[\left(\frac{(\ln(3\tau_2/\tau_1))^{1/3}}{2.03}\right)\left(\frac{1}{2\pi\tau_2}\right)\right],\quad S_{gg}\left[\left(\frac{(\ln(3\tau_2/\tau_1))^{1/3}}{2.03}\right)\left(\frac{1}{2\pi\tau_2}\right)\right]\right)$$

and:

$$\left(\max\left[2\left(\frac{\tau_2-\tau_1}{2\pi\tau_2\tau_1}\right),\quad \left(\frac{2.03}{(\ln(3\tau_2/\tau_1))^{1/3}}\right)\left(\frac{1}{2\pi\tau_1}\right)\right],\right.$$
$$\left. S_{gg}\left\{\max\left[2\left(\frac{\tau_2-\tau_1}{2\pi\tau_2\tau_1}\right),\quad \left(\frac{2.03}{(\ln(3\tau_2/\tau_1))^{1/3}}\right)\left(\frac{1}{2\pi\tau_1}\right)\right]\right\}\right)$$

this line will be below $S_{gg}(f)$ for all f in the mid-range. These functions are given by, respectively:

$$\left(\frac{1}{f}\right)\left(\frac{(\ln(3\tau_2/\tau_1))^{1/3}}{2.03}\right)\left(\frac{1}{2\pi\tau_2}\right) \times S_{gg}\left[\left(\frac{(\ln(3\tau_2/\tau_1))^{1/3}}{2.03}\right)\left(\frac{1}{2\pi\tau_2}\right)\right] \triangleq \left(\frac{1}{f}\right)\text{(quantity a)}$$

and:

$$\left(\frac{1}{f}\right)\left(\max\left[2\left(\frac{\tau_2-\tau_1}{2\pi\tau_2\tau_1}\right),\quad \left(\frac{2.03}{(\ln(3\tau_2/\tau_1))^{1/3}}\right)\left(\frac{1}{2\pi\tau_1}\right)\right]\right) \times$$
$$\times\; S_{gg}\left[\max\left[2\left(\frac{\tau_2-\tau_1}{2\pi\tau_2\tau_1}\right),\quad \left(\frac{2.03}{(\ln(3\tau_2/\tau_1))^{1/3}}\right)\left(\frac{1}{2\pi\tau_1}\right)\right]\right] \triangleq \left(\frac{1}{f}\right)\text{(quantity b)}$$

Thus the error ratio is bounded by:

$$\frac{e_{\delta\leq 2}(f)}{S_{gg}(f)} \leq \frac{\left(\frac{\ln(3\tau_2/\tau_1)}{\ln(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\left(\frac{1}{2}\right)\delta^2}{\left(\frac{1}{f}\right)\{\min(\text{quantity a, quantity b})\}} \tag{326}$$

Equation (326) is our mid-frequency range error ratio bound. Thus, for a given (τ_2/τ_1) ratio, we select the largest of Eq. (320), (325), and (326) as our full frequency bound on the error ratio $(e_{\delta\leq 2}(f)/S_{gg}(f))$. This will give us:

$$\frac{e(f)}{S_{gg}(f)} \leq [K_{\delta\leq 2}(\tau_2/\tau_1)]\delta^2 \ ; \quad (\forall\delta \leq 2) \tag{327}$$

So that (the first inequality is from Eq. (296)):

$$S_{gg}(f) \leq S_{ggapx}(f) \leq (1 + [K_{\delta\leq 2}(\tau_2/\tau_1)]\delta^2)S_{gg}(f) \tag{328}$$

The digital power spectral densities resulting from the sampled systems are merely aliased versions of those in Eq. (328), so that:

$$S_{gg}(\omega_d) \leq S_{ggapx}(\omega_d) \leq (1 + [K_{\delta\leq 2}(\tau_2/\tau_1)]\delta^2)S_{gg}(\omega_d) \ ; \quad (\forall\delta \leq 2) \tag{329}$$

The corresponding numbers to Eq. (320), (325), and (326) for $(\delta \leq 0.5)$ are:

$$\frac{e_{\delta\leq 0.5}(f)}{S_{gg}(f)} \leq \frac{e'_{\delta\leq 0.5}\left|(0.602)(\ell n(1.5\tau_2/\tau_1))^{1/3}\ (\frac{1}{2\pi\tau_2})\right|}{S_{gg}\left|(0.602)(\ell n(1.5\tau_2/\tau_1))^{1/3}\ (\frac{1}{2\pi\tau_2})\right|}$$

$$= \left[\frac{\left(\frac{(\ell n(1.5\tau_2/\tau_1))^{2/3}}{\ell n(\tau_2/\tau_1)}\right)(0.519)(\tau_2)\delta^2}{S_{gg}\left|(0.602)(\ell n(1.5\tau_2/\tau_1))^{1/3}\ (\frac{1}{2\pi\tau_2})\right|}\right] \ ;$$

$$f \ \varepsilon [0,\ (0.602)(\ell n(1.5\tau_2/\tau_1))^{1/3}\ (\frac{1}{2\pi\tau_2})] \tag{330}$$

$$\frac{e_{\delta\leq 0.5}(f)}{S_{gg}(f)} \leq (0.238)\left(\frac{1}{1 - (\tau_1/\tau_2)}\right)^3 \left(1 + \left|\frac{1}{4((\tau_2/\tau_1) - 1)(1 - (\tau_1/\tau_2))}\right|\right)\delta^2 \ ; \tag{331}$$

$$f \geq \max\left|2\left(\frac{\tau_2-\tau_1}{2\pi\tau_2\tau_1}\right) \ , \ \left(\frac{1}{(0.602)(\ell n(1.5\tau_2/\tau_1))^{1/3}}\right)(\frac{1}{2\pi\tau_1})\right|$$

$$\frac{e_{\delta\le 0.5}(f)}{S_{gg}(f)} \le \frac{\left[\left(\frac{\ell n(1.5\tau_2/\tau_1)}{\ell n(\tau_2/\tau_1)}\right)(\frac{1}{2\pi})(\frac{5}{16})\delta^2\right]}{\left[\min\left\{\begin{matrix}(0.602)(\ell n(1.5\tau_2/\tau_1))^{1/3}(\frac{1}{2\pi\tau_2})S_{gg}[(0.602)(\ell n(1.5\tau_2/\tau_1))^{1/3}(\frac{1}{2\pi\tau_2})], \\ \max\left[2\left(\frac{\tau_2-\tau_1}{2\pi\tau_2\tau_1}\right), \left(\frac{1}{(0.602)(\ell n(1.5\tau_2/\tau_1))^{1/3}}\right)(\frac{1}{2\pi\tau_1})\right] \times \\ \times S_{gg}\left\{\max\left[2\left(\frac{\tau_2-\tau_1}{2\pi\tau_2\tau_1}\right), \left(\frac{1}{(0.602)(\ell n(1.5\tau_2/\tau_1))^{1/3}}\right)(\frac{1}{2\pi\tau_1})\right]\right\}\end{matrix}\right\}\right]} \tag{332}$$

$$f\varepsilon\left[(0.602)(\ell n(1.5\tau_2/\tau_1))^{1/3}(\frac{1}{2\pi\tau_2}),\ \max\left\{2\left(\frac{\tau_2-\tau_1}{2\pi\tau_2\tau_1}\right), \left[\left(\frac{1}{(0.602)(\ell n(1.5\tau_2/\tau_1))^{1/3}}\right)(\frac{1}{2\pi\tau_1})\right]\right\}\right]$$

Given the ratio (τ_2/τ_1), we select the largest of the three numbers in Eq. (330), (331), and (332) and call this $K_{\delta\le 0.5}(\tau_2/\tau_1)$. We then obtain a tighter bound than in Eq. (329):

$$S_{gg}(\omega_d) \le S_{ggapx}(\omega_d) \le (1 + [K_{\delta\le 0.5}(\tau_2/\tau_1)]\delta^2)S_{gg}(\omega_d) \quad ; \quad (\delta \le 0.5) \tag{333}$$

Plots of $K_{\delta\le 0.5}(\tau_2/\tau_1)$ and $K_{\delta\le 2}(\tau_2/\tau_1)$ are displayed in Figure 12. Note that the plots are done for time constant density function endpoint ratios $[(\tau_2/\tau_1) \ge 5]$. For values of (τ_2/τ_1) less than this we observe that $K_{\delta\le 2}(\tau_2/\tau_1)$ and $K_{\delta\le 0.5}(\tau_2/\tau_1)$ actually increase as (τ_2/τ_1) approaches unity. The increase in the high frequency range bounds (Eq.(325) and (331)) is lessened by considering in their derivations the second frequencies in their ranges, i.e.

$$\left[f \ge \left(\frac{2.02}{(\ell n(3\tau_2/\tau_1))^{1/3}}\right)(\frac{1}{2\pi\tau_1})\right]$$

and

$$\left[f \ge \left(\frac{1}{0.602(\ell n(1.5\tau_2/\tau_1))^{1/3}}\right)(\frac{1}{2\pi\tau_1})\right] ,$$

respectively, instead of merely using

$$\left[f \ge 2\left(\frac{\tau_2-\tau_1}{2\pi\tau_2\tau_1}\right)\right] .$$

When this is done, we find that $K_{\delta\le 2}(\tau_2/\tau_1)$ and $K_{\delta\le 0.5}(\tau_2/\tau_1)$ behave proportionally to $(1/[(\tau_2/\tau_1)-1])$ for (τ_2/τ_1) close to 1. Of course

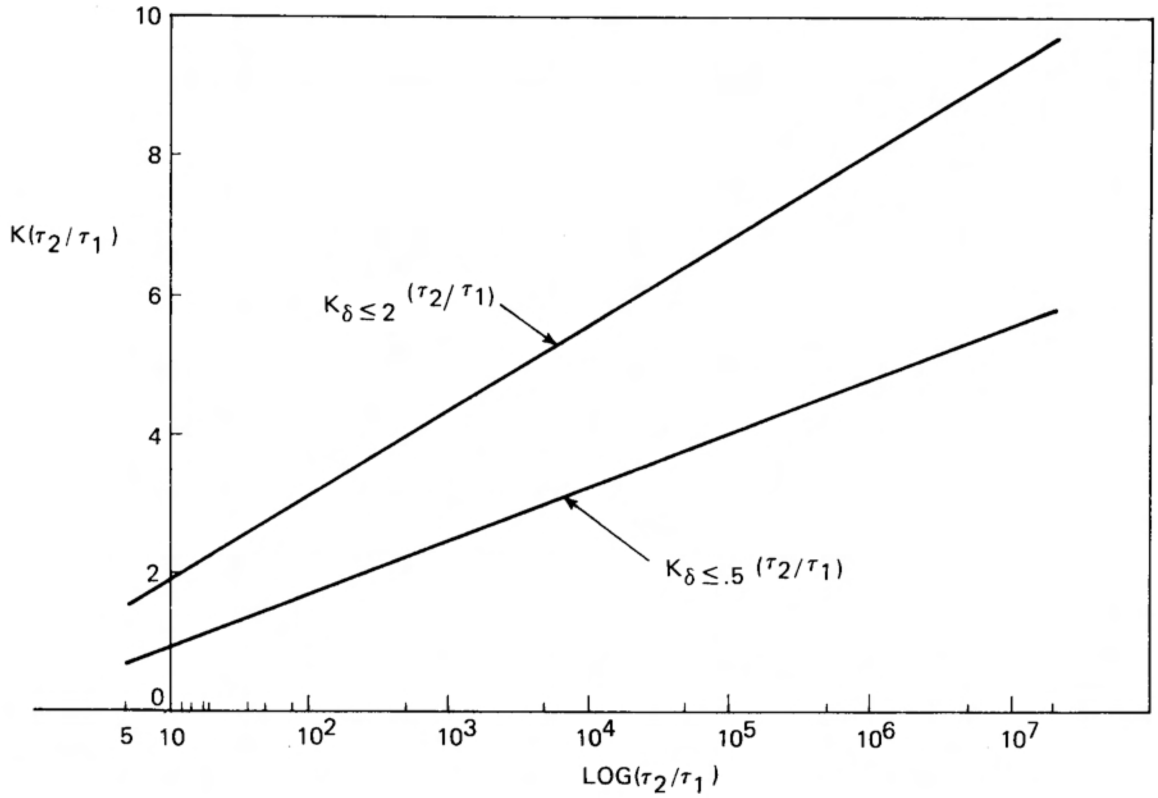

Figure 12. Plots of $K_{\delta\leq 2}(\tau_2/\tau_1)$ and $K_{\delta\leq 0.5}(\tau_2/\tau_1)$ versus (τ_2/τ_1). The scale is semilog, and shows that for $((\tau_2/\tau_1) \geq 5)$ we have that:

$$K_{\delta\leq 2}(\tau_2/\tau_1) \simeq (0.539)\ell n(\tau_2/\tau_1) + 0.602$$

$$K_{\delta\leq 0.5}(\tau_2/\tau_1) \simeq (0.337)\ell n(\tau_2/\tau_1) + 0.142$$

we always have $[\delta \leq (\tau_2/\tau_1) - 1]$, so that, as desired, the quantity $[(K(\tau_2/\tau_1))\delta^2]$ falls to zero as $(\tau_2/\tau_1) \to 1$. Thus of necessity our approximation always gets better as $\tau_2 \to \tau_1$. We concentrate on larger values of (τ_2/τ_1) because the (1/f) asymptote is not observed for small ratios. We may combine Eq. (257), (267), and (333) to obtain:

$$\left(\frac{1}{1 + [K_{\delta\leq 0.5}(\tau_2/\tau_1)]\delta^2}\right)\overline{e^2(k)}_{S_{ggapx},\text{opt } S_{ggapx} \text{ filter}} \leq \overline{e^2(k)}_{S_{gg},\text{opt } S_{gg} \text{ filter}}$$

$$\leq \overline{e^2(k)}_{S_{gg},\text{opt } S_{ggapx} \text{ filter}} \leq \overline{e^2(k)}_{S_{ggapx},\text{opt } S_{ggapx} \text{ filter}} \quad ; \quad (\forall k) \quad (334)$$

Similarly, for the steady-state mean-squared estimation errors we obtain from Eq. (257), (267a), and (333):

$$\left(\frac{1}{1 + [K_{\delta\leq 0.5}(\tau_2/\tau_1)]\delta^2}\right)\overline{e^2}_{S_{ggapx},\text{opt } S_{ggapx} \text{ filter}} \leq \overline{e^2}_{S_{gg},\text{opt } S_{gg} \text{ filter}}$$

$$\leq \overline{e^2}_{S_{gg},\text{opt } S_{ggapx} \text{ filter}} \leq \overline{e^2}_{S_{ggapx},\text{opt } S_{ggapx} \text{ filter}} \qquad (334a)$$

Observing the behavior of $[K(\tau_2/\tau_1)]$ in Figure 12, we see that we may write:

$$\left(\frac{1}{1 + 10\delta^2}\right)\overline{e^2(k)}_{S_{ggapx},\text{opt } S_{ggapx} \text{ filter}} \leq \overline{e^2(k)}_{S_{gg},\text{opt } S_{gg} \text{ filter}} \leq \overline{e^2(k)}_{S_{gg},\text{opt } S_{ggapx} \text{ filter}}$$

$$\leq \overline{e^2(k)}_{S_{ggapx},\text{opt } S_{ggapx} \text{ filter}} \quad ; \quad (\forall k), \ (\forall \delta \leq 2), \ (\forall 5 \leq (\tau_2/\tau_1) \leq 10^7) \qquad (335)$$

Similarly, for the steady-state mean-squared estimation errors:

$$\left(\frac{1}{1 + 10\delta^2}\right)\overline{e^2_{S_{ggapx},\text{opt } S_{ggapx} \text{ filter}}} \leq \overline{e^2_{S_{gg},\text{opt } S_{gg} \text{ filter}}} \leq \overline{e^2_{S_{gg},\text{opt } S_{ggapx} \text{ filter}}}$$

$$\leq \overline{e^2_{S_{ggapx},\text{opt } S_{ggapx} \text{ filter}}};$$

$$(\forall \delta \leq 2), \ (\forall 5 \leq (\tau_2/\tau_1) \leq 10^7) \qquad (335a)$$

Equations (335) and (335a) may be used without referring to Figure 12 in order to bound the mean-squared estimation errors of interest.

Equation (334) relates the mean-squared filtering error to δ, the factor of approximation of the time constant density function, and (τ_2/τ_1), the given ratio of the endpoints of the time constant density function. It is clear from Eq. (334) that the smallness (relative to unity) of the quantity $([K_{\delta\leq 0.5}(\tau_2/\tau_1)]\delta^2)$ determines the "goodness" of our approximation. We shall now discuss how this quantity will decrease with the number of approximating systems used. For a given ratio (τ_2/τ_1) and a given number N of systems, we see that:

$$N \leq 1 + \log_{(1+\delta)}(\tau_2/\tau_1)$$

$$N \leq 1 + \left(\frac{\ln(\tau_2/\tau_1)}{\ln(1 + \delta)}\right)$$

$$\ln(1 + \delta) \leq \left(\frac{\ln(\tau_2/\tau_1)}{N - 1}\right) = \ln\left[(\tau_2/\tau_1)^{\left(\frac{1}{N-1}\right)}\right]$$

Thus:

$$\delta \leq \left[(\tau_2/\tau_1)^{(\frac{1}{N-1})} \right] - 1 \tag{336}$$

and hence (if $\delta(N)$ in Eq. (336) is less than (0.5)):

$$[K_{\delta \leq 0.5}(\tau_2/\tau_1)]\delta^2 \leq [K_{\delta \leq 0.5}(\tau_2/\tau_1)] \left[\left\{ (\tau_2/\tau_1)^{(\frac{1}{N-1})} \right\} - 1 \right]^2 \tag{337}$$

Equation (337) relates the quantity of interest to the number of approximating systems (N).

Having discussed the mean-squared error filter effects of using the finite-dimensional Kalman filter resulting from the approximate gyro noise model, we next discuss the exact form which the model takes in discrete time. From Eq. (292) and (294) we see that the approximate model, in continuous time, is the sum of the outputs of N first-order linear systems, each output having power spectral density given by:

$$S^i_{ggapx}(f) = \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{\delta\tau^i}{1 + 4\pi^2 f^2 (1+\delta)(\tau^i)^2}\right) \; ; \quad (i=1, \ldots, N-1) \tag{338}$$

where:

$$\tau^1 = \tau_1, \; \tau^{i+1} = (1+\delta)\tau^i \; ; \quad (i=1, \ldots, N-1) \tag{339}$$

and for the last system:

$$S^N_{ggapx}(f) = \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{((\tau_2/\tau^N)-1)\tau^N}{1 + 4\pi^2 f^2 \tau_2 \tau^N}\right) \tag{340}$$

For the most part we shall develop our expressions for Eq. (338). Those for the last system (Eq. (340)) follow trivially with δ replaced by $((\tau_2/\tau_N) - 1)$. In order to generate a power spectral density as in Eq. (338) we first factor as follows:

$$S^i_{ggapx}(f) = \left[\left(\sqrt{\frac{2}{\ell n(\tau_2/\tau_1)}(\frac{\delta}{1+\delta})(\frac{1}{\tau_i})}\right)\left(\frac{\tau_i\sqrt{1+\delta}}{1+\sqrt{1+\delta}\;\tau_i s}\right)\right]\left[\left(\sqrt{\frac{2}{\ell n(\tau_2/\tau_1)}(\frac{\delta}{1+\delta})(\frac{1}{\tau_i})}\right)\left(\frac{\tau_i\sqrt{1+\delta}}{1-\sqrt{1+\delta}\;\tau_i s}\right)\right] \tag{341}$$

The system which generates this noise can then be regarded as having transfer function:

$$\left(\frac{\tau_i\sqrt{1+\delta}}{1+\tau_i\sqrt{1+\delta}\;s}\right) \tag{342}$$

and being fed by white noise of covariance:

$$E[w_i(t)w_j(s)] = \begin{cases} \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{\delta}{1+\delta}\right)\left(\frac{1}{\tau^i}\right)\delta(t-s)\,\delta_{i,j} \;; & \begin{cases} i=1, \ldots, N-1 \\ j=1, \ldots, N-1 \end{cases} \\ \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{(\tau_2/\tau^N)-1}{(\tau_2/\tau^N)}\right)\left(\frac{1}{\tau^N}\right)\delta(t-s)\,\delta_{i,N}\delta_{j,N} \;; & \begin{cases} i=N;\ j=1, \ldots, N \\ \text{or } j=N;\ i=1, \ldots, N \end{cases} \end{cases} \tag{343}$$

We may compare Eq. (342) and (343) with Eq. (271) and (271a), the model equations for the actual gyroscopic noise. Note that Eq. (271a) is a "two dimensional" white noise, whereas once the time constant density function is discretized one Dirac delta function in Eq. (271a) is replaced by a Kronecker delta function in Eq. (343). The difference in factors of $(\ell n(\tau_2/\tau_1))$ is caused by the fact that gyro noise is obtained by weighting and integrating the outputs of the linear systems in Eq. (271b) whereas the approximation is obtained by directly summing (without a weighting function) the outputs of the systems driven by the noise in Eq. (343). The outputs of the systems in Eq. (342) are now sampled in time. The discrete-time difference equations which represent the resulting systems are:

$$\begin{aligned} y(n+1) &= \left(e^{-\tilde{T}/\tau^i\sqrt{1+\delta}}\right)y(n) + w_i(n) \quad ; \quad (i=1, \ldots, N-1) \\ y(n+1) &= \left(e^{-\tilde{T}/\sqrt{\tau_2\tau^N}}\right)y(n) + w_N(n) \quad ; \quad i=N \end{aligned} \tag{344}$$

where $\tilde{T}$ is the sampling time, and the covariance of $w(n)$ is given by:

$$E[w_i(n)w_j(m)] = \begin{cases} \left(\frac{1}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{\delta}{\sqrt{1+\delta}}\right)\left(1 - e^{-2\tilde{T}/\tau^i\sqrt{1+\delta}}\right)\delta_{i,j}\delta_{n,m} \;; & i,j=1, \ldots, N-1 \\ \left(\frac{1}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{(\tau_2/\tau^N)-1}{(\tau_2/\tau^N)^{1/2}}\right)\left(1-e^{-2\tilde{T}/\sqrt{\tau_2\tau^N}}\right)\delta_{n,m}\delta_{N,j}\delta_{N,i} \;; & \begin{cases} i=N,\ j=1,\ldots,N \\ \text{or} \\ j=N,\ i=1,\ldots,N \end{cases} \end{cases} \tag{345}$$

Thus the approximation to sampled gyroscopic noise is obtained by summing system outputs from the N difference equation systems of Eq. (344). Each system output has steady-state variance given by:

$$\text{Variance(steady state)} = \left(\frac{1}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{\delta}{\sqrt{1+\delta}}\right) \;; \quad i=1, \ldots, N-1 \tag{346}$$

Note that Eq. (344) is the same difference equation as that used for the systems which generate actual gyro noise. These were given by (Eq. (34) and (35)):

$$x(\tau,\ n+1) = e^{-\tilde{T}/\tau}\ x(\tau,\ n) + w(\tau,\ n) \tag{347}$$

Where the covariance of $w(\tau,\ n)$ was (Eq. (37)):

$$E[w(\tau,\ n)w(\gamma,\ m)] = [\ln(\tau_2/\tau_1)](\tau)(1 - e^{-2\tilde{T}/\tau})\delta_{n,m}\delta(\tau - \gamma) \tag{348}$$

Because the time constant density function is now discretized, the covariance of the driving noise (Eq. (345)) now has a Kronecker delta function instead of a Dirac delta function in the time constant variable. As for the power spectral density, the PSD of the sampled gyroscopic noise is given by (Eq. (269a)):

$$S_{gg}(\omega_d) = \left(\frac{1}{\tilde{T}}\right)\sum_{k=-\infty}^{\infty} S_{gg}\left(\frac{\omega_d}{2\pi\tilde{T}} + \frac{k}{\tilde{T}}\right) \ ; \quad \omega_d \in [-\pi,\ \pi] \tag{349}$$

The power spectra of the outputs of the discrete systems in Eq. (344) are found as follows: The z-transforms of the systems are:

$$H(z) = \frac{1}{1 - \left(e^{-\tilde{T}/\tau^i\sqrt{1+\delta}}\right)z} \ ; \quad (i=1,\ \ldots,\ N-1) \tag{350}$$

The PSD, as a function of z, of the output of a discrete system of z-transform H(z) fed by white noise of variance M is given by:

$$PSD(z) = M\, H(z)H(z^{-1}) \tag{351}$$

Thus on the unit circle in the z-plane ($z = e^{j\omega_d}$) from Eq. (345), (350), and (351) we have that:

$$S^i_{ggapx}(\omega_d) = \left(\frac{1}{\ln(\tau_2/\tau_1)}\right)\left(\frac{\delta}{\sqrt{1+\delta}}\right)\left(1-e^{-2\tilde{T}/\tau^i\sqrt{1+\delta}}\right)\left(\frac{1}{1-\left(e^{-\tilde{T}/\tau^i\sqrt{1+\delta}}\right)e^{j\omega_d}}\right)\left(\frac{1}{1-\left(e^{-\tilde{T}/\tau^i\sqrt{1+\delta}}\right)e^{-j\omega_d}}\right) \tag{352}$$

Equation (352) simplifies to:

$$S^i_{ggapx}(\omega_d) = \left(\frac{1}{\ln(\tau_2/\tau_1)}\right)\left(\frac{\delta}{\sqrt{1+\delta}}\right)\left(1-e^{-2\tilde{T}/\tau^i\sqrt{1+\delta}}\right)\left(\frac{1}{1+\left(e^{-2\tilde{T}/\tau^i\sqrt{1+\delta}}\right)-2\left(e^{-\tilde{T}/\tau^i\sqrt{1+\delta}}\right)\cos\,\omega_d}\right) \tag{353}$$

Because the approximation to gyro noise is given by the sum of the outputs of the systems in Eq. (344), and the driving noises are independent, we see that we may sum the power spectral densities to obtain:

$$S_{ggapx}(\omega_d) = \sum_{i=1}^{N-1} \left[\left(\frac{1}{\ln(\tau_2/\tau_1)}\right) \left(\frac{\delta}{\sqrt{1+\delta}}\right) \left(1 - e^{-2\tilde{T}/\tau^i\sqrt{1+\delta}}\right) \left(\frac{1}{1 + (e^{-2\tilde{T}/\tau^i\sqrt{1+\delta}}) - 2(e^{-\tilde{T}/\tau^i\sqrt{1+\delta}}) \cos \omega_d} \right) \right]$$

$$+ \left(\frac{1}{\ln(\tau_2/\tau_1)}\right) \left(\frac{(\tau_2/\tau^N) - 1}{(\tau_2/\tau^N)^{1/2}} \right) \left(1 - e^{-2\tilde{T}/\sqrt{\tau_2\tau^N}}\right) \left[\frac{1}{1 + \left(e^{-2\tilde{T}/\sqrt{\tau_2\tau^N}}\right) - 2\left(e^{-\tilde{T}/\sqrt{\tau_2\tau^N}}\right) \cos \omega_d} \right] \tag{354}$$

We now give an example of the foregoing discussion. Let $\tau_1 = 0.01$ s and $\tau_2 = 1.0$ s be the endpoints of the time constant density function. The continuous-time PSD of gyroscopic noise (variance normalized to unity) with these time constants is plotted in Figure 13. The digital PSD of

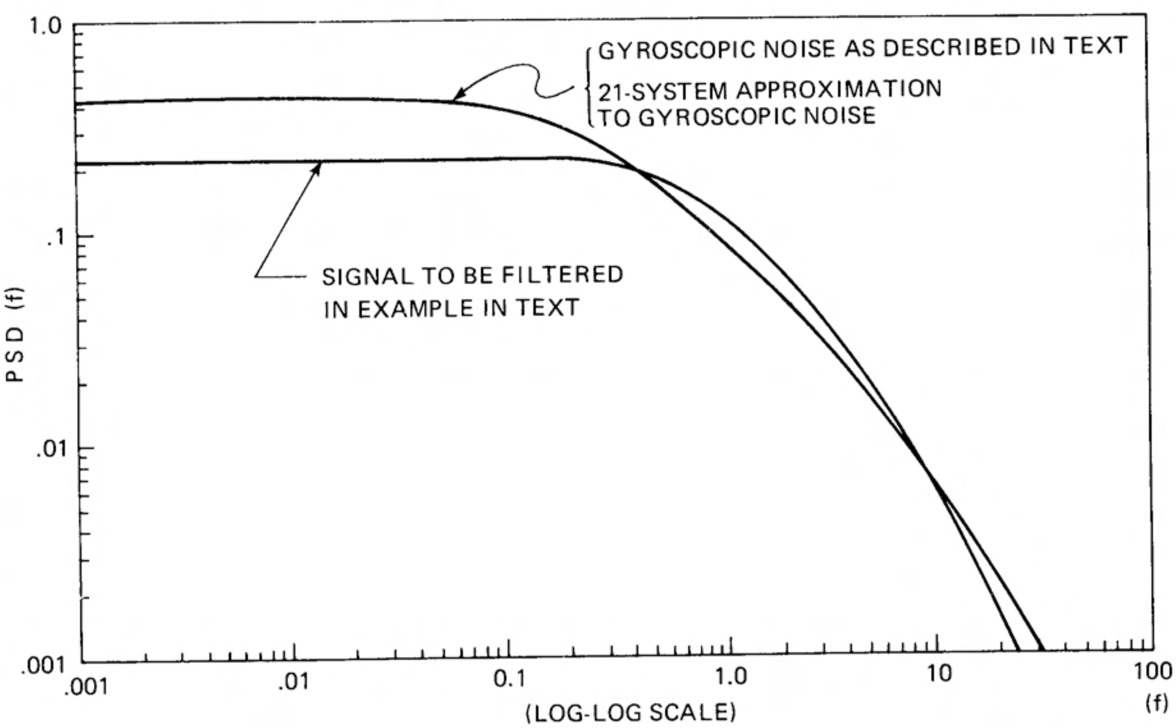

Figure 13. Power spectral densities of the following processes: (continuous time)

(1) Gyroscopic noise as described in text.
(2) 21-system approximation to gyroscope noise.
(3) Signal to be filtered in example in text.

this noise sampled at 0.01 s intervals ($\tilde{T} = 0.01$) is given (using Eq. (349)) in Figure 14. Letting $\delta = 0.25$, we make a 21-system approximation to this gyro noise. The PSD of this continuous-time process is compared against

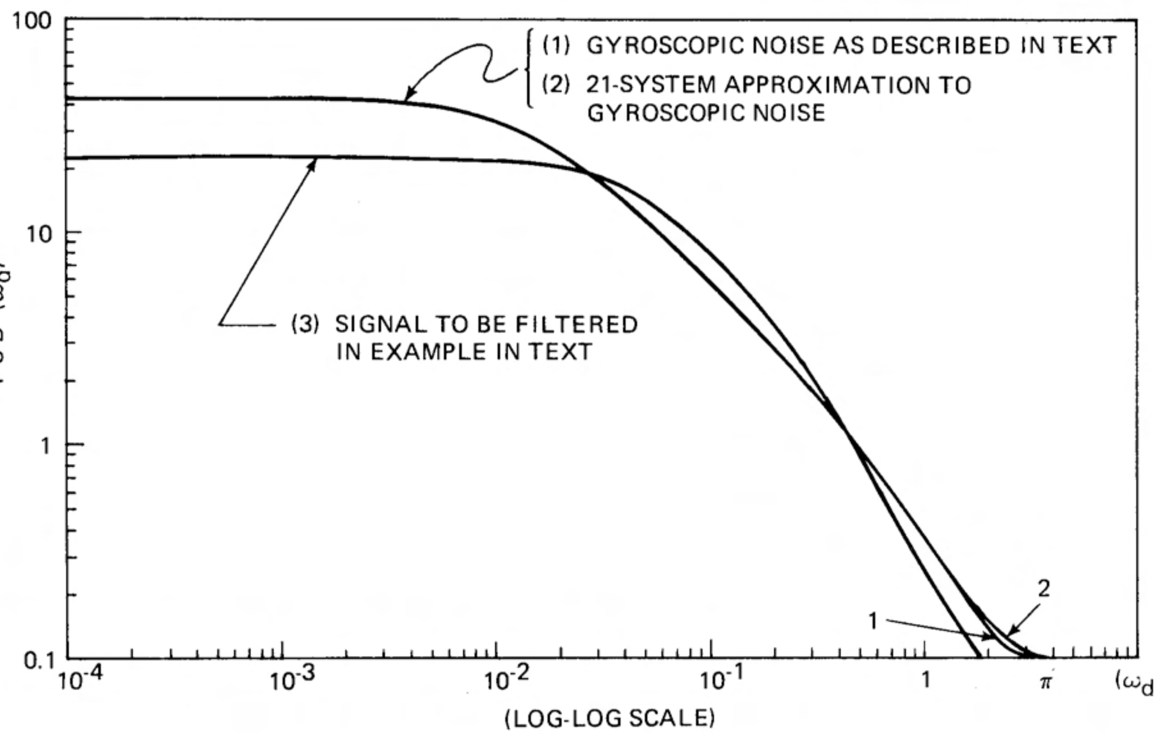

Figure 14. Digital PSDs (sample time = 0.01 s) of the following processes:

(1) Gyroscopic noise as described in text.
(2) 21-system approximation to gyroscope noise.
(3) Signal to be filtered in example in text.

that of the gyro noise in Figure 13. The digital PSD of the sampled version of this approximation is given in Figure 14, to be compared against the PSD of sampled gyro noise. We let the white noise in Figure 9 have variance 0.25, and let the signal to be recovered be modeled as the output of the following vector difference equation to a white noise input:

$$\begin{bmatrix} a_1(n+1) \\ a_2(n+1) \end{bmatrix} = \begin{bmatrix} e^{-\tilde{T}/\tau_a} & 0 \\ 0 & e^{-\tilde{T}/\tau_b} \end{bmatrix} \begin{bmatrix} a_1(n) \\ a_2(n) \end{bmatrix} + \begin{bmatrix} u_1(n) \\ u_2(n) \end{bmatrix}$$

$$p_1(n) = [1 \;\; 1] \begin{bmatrix} a_1(n) \\ a_2(n) \end{bmatrix} \tag{355}$$

where $\tau_a = 0.175$ s, $\tau_b = 0.05$ s, and the covariance of $\underline{u}(n)$ is given by:

$$E(\underline{u}(n)\underline{u}'(m)) \triangleq \underline{Q}_1(n)\delta_{n,m} = \begin{bmatrix} 0.054 & 0 \\ 0 & 0.165 \end{bmatrix} \qquad (355a)$$

This gives the signal $p_1(n)$ a steady-state variance of unity. The digital PSD of the signal $p_1(n)$ is plotted in Figure 14, while the PSD of the continuous-time 2-pole system output, of which $p_1(n)$ is samples, is plotted in Figure 13. The steady-state variance of the estimated value of the signal is found by allowing the (23×23) covariance matrix Riccati equations to iteratively approach steady state. This value is found to be 0.5351. Thus, by Eq. (334a) we may bound both the steady-state error which would be achieved in filtering the gyroscopic noise using the optimal gyro noise filter and the steady-state error which would be achieved in filtering the gyroscopic noise using the approximate filter as follows:

$$\left(\frac{1}{1+[K_{\delta\leq 0.5}(\tau_2/\tau_1)]\delta^2}\right)(0.5351) \leq \overline{e^2_{S_{gg}}},_{opt\ S_{gg}\ filter} \leq \overline{e^2_{S_{gg}}},_{opt\ S_{ggapx}\ filter} \leq (0.5351) \qquad (356)$$

We may find that $K_{\delta\leq 0.5}(100) = 1.692$ from Figure 12. We have already set $\delta = 0.25$, so we have that:

$$(0.4839) \leq \overline{e^2_{S_{gg}}},_{opt\ S_{gg}\ filter} \leq \overline{e^2_{S_{gg}}},_{opt\ S_{ggapx}\ filter} \leq (0.5351) \qquad (357)$$

The above followed because Eq. (257) was satisfied with

$$\beta = [K_{\delta\leq 0.5}(\tau_2/\tau_1)]\delta^2 = 0.1058$$

From the computer-evaluated plot of Figure 14 we find that in fact with our approximate gyro noise model Eq. (257) is satisfied with ($\beta \doteq 0.0336$). Thus we have demonstrated that our bounding technique works. In fact, we thus may replace (0.4839) in Eq. (357) with (0.5177).

We now turn our attention to the second problem to be addressed, the effects on filtering performance which would result from having errors in τ_1 or τ_2, the time constant density function bounds of Eq. (270). Because the (1/f) portion of the gyro noise spectrum is observed, we consider the case where we observe a given spectrum (i.e. Eq. (269)) for the gyroscopic noise when in fact the actual noise has the same functional form (with a different constant multiplier and different values of τ_1 and

τ_2) and a (1/f) portion passing closely through the (1/f) portion of the observed PSD. For example, this type of situation is pictured in Figure 15. Our desire is to bound the error in filtering the actual gyroscopic

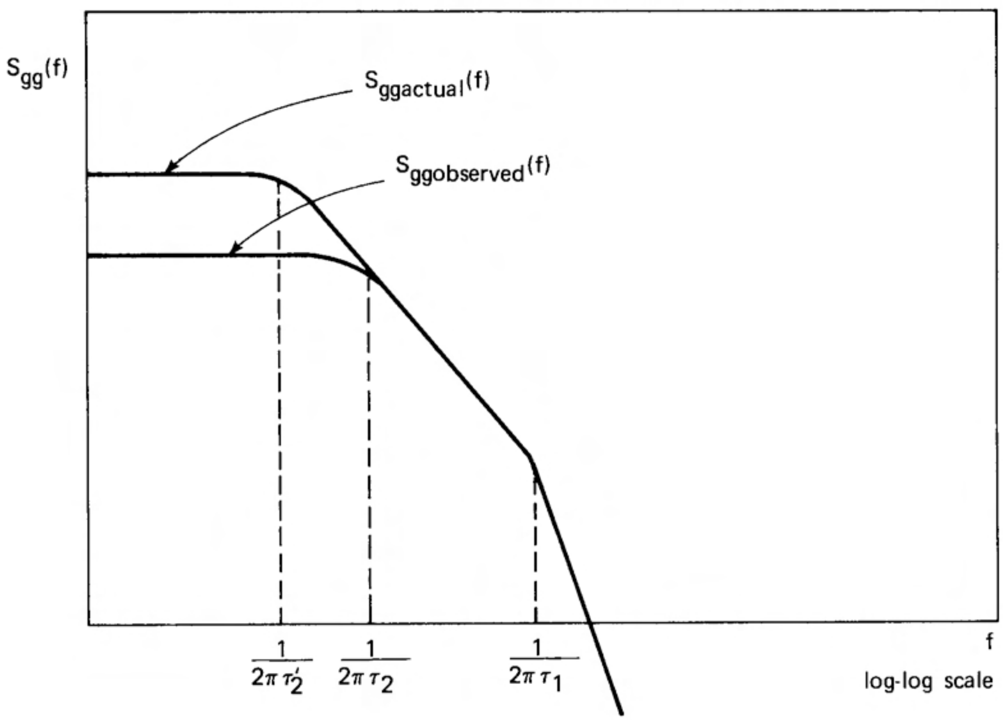

Figure 15. Observed and actual gyroscopic noise PSD.

noise with a filter resulting from the observed power spectral density. Notationally, when τ_1 is in actuality other than its observed value, we shall call the observed value τ_1 and the actual value τ_1'; the same will be true of τ_2 and τ_2'. We first note an interesting fact. In Figure 16 we asymptotically plot a gyroscopic noise PSD in addition to the PSD of the output of a first-order linear system driven with white noise. From Eq. (269) we may easily verify that the low frequency and high frequency asymptotes of the two PSDs coincide. Thus, given an observed gyro PSD as in Figure 15 and the time constant density function upper bound (τ_2') of the actual gyroscopic noise, we would produce the low frequency asymptote (height at f = 0) of the actual PSD by fixing τ_1 and changing the break frequency of the first-order output in Figure 16 to ($f = 1/2\pi\sqrt{\tau_1\tau_2'}$).

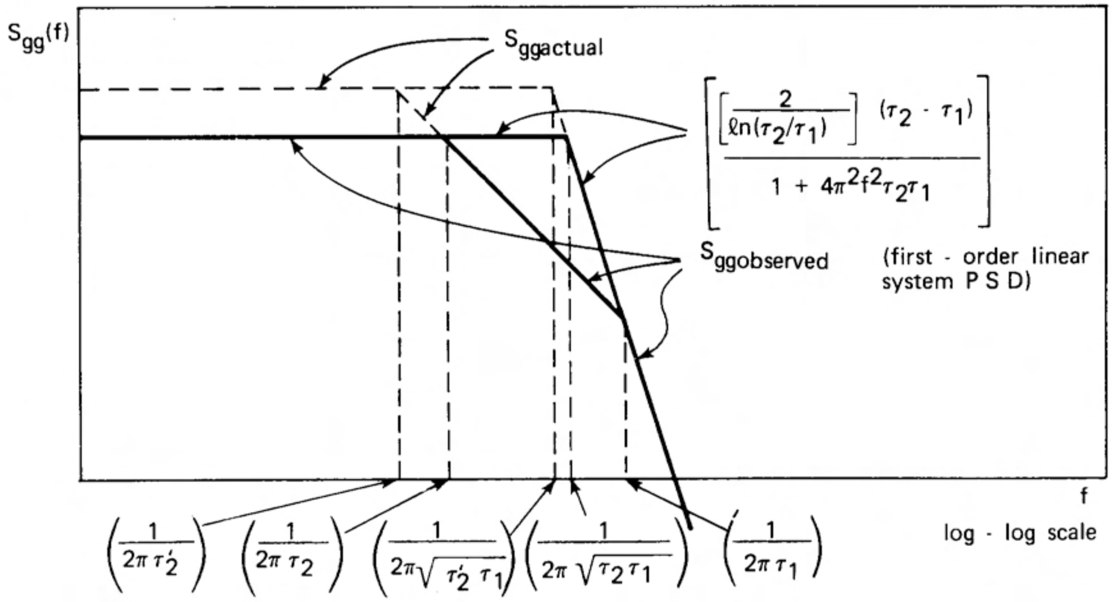

Figure 16. Solid lines are asymptotes of observed gyro noise PSD compared to those of output of first-order linear system. Dotted lines are used to find asymptotes of actual gyroscopic noise, as described in text.

We first discuss four cases, in each of which one time constant density function bound is observed correctly, one incorrectly. The cases are:

$$\begin{aligned}
&(1) \quad \tau_2' > \tau_2 \;;\; \tau_1 = \tau_1' \\
&(2) \quad \tau_2' < \tau_2 \;;\; \tau_1 = \tau_1' \\
&(3) \quad \tau_1' < \tau_1 \;;\; \tau_2 = \tau_2' \\
&(4) \quad \tau_1' > \tau_1 \;;\; \tau_2 = \tau_2'
\end{aligned} \qquad (358)$$

Case (1): This is the situation shown in Figure 15. Consider the following power spectral density. Make a finite-dimensional approximation to the observed PSD and add systems with high time constants until the high end of the time constant density function is at τ_2'. If the constant δ of the discrete time constant density function is small enough, the power spectral density due to this density function will be very close to that given by a gyroscopic noise with time constants τ_1 and τ_2'. The actual PSD is in fact such a gyroscopic noise (having time constant density function bounds τ_1 and τ_2', but a different constant multiplier). We shall show that the constant multiplier of the PSD just constructed (call this

$S_{ggcon}(f))$ is greater than that of $S_{ggactual}(f)$. The height of S_{ggcon} at $(f = 0)$ is:

$$S_{ggcon}(f = 0) \simeq \frac{2(\tau_2'-\tau_1)}{\ell n(\tau_2/\tau_1)} \tag{359}$$

this is because if the variances of the white noise inputs to the approximating systems were scaled by $(\ell n(\tau_2'/\tau_1)/\ell n(\tau_2/\tau_1))$ (see Eq. (343)) then S_{ggcon} would approximate:

$$(\frac{2}{\ell n(\tau_2'/\tau_1)})(\frac{1}{2\pi f})\tan^{-1}\left[\frac{2\pi f(\tau_2'-\tau_1)}{1 + 4\pi^2 f^2 \tau_2'\tau_1}\right] \tag{360}$$

which has a height (see Figure 13) of $(2(\tau_2' - \tau_1)/\ell n(\tau_2'/\tau_1))$ at $(f = 0)$. The height of $S_{ggactual}$ at $(f = 0)$ can be found in either of two ways (both of which, incidentally, lead to the same answer). We can either extrapolate the $(1/f)$ asymptote of S_{ggobs} (the observed PSD) back to find its value at $(f = 1/2\pi\tau_2')$, or we can evaluate the value of the asymptotes of the first-order system PSD in Figure 16 at its break frequency

$$f = 1/(2\pi\sqrt{\tau_2'\tau_1})$$

The second method is the one referred to previously. Either way we obtain:

$$S_{ggactual}(f = 0) = \frac{2(\tau_2' - \tau_1(\tau_2'/\tau_2))}{\ell n(\tau_2/\tau_1)} \tag{361}$$

The quantity in Eq. (361) is smaller than that in Eq. (359). Thus, since S_{ggcon} and $S_{ggactual}$ have the same form except for a multiplicative constant, we have that:

$$S_{ggcon}(f = 0) > S_{ggactual}(f = 0) \Rightarrow S_{ggcon}(f) > S_{ggactual}(f) \quad (\forall f) \tag{362}$$

The height of S_{ggobs} at $(f = 0)$ is given by (see Figure 13):

$$S_{ggobs}(f = 0) = \frac{2(\tau_2 - \tau_1)}{\ell n(\tau_2/\tau_1)} \tag{363}$$

Because we've added systems to S_{ggobs} with time constants longer (PSD frequency breakpoints lower) than those of S_{ggobs}, the following is true:

$$\left(\frac{\tau_2'-\tau_1}{\tau_2-\tau_1}\right) = \frac{S_{ggcon}(0)}{S_{ggobs}(0)} \geq \frac{S_{ggcon}(f)}{S_{ggobs}(f)}; \; (\forall f) \tag{364}$$

By Eq. (362) and (364) we have the right-hand side of:

$$S_{ggobs}(f) \le S_{ggactual}(f) \le \left(\frac{\tau_2'-\tau_1}{\tau_2-\tau_1}\right) S_{ggobs}(f) \qquad (365)$$

where the left-hand inequality is clear from Figure 15 and the fact that $S_{ggactual}$ is closer to its asymptote at $(f = 1/2\pi\tau_2)$ than is S_{ggobs}. Using the same arguments which led to Eq. (267) (we may do this because the aliased PSDs have the same relationship as do the continuous-time PSDs in Eq. (365)) we obtain:

$$\begin{aligned}
\overline{e^2(k)}_{S_{ggobs},\text{opt } S_{ggobs} \text{ filter}} &\le \overline{e^2(k)}_{S_{ggactual},\text{opt } S_{ggactual} \text{ filter}} \\
&\le \overline{e^2(k)}_{S_{ggactual},\text{opt } S_{ggobs} \text{ filter}} \\
&\le \overline{e^2(k)}_{\left(\frac{\tau_2'-\tau_1}{\tau_2-\tau_1}\right) S_{ggobs},\text{opt } S_{ggobs} \text{ filter}} \\
&\le \left(\frac{\tau_2'-\tau_1}{\tau_2-\tau_1}\right) \overline{e^2(k)}_{S_{ggobs},\text{opt } S_{ggobs} \text{ filter}} \quad ; \quad (\forall k)
\end{aligned} \qquad (366)$$

Case (2): In this case, we again make a finite-dimensional approximation to $S_{ggobs}(f)$, but here we delete systems with long time constants until the time constant density function ranges from τ_1 to τ_2', where $\tau_2' < \tau_2$. We again find that:

$$S_{ggcon}(f = 0) \simeq \frac{2(\tau_2'-\tau_1)}{\ell n(\tau_2/\tau_1)} \; ; \quad S_{ggactual}(f = 0) = \frac{2(\tau_2'-\tau_1(\tau_2'/\tau_1))}{\ell n(\tau_2/\tau_1)} \qquad (367)$$

so that in this case:

$$S_{ggcon}(f) \le S_{ggactual}(f) \le S_{ggobs}(f) \; ; \quad (\forall f) \qquad (368)$$

Further:

$$\left(\frac{\tau_2'-\tau_1}{\tau_2-\tau_1}\right) = \frac{S_{ggcon}(f=0)}{S_{ggobs}(f=0)} \le \frac{S_{ggcon}(f)}{S_{ggobs}(f)} \qquad (369)$$

From Eq. (368) and (369) we obtain:

$$\left(\frac{\tau_2'-\tau_1}{\tau_2-\tau_1}\right) S_{ggobs}(f) \le S_{ggactual}(f) \le S_{ggobs}(f) \qquad (370)$$

As in Eq. (366), we now obtain that:

$$\left(\frac{\tau_2'-\tau_1}{\tau_2-\tau_1}\right)\overline{e^2(k)}_{S_{ggobs},\text{opt } S_{ggobs}\text{ filter}} \le \left(\frac{\tau_2'-\tau_1}{\tau_2-\tau_1}\right)\overline{e^2(k)}_{S_{ggobs},\text{opt } S_{ggactual}\text{ filter}}$$

$$\le \overline{e^2(k)}_{\left(\frac{\tau_2'-\tau_1}{\tau_2-\tau_1}\right)S_{ggobs},\text{opt } S_{ggactual}\text{ filter}}$$

$$\le \overline{e^2(k)}_{S_{ggactual},\text{opt } S_{ggactual}\text{ filter}}$$

$$\le \overline{e^2(k)}_{S_{ggactual},\text{opt } S_{ggobs}\text{ filter}}$$

$$\le \overline{e^2_S(k)}_{S_{ggobs},\text{opt } S_{ggobs}\text{ filter}} \; ; \quad (\forall k) \qquad (371)$$

Condensing the above, we have that for $\tau_1' = \tau_1$, $\tau_2' < \tau_2$:

$$\left(\frac{\tau_2'-\tau_1}{\tau_2-\tau_1}\right)\overline{e^2(k)}_{S_{ggobs},\text{opt } S_{ggobs}\text{ filter}} \le \overline{e^2(k)}_{S_{ggactual}\text{ opt } S_{ggactual}\text{ filter}}$$

$$\le \overline{e^2(k)}_{S_{ggactual},\text{opt } S_{ggobs}\text{ filter}}$$

$$\le \overline{e^2(k)}_{S_{ggobs},\text{opt } S_{ggobs}\text{ filter}} \; ; \quad (\forall k) \qquad (371a)$$

Case (3): This situation is shown in Figure 17. We again construct a new power spectral density, this time by approximating S_{ggobs} by a finite-dimensional system output with small δ, and adding systems with short time constants until the discrete time constant density function starts at τ_1'. This system will be characterized by a PSD S_{ggcon}, which is very close (let δ be very small) to:

$$S_{ggcon}(f) \approx \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\tan^{-1}\left[\frac{2\pi f(\tau_2-\tau_1')}{1+4\pi^2 f^2\tau_2\tau_1'}\right] \qquad (372)$$

At high frequencies, this PSD behaves like:

$$S_{ggcon}(f)\Big|_{f\text{ high}} \approx \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{\tau_2-\tau_1'}{4\pi^2 f^2\tau_2\tau_1'}\right) \qquad (373)$$

The observed PSD is of form:

$$S_{ggobs}(f) = \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\tan^{-1}\left[\frac{2\pi f(\tau_2-\tau_1)}{1 + 4\pi^2 f^2 \tau_2\tau_1}\right] \tag{374}$$

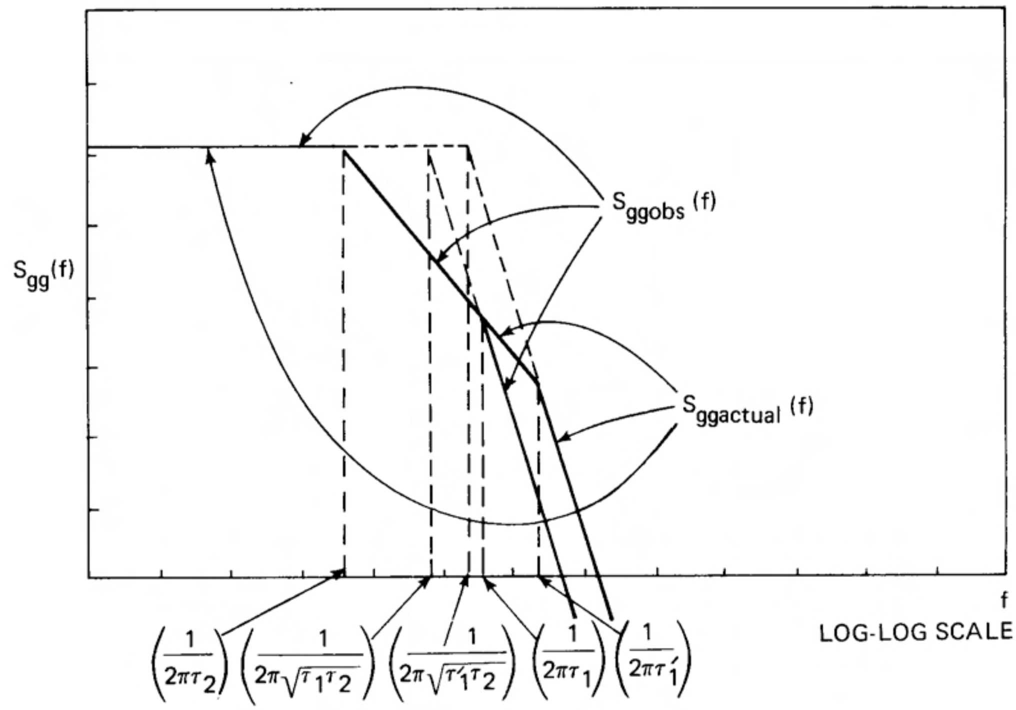

Figure 17. The actual power spectral density has $\tau_1' < \tau_1$; $\tau_2' = \tau_2$. The dotted lines are the first-order system asymptotes which coincide with those of S_{ggobs} and $S_{ggactual}$.

At high frequencies, this looks like:

$$S_{ggobs}(f)\Big|_{f\ high} \approx \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{\tau_2-\tau_1}{4\pi^2 f^2 \tau_2\tau_1}\right) \tag{375}$$

The high frequency asymptotes of $S_{ggactual}(f)$ may be found by using the first-order linear system PSD diagrammed in Figure 17. This asymptote is found to be:

$$S_{ggactual}(f)\Big|_{f\ high} \approx \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{\tau_2-\tau_1}{4\pi^2 f^2 \tau_2\tau_1'}\right) \tag{376}$$

Because the quantity in Eq. (373) is greater than that in Eq. (376) and

$S_{ggactual}(f)$ and $S_{ggcon}(f)$ have the same functional form except for a multiplicative constant, we see that:

$$S_{ggcon}(f) \geq S_{ggactual}(f) \geq S_{ggobs}(f) \; ; \quad (\forall f) \tag{377}$$

where the right-hand side comes from Figure 17 and the fact that $S_{ggactual}$ is closer to its asymptotic value at $(f = 1/2\pi\tau_1)$ than is S_{ggobs}. Also, since systems with short time constants (high frequency poles) have been added to $S_{ggobs}(f)$, we have the following:

$$\left(\frac{\tau_2-\tau_1'}{\tau_2-\tau_1}\right) = \frac{S_{ggcon}(f)}{S_{ggobs}(f)}\bigg|_{f\ high} \geq \frac{S_{ggcon}(f)}{S_{ggobs}(f)} \; ; \quad (\forall f) \tag{378}$$

From Eq. (377) and (378) we obtain:

$$S_{ggobs}(f) \leq S_{ggactual}(f) \leq \left(\frac{\tau_2-\tau_1'}{\tau_2-\tau_1}\right) S_{ggobs}(f) \tag{379}$$

From Eq. (379) we obtain, exactly as in Eq. (366):

$$\begin{aligned} \overline{e^2(k)}_{S_{ggobs},\text{opt } S_{ggobs}\text{ filter}} &\leq \overline{e^2(k)}_{S_{ggactual},\text{opt } S_{ggactual}\text{ filter}} \\ &\leq \overline{e^2(k)}_{S_{ggactual},\text{opt } S_{ggobs}\text{ filter}} \\ &\leq \left(\frac{\tau_2-\tau_1'}{\tau_2-\tau_1}\right) \overline{e^2(k)}_{S_{ggobs},\text{opt } S_{ggobs}\text{ filter}} \; ; \quad (\forall k) \end{aligned} \tag{380}$$

Case (4): In this case we construct S_{ggcon} by deleting systems with short time constants until the time constant density function ranges from τ_1' to τ_2, where $\tau_1' > \tau_1$. We again find that:

$$\begin{aligned} S_{ggcon}(f)\bigg|_{f\ high} &\approx \left(\frac{2}{\ln(\tau_2/\tau_1)}\right)\left(\frac{\tau_2-\tau_1'}{4\pi^2 f^2 \tau_2\tau_1'}\right) \; ; \\ S_{ggactual}(f)\bigg|_{f\ high} &\approx \left(\frac{2}{\ln(\tau_2/\tau_1)}\right)\left(\frac{\tau_2-\tau_1}{4\pi^2 f^2 \tau_2\tau_1'}\right) \end{aligned} \tag{381}$$

Further:

$$S_{ggcon}(f) \leq S_{ggactual}(f) \leq S_{ggobs}(f) \tag{382}$$

Also, since systems with high frequency poles have been deleted from $S_{ggobs}(f)$, we have that:

$$\left(\frac{\tau_2-\tau_1'}{\tau_2-\tau_1}\right) = \left.\frac{S_{ggcon}(f)}{S_{ggobs}(f)}\right|_{f\ high} \leq \frac{S_{ggcon}(f)}{S_{ggobs}(f)} \; ; \quad (\forall f) \tag{383}$$

From Eq. (382) and (383) we obtain:

$$\left(\frac{\tau_2-\tau_1'}{\tau_2-\tau_1}\right) S_{ggobs}(f) \leq S_{ggactual}(f) \leq S_{ggobs}(f) \tag{384}$$

Thus, exactly as in Eq. (371a) we have that:

$$\begin{aligned}\left(\frac{\tau_2-\tau_1'}{\tau_2-\tau_1}\right)\overline{e^2(k)}_{S_{ggobs},\text{opt } S_{ggobs} \text{ filter}} &\leq \overline{e^2(k)}_{S_{ggactual},\text{opt } S_{ggactual} \text{ filter}} \\ &\leq \overline{e^2(k)}_{S_{ggactual},\text{opt } S_{ggobs} \text{ filter}} \\ &\leq \overline{e^2(k)}_{S_{ggobs},\text{opt } S_{ggobs} \text{ filter}} \; ; \quad (\forall k)\end{aligned} \tag{385}$$

Thus we have bounded the filter mean-squared errors in the four cases given. Naturally, in the general case we will have $\tau_1' \neq \tau_1$ <u>and</u> $\tau_2' \neq \tau_2$. We shall now deal with that event. Denote by $S_{ggactual,1}(f)$ the PSD which results from correcting $S_{ggobs}(f)$ only with respect to its error in τ_1. As before, let $S_{ggactual}$ denote the actual gyro noise PSD. The method here is to first adjust τ_1 to its proper value τ_1', then bound $S_{ggactual,1}$ with respect to S_{ggobs}, then adjust τ_2 to its proper value τ_2', and finally bound $S_{ggactual}$ with respect to S_{ggobs}. The following combinations of cases (1) to (4) are possible:

(a) Case (1) and case (3); (i.e. $\tau_1' < \tau_1$, $\tau_2' > \tau_2$)

(b) Case (1) and case (4); (i.e. $\tau_1' > \tau_1$, $\tau_2' > \tau_2$)

(c) Case (2) and case (3); (i.e. $\tau_1' < \tau_1$, $\tau_2' < \tau_2$) (386)

(d) Case (2) and case (4); (i.e. $\tau_1' > \tau_1$, $\tau_2' < \tau_2$)

We now deal with these cases.

Case (a): We first take into account τ_1, to obtain, as in Eq. (365):

$$S_{ggobs}(f) \le S_{ggactual_1}(f) \le \left(\frac{\tau_2{'}-\tau_1}{\tau_2-\tau_1}\right) S_{ggobs}(f) \qquad (387)$$

We next make our modification of $S_{ggactual,1}(f)$ to $S_{ggactual}(f)$ by adjusting τ_2. So we obtain from Eq. (379):

$$S_{ggactual_1}(f) \le S_{ggactual}(f) \le \left(\frac{\tau_2-\tau_1{'}}{\tau_2-\tau_1}\right) S_{ggactual_1}(f) \qquad (388)$$

We finally have, from Eq. (387) and (388):

$$S_{ggobs}(f) \le S_{ggactual}(f) \le \left(\frac{\tau_2-\tau_1{'}}{\tau_2-\tau_1}\right)\left(\frac{\tau_2{'}-\tau_1}{\tau_2-\tau_1}\right) S_{ggobs}(f)$$

Case (a) makes it obvious how the other cases are to be treated. We have, for all four cases:

$$\left[\min\left(1,\frac{\tau_2-\tau_1{'}}{\tau_2-\tau_1}\right)\right]\left[\min\left(1,\frac{\tau_2{'}-\tau_1}{\tau_2-\tau_1}\right)\right] S_{ggobs}(f) \le S_{ggactual}(f)$$
$$\le \left[\max\left(1,\frac{\tau_2-\tau_1{'}}{\tau_2-\tau_1}\right)\right]\left[\max\left(1,\frac{\tau_2{'}-\tau_1}{\tau_2-\tau_1}\right)\right] S_{ggobs}(f) \qquad (389)$$

As in Eq. (366) and Eq. (371) we obtain finally that:

$$\left[\min\left(1,\frac{\tau_2-\tau_1{'}}{\tau_2-\tau_1}\right)\right]\left[\min\left(1,\frac{\tau_2{'}-\tau_1}{\tau_2-\tau_1}\right)\right] \overline{e^2(k)}_{S_{ggobs},\text{opt } S_{ggobs} \text{ filter}}$$
$$\le \overline{e^2(k)}_{S_{ggactual},\text{opt } S_{ggactual} \text{ filter}}$$
$$\le \overline{e^2(k)}_{S_{ggactual},\text{opt } S_{ggobs} \text{ filter}}$$
$$\le \left[\max\left(1,\frac{\tau_2-\tau_1{'}}{\tau_2-\tau_1}\right)\right]\left[\max\left(1,\frac{\tau_2{'}-\tau_1}{\tau_2-\tau_1}\right)\right] \overline{e^2(k)}_{S_{ggobs},\text{opt } S_{ggobs} \text{ filter}} \quad ; \ (\forall k) \qquad (390)$$

Thus we see the effects on filter performance of errors in the bounds of the system time constant density function used to construct the filter.

As an example to illustrate Eq. (390), say we observe the power spectral density of the noise associated with a given gyroscope and find that (τ_1 = 0.01, τ_2 = 1.0). In addition, let the tolerance of the time constant determination be such that the actual time constant density function bounds (τ_1', τ_2') are given by:

$$
\begin{aligned}
0.9\tau_1 &\le \tau_1' \le 1.1\tau_1 \\
0.9\tau_2 &\le \tau_2' \le 1.1\tau_2
\end{aligned}
\tag{391}
$$

Then we find from Eq. (390) that:

$$
\begin{aligned}
&\left|\min\left(1, \frac{(1.0)-(1.1)(0.01)}{(1.0)-(0.01)}\right)\right| \left|\min\left(1, \frac{(0.9)(1.0)-(0.01)}{(1.0)-(0.01)}\right)\right| \overline{e^2(k)}_{S_{ggobs}, \text{opt } S_{ggobs} \text{ filter}} \\
&\quad \le \overline{e^2(k)}_{S_{ggactual}, \text{opt } S_{ggactual} \text{ filter}} \le \overline{e^2(k)}_{S_{ggactual}, \text{opt } S_{ggobs} \text{ filter}} \\
&\quad \le \left|\max\left(1, \frac{(1.0)-(0.9)(0.01)}{(1.0)-(0.01)}\right)\right| \times \\
&\quad \times \left|\max\left(1, \frac{(1.1)(1.0)-(0.01)}{(1.0)-(0.01)}\right)\right| \overline{e^2(k)}_{S_{ggobs}, \text{opt } S_{ggobs} \text{ filter}} \quad ; \quad (\forall k)
\end{aligned}
\tag{392}
$$

Equation (392) simplifies to:

$$
\begin{aligned}
(0.8981)\overline{e^2(k)}_{S_{ggobs}, \text{opt } S_{ggobs} \text{ filter}} &\le \overline{e^2(k)}_{S_{ggactual}, \text{opt } S_{ggactual} \text{ filter}} \\
&\le \overline{e^2(k)}_{S_{ggactual}, \text{opt } S_{ggobs} \text{ filter}} \\
&\le (1.102)\overline{e^2(k)}_{S_{ggobs}, \text{opt } S_{ggobs} \text{ filter}} \quad ; \quad (\forall k)
\end{aligned}
\tag{393}
$$

Equation (393) is the result we sought.

We now summarize the results of this section. First we described a method to generate a finite-dimensional approximation to gyroscopic noise. We showed (in Eq. (334)) that the mean-squared error obtained in filtering a statistically described signal from additive gyroscopic noise using the discrete recursive Kalman filter which results from the finite-dimensional model approaches the minimal error as the number of dimensions of the model is permitted to grow. Second (in Eq. (390)) we showed the effects on filter performance which result from errors in the bounds on the linear system time constant density function for the generation of the gyroscopic noise.

SECTION 6

SUMMARY AND SUGGESTIONS FOR FUTURE WORK

We now summarize the results of this thesis. In Section 2 we developed an infinite-dimensional model to generate gyroscopic noise. In Section 3 we derived the optimal filter to separate a statistically described signal from additive gyroscopic noise. This filter was expressed as a discrete-time infinite-dimensional Kalman filter with an associated Riccati covariance operator equation.

In Section 4 we discussed general properties of Hilbert space Kalman filters and Riccati operator equations. We presented sufficient conditions for asymptotic stability of the Kalman filter in the time-varying model case, and sufficient conditions in the time-invariant model case for:

(1) Asymptotic stability of the Kalman filter.

(2) Weak convergence of the Riccati covariance operator, from arbitrary positive semidefinite initial conditions, to a unique positive definite (bounded below) operator, which is the unique positive semidefinite solution of the steady-state Riccati operator equation.

(3) Asymptotic stability of the steady-state Kalman filter.

The sufficient conditions were then specified to the gyroscopic noise filtering problem, and expressed as conditions on the system generating the signal to be recovered.

Finally, in Section 5 we discussed finite-dimensional approximate gyroscopic noise models, which lead to finite-dimensional Kalman filters for our estimation problem. We discussed in detail a particular approximate model, and showed that the mean-squared estimation error incurred by using the Kalman filter of the finite-dimensional approximate model approaches the mean-squared estimation error of the optimal infinite-dimensional Kalman filter as the number of dimensions in the finite-dimensional gyroscopic noise approximation is permitted to grow. This technique should find applicability in data processing for inertial navigation systems.

The author would like to suggest the following areas for further exploration:

(1) An investigation into possible further properties (e.g. almost sure sample function continuity) of the two-dimensional Wiener process, discussed in Appendix B, might be of use in future applications of the theory of this stochastic process.

(2) As mentioned in Appendix B, our discussion of the two-dimensional Wiener process is easily extended to (n) dimensions. In what types of physical problems might such mathematical models be useful?

(3) Further investigation of Hilbert space Kalman filters might yield alternate or even weaker conditions to guarantee the properties discussed in Section 4.

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APPENDIX A

SOME DETAILED DERIVATIONS

In this appendix are contained derivations referred to in the text of the thesis.

PART A

Substituting Eq. (4) and Eq. (5) into Eq. (6), we find that:

$$m(t) = \frac{K}{\ell n(\tau_2/\tau_1)} \int_{\tau_1}^{\tau_2} (1/\tau)(e^{-t/\tau})d\tau \qquad \text{(A-1)}$$

changing variables, we obtain:

$$m(t) = \frac{K}{\ell n(\tau_2/\tau_1)} \int_{t/\tau_2}^{t/\tau_1} (e^{-y}/y)dy \qquad \text{(A-2)}$$

where we have made the substitution:

$$y = t/\tau \qquad \text{(A-3)}$$

Finally, we obtain:

$$m(t) = \frac{K}{\ell n(\tau_2/\tau_1)}[E_1(t/\tau_2) - E_1(t/\tau_1)] \qquad \text{(A-4)}$$

Where $E_1(z)$ is the exponential integral, defined by:

$$E_1(z) = \int_z^{\infty} (\frac{e^{-u}}{u})du \qquad \text{(A-5)}$$

PART B

In Eq. (15) we have modeled gyroscopic noise, $g(t)$, as the weighted integral of the outputs, $x(\tau, t)$, of the first-order linear systems of Eq. (3) from inputs of two-dimensional white noise with covariance function given in Eq. (13). Thus our model for gyroscopic noise is given by:

$$g(t) = \int_0^{\infty} x(\tau, t) p_d(\tau) d\tau \qquad \text{(A-6)}$$

The autocorrelation function of the noise is:

$$R_{gg}(\alpha) = E\{[\int_0^{\infty} x(\tau, t) p_d(\tau) d\tau][\int_0^{\infty} x(\gamma, t-\alpha) p_d(\gamma) d\gamma]\} \qquad \text{(A-7)}$$

Bringing the expectation within the integrals, we obtain:

$$R_{gg}(\alpha) = \int_0^{\infty}\int_0^{\infty} E\{x(\tau, t) x(\gamma, t-\alpha)\} p_d(\tau) p_d(\gamma) d\gamma d\tau \qquad \text{(A-8)}$$

From Eq. (14), we have:

$$E\{x(\tau, t) x(\gamma, t-\alpha)\} = \frac{W\tau}{2} e^{-|\alpha|/\tau} \delta(\tau - \gamma) \qquad \text{(A-9)}$$

Substituting this into the integral above yields:

$$R_{gg}(\alpha) = \int_0^{\infty} [\int_0^{\infty} (\frac{W\tau}{2} e^{-|\alpha|/\tau} \delta(\tau - \gamma)) p_d(\gamma) d\gamma] p_d(\tau) d\tau \qquad \text{(A-10)}$$

Integrating, we have:

$$R_{gg}(\alpha) = \int_0^{\infty} \frac{W\tau}{2} e^{-|\alpha|/\tau} [p_d(\tau)]^2 d\tau \qquad \text{(A-11)}$$

A Fourier transformation of the above autocorrelation function yields, for the power spectral density of the noise:

$$S_{gg}(f) = \int_0^{\infty} \left(\frac{W\tau^2}{1 + 4\pi^2 f^2 \tau^2}\right) [p_d(\tau)]^2 d\tau \qquad \text{(A-12)}$$

Substituting Eq. (4) into Eq. (A-12), we find that:

$$S_{gg}(f) = \left(\frac{1}{\ln(\tau_2/\tau_1)}\right)^2 \int_{\tau_1}^{\tau_2} (1/\tau)^2 \left(\frac{W\tau^2}{1 + 4\pi^2 f^2 \tau^2}\right) d\tau \qquad \text{(A-13)}$$

Making use of the relation:

$$\int \left(\frac{dx}{a^2 + b^2 x^2}\right) = (\frac{1}{ab}) \tan^{-1}(\frac{bx}{a}) \qquad \text{(A-14)}$$

We obtain:

$$S_{gg}(f) = \left(\frac{1}{\ln(\tau_2/\tau_1)}\right)^2 (\frac{W}{2\pi f}) [\tan^{-1}(2\pi f \tau_2) - \tan^{-1}(2\pi f \tau_1)] \qquad \text{(A-15)}$$

We simplify Eq. (A-15) with the following trigonometric identity:

$$\tan^{-1}(A) - \tan^{-1}(B) = \tan^{-1}\left|\frac{A - B}{1 + AB}\right| \tag{A-16}$$

We thus obtain:

$$S_{gg}(f) = \left(\frac{1}{\ell n(\tau_2/\tau_1)}\right)^2 \left(\frac{W}{2\pi f}\right)\tan^{-1}\left[\frac{2\pi f(\tau_2 - \tau_1)}{1 + 4\pi^2 f^2 \tau_2 \tau_1}\right] \tag{A-17}$$

Normalization of Eq. (A-17) so that:

$$\text{Variance } (g(t)) = R_{gg}(0) = \int_{-\infty}^{\infty} S_{gg}(f)df = 1 \tag{A-18}$$

requires:

$$W = 2\ell n(\tau_2/\tau_1) \tag{A-19}$$

We thus obtain:

$$S_{gg}(f) = \left(\frac{2}{\ell n(\tau_2/\tau_1)}\right)\left(\frac{1}{2\pi f}\right)\tan^{-1}\left[\frac{2\pi f(\tau_2 - \tau_1)}{1 + 4\pi^2 f^2 \tau_2 \tau_1}\right] \tag{A-20}$$

PART C

The operator $H_s(n)$ was defined in the text (Eq. (82)) to be:

$$H_s(n) : V_s \to R ; \quad H_s(n)\begin{bmatrix} x(\tau) \\ \underline{a} \end{bmatrix} = \int_{\tau_1}^{\tau_2} p(\tau)(x(\tau))d\tau + \underline{H}(n)\underline{a} \tag{A-21}$$

The adjoint operator $H_s^*(n)$ is defined by ($\beta \in R$, $x \in V_s$):

$$H_s^*(n) : R \to V_s \ni. \langle\beta, H_s(n)x\rangle_R = \langle H_s^*(n)\beta, x\rangle_{V_s} \tag{A-22}$$

From Eq. (A-21) we obtain:

$$\langle\beta, H_s(n)x\rangle_R = \beta[\int_{\tau_1}^{\tau_2} p(\tau)x(\tau)d\tau + \underline{H}(n)\underline{a}] \tag{A-23}$$

Thus from Eq. (A-22) we have that:

$$\langle H_s^*(n)\beta, x\rangle_{V_s} = \beta[\int_{\tau_1}^{\tau_2} p(\tau)x(\tau)d\tau + \underline{H}(n)\underline{a}] \tag{A-24}$$

By the definition (Eq. (80)) of $\langle\cdot,\cdot\rangle_{V_s}$, we see that:

$$\langle H_s^*(n)\beta,\ x\rangle_{V_s} = \begin{bmatrix} p(\tau)\beta \\ \underline{H}'(n)\beta \end{bmatrix} ⓢ \begin{bmatrix} x(\tau) \\ \underline{a} \end{bmatrix} \tag{A-25}$$

Thus, we have:

$$H_s^*(n) : R \to V_s \ ; \quad H_s^*(n)(\beta) = \begin{bmatrix} p(\tau)\beta \\ \underline{H}'(n)\beta \end{bmatrix} \tag{A-26}$$

PART D

Grouping Eq. (65) through (68), we have:

$$\begin{bmatrix} \Sigma_{\tau,\gamma}(n+1|n) \ , & \Sigma_{\tau,\underline{a}}(n+1|n) \\ \Sigma_{\underline{a},\gamma}(n+1|n) \ , & \Sigma_{\underline{a},\underline{a}}(n+1|n) \end{bmatrix} = \begin{bmatrix} \Phi_\tau \Sigma_{\tau,\gamma}(n|n)\Phi_\gamma \ , & \Phi_\tau \Sigma_{\tau,\underline{a}}(n|n)\underline{\Phi}'(n) \\ \underline{\Phi}(n)\Sigma_{\underline{a},\gamma}(n|n)\Phi_\gamma \ , & \underline{\Phi}(n)\Sigma_{\underline{a},\underline{a}}(n|n)\underline{\Phi}'(n) \end{bmatrix}$$

$$+ \begin{bmatrix} Q(\gamma)\delta(\gamma-\tau) \ , & \underline{0} \\ \underline{0} \ , & \underline{B}(n)\underline{Q}_1(n)\underline{B}'(n) \end{bmatrix} \tag{A-27}$$

Convert the first column to an integral operator, and expand the first term on the right:

$$\begin{bmatrix} \int_{\tau_1}^{\tau_2} \Sigma_{\tau,\gamma}(n+1|n)(\cdot)d\gamma \ , & \Sigma_{\tau,\underline{a}}(n+1|n) \\ \int_{\tau_1}^{\tau_2} \Sigma_{\underline{a},\gamma}(n+1|n)(\cdot)d\gamma \ , & \Sigma_{\underline{a},\underline{a}}(n+1|n) \end{bmatrix} = \begin{bmatrix} \Phi_\tau \ , & \underline{0} \\ \underline{0} \ , & \underline{\Phi}(n) \end{bmatrix} \times$$

$$\times \begin{bmatrix} \int_{\tau_1}^{\tau_2} \Sigma_{\tau,\gamma}(n|n)(\cdot)d\gamma \ , & \Sigma_{\tau,\underline{a}}(n|n) \\ \int_{\tau_1}^{\tau_2} \Sigma_{\underline{a},\gamma}(n|n)(\cdot)d\gamma \ , & \Sigma_{\underline{a},\underline{a}}(n|n) \end{bmatrix} \begin{bmatrix} \Phi_\gamma \ , & \underline{0} \\ \underline{0} \ , & \underline{\Phi}'(n) \end{bmatrix}$$

$$+ \begin{bmatrix} Q(\tau)\int_{\tau_1}^{\tau_2} \delta(\gamma-\tau)(\cdot)d\gamma \ , & \underline{0} \\ \underline{0} \ , & \underline{B}(n)\underline{Q}_1(n)\underline{B}'(n) \end{bmatrix} \tag{A-28}$$

Use Eq. (84), (85), and (89) in the above, to see that:

$$\Sigma_s(n+1|n) = \Phi_s(n)\Sigma_s(n|n)\Phi_s^*(n) + D(n) \tag{A-29}$$

where we define $D(n) : V_s \to V_s$ by:

$$D(n)\left\{\begin{bmatrix} f(\tau) \\ \\ \underline{d} \end{bmatrix}\right\} = \begin{bmatrix} Q(\tau)\int_{\tau_1}^{\tau_2} \delta(\gamma - \tau)f(\gamma)d\gamma \\ \\ \underline{B}(n)\underline{Q}_1(n)\underline{B}'(n)\underline{d} \end{bmatrix} = \begin{bmatrix} Q(\tau)f(\tau) \\ \\ \underline{B}(n)\underline{Q}_1(n)\underline{B}'(n)\underline{d} \end{bmatrix} \varepsilon\ V_s \tag{A-30}$$

APPENDIX B

A TWO-DIMENSIONAL WIENER PROCESS

In this appendix, we define a two-dimensional Wiener process, with covariance:

$$E[\beta(\tau, t)\beta(s, \sigma)] = W \cdot \min(\tau, s) \cdot \min(t, \sigma) \tag{B-1}$$

Note that, formally, the mixed double partial derivative of this process will have the covariance of a two-dimensional white noise, because, formally:

$$E[(\frac{\partial^2}{\partial t \partial \tau} \beta(\tau, t))(\frac{\partial^2}{\partial \sigma \partial s} \beta(s, \sigma))] = \frac{\partial^4}{\partial t \partial \tau \partial \sigma \partial s} E[\beta(\tau, t)\beta(s, \sigma)] \tag{B-2}$$

and from Eq. (B-1) we have that:

$$\frac{\partial^4}{\partial t \partial \tau \partial \sigma \partial s} E[\beta(\tau, t)\beta(s, \sigma)] = W \cdot \delta(\tau - s) \cdot \delta(t - \sigma) \tag{B-3}$$

We shall first show the existence of a (Gaussian) random process with the covariance in Eq. (B-1). We then shall give meaning to a two-dimensional Wiener integral (where $f \in L^2([0, \infty) \times [0, \infty))$):

$$\int_0^\infty \int_0^\infty f(\tau, t)\,d\beta(\tau, t) \tag{B-4}$$

Finally, we shall show that the model we have given for gyroscopic noise in Eq. (23):

$$g(t) = \int_{\tau_1}^{\tau_2} p_d(\tau)e^{-t/\tau}\,d\mu(\tau, 0) + \int_{\tau_1}^{\tau_2} \int_0^t p_d(\tau)e^{-(t-s)/\tau}\,d\beta(\tau, s) \tag{B-5}$$

in fact yields the desired power spectral density (Eq. (17)):

$$S_{gg}(f) = (\frac{2}{\ln(\tau_2/\tau_1)})(\frac{1}{2\pi f})\tan^{-1}\left[\frac{2\pi f(\tau_2 - \tau_1)}{1 + 4\pi^2 f^2 \tau_2 \tau_1}\right] \tag{B-6}$$

We first show the existence of the two-dimensional Wiener process. The argument closely parallels that of J.M.C. Clark[21]. Choose two sets of complete orthonormal functions in $L^2([0, \infty))$: (where $<\cdot,\cdot>$ is dot product notation)

$$\{\psi_i(t)\} \ .\ni.\ <\psi_i, \psi_j> = \int_0^\infty \psi_i(t)\psi_j(t)dt = \begin{cases} 0 ; & i\neq j \\ 1 ; & i=j \end{cases} ; \quad i=1, 2, \cdots \tag{B-7}$$

$$\{\phi_i(t)\} \ .\ni.\ <\phi_i, \phi_j> = \int_0^\infty \phi_i(\tau)\phi_j(\tau)d\tau = \begin{cases} 0 ; & i\neq j \\ 1 ; & i=j \end{cases} ; \quad i=1, 2, \cdots \tag{B-8}$$

(Note that $\{\psi_i(t)\}$ and $\{\phi_i(\tau)\}$ may be the same set of functions.) Next, define a sequence of doubly-indexed Gaussian random variables:

$$\{a_{ij}\} \ .\ni.\ E[a_{ij}a_{mq}] = W \cdot \delta_{im} \cdot \delta_{jq} ; \begin{pmatrix} i=1, 2, \ldots \\ j=1, 2, \ldots \end{pmatrix} \tag{B-9}$$

We now define a sequence of random processes $\{\beta^N(\tau, t)\}$. Each random process is a function of two variables, (τ) and (t). The definition is given by:

$$\beta^N(\tau, t) = \sum_{i=0}^{N} \sum_{j=0}^{N} a_{ij} \int_0^t \int_0^\tau \psi_i(\gamma)\phi_j(\alpha)d\alpha d\gamma \tag{B-10}$$

Fix (τ) and (t). We claim that $\{B^N(\tau, t)\}$ is a quadratic mean Cauchy convergent sequence of random variables. Observe that: (say $M > N$)

$$E[(\beta^M(\tau,t) - \beta^N(\tau,t))^2] = E\left[\sum_{i=N+1}^{M} \sum_{j=N+1}^{M} a_{ij} \int_0^t \int_0^\tau \psi_i(\gamma)\phi_j(\alpha)d\alpha d\gamma \times \right.$$
$$\left. \times \sum_{k=N+1}^{M} \sum_{q=N+1}^{M} a_{kq} \int_0^t \int_0^\tau \psi_k(\lambda)\phi_q(\eta)d\eta d\lambda \right] \tag{B-11}$$

By Eq. (B-9), we obtain:

$$E[(\beta^M(\tau,t) - \beta^N(\tau,t))^2] = \sum\sum\sum\sum_{i,j,k,q=N+1}^{M} E[a_{ij}a_{kq}] \int_0^t \int_0^\tau \int_0^t \int_0^\tau \psi_i(\gamma)\phi_j(\alpha)\psi_k(\lambda)\phi_q(\eta)d\alpha d\gamma d\eta d\lambda$$

$$= \sum\sum\sum\sum_{i,j,k,q=N+1}^{M} \delta_{ik}\delta_{jq} \int_0^t \int_0^\tau \int_0^t \int_0^\tau \psi_i(\gamma)\phi_j(\alpha)\psi_k(\lambda)\phi_q(\eta)d\alpha d\gamma d\eta d\lambda$$

$$= W \cdot \sum_{i=N+1}^{M} \sum_{j=N+1}^{M} \left(\int_0^t \psi_i(\gamma)d\gamma\right)^2 \left(\int_0^\tau \phi_j(\lambda)d\lambda\right)^2$$

$$E[(\beta^M(\tau,t) - \beta^N(\tau,t))^2] = W \cdot \left[\sum_{i=N+1}^{M} (\int_0^t \psi_i(\gamma)d\gamma)^2\right] \cdot \left[\sum_{j=N+1}^{M} (\int_0^\tau \phi_j(\lambda)d\lambda)^2\right] \tag{B-12}$$

Define the following function:

$$I_t(\gamma) = \begin{Bmatrix} 1 ; & 0 \le \gamma \le t \\ 0 ; & \gamma > t \end{Bmatrix} \tag{B-13}$$

We may now express Eq. (B-12) in dot product notation:

$$E[(\beta^M(\tau,t) - \beta^N(\tau,t))^2] = W \cdot \left[\sum_{i=N+1}^{M} (<\psi_i, I_t>)^2\right] \cdot \left[\sum_{j=N+1}^{M} (<\phi_j, I_\tau>)^2\right] \tag{B-14}$$

By the orthonormality of the sequences $\{\psi_i(\gamma)\}$ and $\{\phi_j(\lambda)\}$ we have that each factor on the right of Eq. (B-14) approaches zero as $N \to \infty$. Thus $\{\beta^N(\tau, t)\}$ is a quadratic mean Cauchy convergent sequence of random variables. Call the limit $\beta(\tau, t)$. Thus $\beta(\tau, t)$ is a random process in the variables (τ) and (t). We now demonstrate that $\beta(\tau, t)$ has covariance as in Eq. (B-1):

$$E[\beta(\tau,t)\beta(s,\sigma)] = E\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij} \int_0^t\int_0^\tau \psi_i(\gamma)\phi_j(\alpha)d\alpha d\gamma \sum_{k=0}^{\infty}\sum_{q=0}^{\infty} a_{kq} \int_0^\sigma\int_0^s \psi_k(\lambda)\phi_q(\eta)d\eta d\lambda\right]$$

$$= W \cdot \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} <\psi_i, I_t><\psi_i, I_\sigma><\phi_j, I_\tau><\phi_j, I_s>$$

$$= W \cdot \left[\sum_{i=0}^{\infty} <\psi_i, I_t><\psi_i, I_\sigma>\right] \cdot \left[\sum_{j=0}^{\infty} <\phi_j, I_\tau><\phi_j, I_s>\right] \tag{B-15}$$

By Parseval's theorem, we have that:

$$E[\beta(\tau,t)\beta(s,\sigma)] = W \cdot <I_t, I_\sigma><I_\tau, I_s>$$

$$= W \cdot \left(\int_0^\infty I_t(\gamma)I_\sigma(\gamma)d\gamma\right)\left(\int_0^\infty I_\tau(\gamma)I_s(\gamma)d\gamma\right)$$

$$E[\beta(\tau,t)\beta(s,\sigma)] = W \cdot \min(t, \sigma) \cdot \min(\tau,s) \tag{B-16}$$

We have thus established the existence of the two-dimensional Wiener process.

We now wish to define a two-dimensional Wiener integral. We first develop an analog of the "orthogonal increment" property of Brownian motion. For a Brownian motion, $\mu(t)$, it is well known[(22)] that:

($\lambda(\cdot)$ denotes length)

$$E[(\mu(t) - \mu(s))(\mu(q) - \mu(r))] = K \cdot \lambda([s,t)\cap[r,q)) \tag{B-17}$$

Define the following function, over a box $([\tau_1, \tau_2) \times [t_1, t_2))$ in the first quadrant (no generality is lost by defining our process for $\tau \geq 0$, $t \geq 0$) of the R^2 plane:

$$F(\tau_1,\tau_2,t_1,t_2) = \beta(\tau_2,t_2) - \beta(\tau_2,t_1) - \beta(\tau_1,t_2) + \beta(\tau_1,t_1) \tag{B-18}$$

Note, incidentally, that if a deterministic function $q(\tau,t) \varepsilon C^2$ had its mixed double partial derivative integrated over this domain, we would obtain:

$$\int_{\tau_1}^{\tau_2}\int_{t_1}^{t_2} \frac{\partial^2 q(\tau,t)}{\partial\tau\partial t} dt d\tau = q(\tau_2,t_2) - q(\tau_2,t_1) - q(\tau_1,t_2) + q(\tau_1,t_1) \tag{B-19}$$

This can be taken as motivation for Eq. (B-18). It is easily seen, using only Eq. (B-16), that: ($A(\cdot)$ denotes area)

$$E[F(\tau_1,\tau_2,t_1,t_2)F(\tau_3,\tau_4,t_3,t_4)] = W \cdot A[([\tau_1,\tau_2)\times[t_1,t_2))\cap([\tau_3,\tau_4)\times[t_3,t_4))] \tag{B-20}$$

Equation (B-20) is the analog to Eq. (B-17). Using this property, we proceed, as Wong(22) does for a one-dimensional orthogonal increment process, to define the two-dimensional Wiener integral.

(1) If $f = I_{[a_1,a_2)\times[b_1,b_2)}$, the indicator function of the rectangle $[a_1,a_2)\times[b_1,b_2)$, we set:

$$\int_0^\infty\int_0^\infty f(\tau, t)d\beta(\tau, t) = F(a_1, a_2, b_1, b_2) \tag{B-21}$$

(2) If $f = \sum_{\nu=1}^{n} a_\nu f_\nu$, with f_ν functions as in (1), we set:

$$\int_0^\infty\int_0^\infty f(\tau, t)d\beta(\tau, t) = \sum_{\nu=1}^{n} a_\nu \int_0^\infty\int_0^\infty f_\nu(\tau, t)d\beta(\tau, t)$$

(3) If $\int_0^\infty\int_0^\infty |f_n(\tau, t) - f(\tau, t)|^2 d\tau dt \to 0$, we set:

$$\int_0^\infty\int_0^\infty f(\tau,t)d\beta(\tau,t) = \underset{n\to\infty}{\ell im \text{ in q.m.}} \int_0^\infty\int_0^\infty f_n(\tau,t)d\beta(\tau,t) \tag{B-22}$$

The class of functions $f(\tau, t)$ for which this is possible is $L^2([0,\infty)\times[0,\infty))$. In addition, as in the one-dimensional case in Wong[22], we find that:

$$E\left\{\left[\int_0^{\infty} f(\tau,t)d\beta(\tau,t)\right]\overline{\left[\int_0^{\infty} g(\tau,t)d\beta(\tau,t)\right]}\right\} = \int_0^{\infty}\int_0^{\infty} f(\tau,t)\overline{g(\tau,t)}d\tau dt \tag{B-23}$$

We shall make use of Eq. (B-23) in showing that our model for gyroscopic noise (Eq. (B-5)) yields the desired power spectral density (Eq. (B-6)). From Eq. (B-5) we have that (for $\alpha \geq 0$):

$$E[g(t)g(t-\alpha)] = \left[\left(\int_{\tau_1}^{\tau_2} p_d(\tau)e^{-t/\tau}\,d\mu(\tau,0) + \int_{\tau_1}^{\tau_2}\int_0^{t} p_d(\tau)e^{-(t-s)/\tau}\,d\beta(\tau,s)\right) \times\right.$$

$$\left.\times\left(\int_{\tau_1}^{\tau_2} p_d(\lambda)e^{-(t-\alpha)/\lambda}\,d\mu(\lambda,0) + \int_{\tau_1}^{\tau_2}\int_0^{\tau-\alpha} p_d(\lambda)e^{-[(t-\alpha)-\sigma]/\lambda}\,d\beta(\lambda,\sigma)\right)\right] \tag{B-24}$$

The process $\mu(\tau,0)$ is an initial "τ-axis-scaled" Brownian motion characterized by (analog of Eq. (B-23) in one dimension[22]):

$$E\left[\left(\int_{\tau_1}^{\tau_2} f(\tau)d\mu(\tau,0)\right)\overline{\left(\int_{\tau_1}^{\tau_2} g(\gamma)d\mu(\gamma,0)\right)}\right] = \int_{\tau_1}^{\tau_2} f(\tau)\overline{g(\tau)}(h(\tau)d\tau) \tag{B-25}$$

(for some $h(\tau)$)

Also, we have that $\beta(\tau,s)$ $(\tau \geq 0,\ s > 0)$ is independent of $\mu(\tau,0)(\tau \geq 0)$. Thus we obtain from Eq. (B-24) that:

$$E[g(t)g(t-\alpha)] = e^{-\alpha/\tau}\int_{\tau_1}^{\tau_2} p_d^2(\tau)e^{-(t-\alpha)/\tau}\,h(\tau)d\tau\ +$$

$$+\ e^{-\alpha/\tau}\int_{\tau_1}^{\tau_2} p_d^2(\tau)e^{-2(t-\alpha)/\tau}\int_0^{t-\alpha} e^{2s/\tau}\,W ds d\tau \tag{B-26}$$

Integrate in Eq. (B-26), to obtain

$$E[g(t)g(t-\alpha)] = e^{-\alpha/\tau}\int_{\tau_1}^{\tau_2} p_d^2(\tau)e^{-(t-\alpha)/\tau}\,h(\tau)d\tau\ +$$

$$+\ e^{-\alpha/\tau}\int_{\tau_1}^{\tau_2}\left(\frac{W\tau}{2}\right)p_d^2(\tau)[1 - e^{-2(t-\alpha)/\tau}]d\tau \tag{B-27}$$

Let $t \to \infty$ in Eq. (B-27), to obtain a stationary random process characterized by:

$$R_{gg}(\alpha) = E[g(t)g(t - \alpha)] = \int_{\tau_1}^{\tau_2} (\frac{Wt}{2}) e^{-|\alpha|/\tau} p_d^2(\tau) d\tau \qquad \text{(B-28)}$$

A Fourier transform of Eq. (B-28) yields:

$$S_{gg}(f) = \int_0^{\infty} \left(\frac{W\tau^2}{1 + 4\pi^2 f^2 \tau^2}\right) [p_d(\tau)]^2 \, d\tau \qquad \text{(B-29)}$$

Substituting Eq. (4) into Eq. (B-29) and using the normalization ($W = 2\ell n(\tau_2/\tau_1)$) of Eq. (16) to have unit variance, we obtain:

$$S_{gg}(f) = (\frac{2}{\ell n(\tau_2/\tau_1)}) (\frac{1}{2\pi f}) \tan^{-1} \left[\frac{2\pi f(\tau_2 - \tau_1)}{1 + 4\pi^2 f^2 \tau_2 \tau_1}\right] \qquad \text{(B-30)}$$

Equation (B-30) is the power spectral density for gyroscopic noise which we had in Eq. (17). Thus our more rigorous derivation yields the same power spectral density as did the heuristic derivation in Section 2. Incidentally, note that our discussion of the two-dimensional Wiener process and two-dimensional Wiener integral can easily be extended to (n) dimensions.

APPENDIX C

DERIVATION OF THE CONDITIONAL EXPECTATION FILTER

In this appendix, the conditional expectation filter is derived for recovering the Guassian scalar output of an n-dimensional linear system from independent additive white noise and gyro noise. The equations defining the problem are given by Eq. (35) through (37) and Eq. (41) through (49). Figure 8 pictorially represents the problem. The states $\underline{a}(n)$ and $x(\tau, n)$, $\tau \varepsilon [\tau_1, \tau_2]$, will be estimated. We will denote the conditional expectation of these quantities using the first (m) values of scalar observation sequence $\{z(i)\}$ as:

$$\hat{x}(\tau, n|m) = E\{x(\tau, n)|(z(1), \ldots, z(m)) = \underline{z}_m\} \tag{C-1}$$

$$\hat{\underline{a}}(n|m) = E\{\underline{a}(n)|\underline{z}_m\} \tag{C-2}$$

The following is a generalization of the finite-dimensional derivation by Schweppe[12]. Define the residual:

$$\delta_z(n+1|n) = z(n+1) - E(z(n+1)|\underline{z}_n) \tag{C-3}$$

Then:

$$E[x(\tau, n+1), \underline{a}(n+1)|\underline{z}_{n+1}] = E[x(\tau, n+1), \underline{a}(n+1)|\underline{z}_n, z(n+1)]$$

$$E[x(\tau, n+1), \underline{a}(n+1)|\underline{z}_{n+1}] = E[x(\tau, n+1), \underline{a}(n+1)|\underline{z}_n, \delta_z(n+1|n)] \tag{C-4}$$

But by orthogonality[12], we have that:

$$E\{\delta_z(n+1|n)\underline{z}_n'\} = \underline{0} \tag{C-5}$$

Thus we may add the conditional expectations in Eq. (C-4)[12], to obtain:

$$E\{x(\tau, n+1), \underline{a}(n+1)|\underline{z}_{n+1}\} = E\{x(\tau, n+1), \underline{a}(n+1)|\underline{z}_n\} + E\{x(\tau, n+1), \underline{a}(n+1)|\delta_z(n+1|n)\} \tag{C-6}$$

So, we have:

$$E\{x(\tau, n+1), \underline{a}(n+1)|n+1\} = E\{x(\tau, n+1), \underline{a}(n+1)|n\} + E\{x(\tau, n+1), \underline{a}(n+1)|\delta_z(n+1|n)\}$$

which is merely:

$$\{\hat{x}(\tau, n+1|n+1), \underline{\hat{a}}(n+1|n+1)\} = \{\hat{x}(\tau, n+1|n), \underline{\hat{a}}(n+1|n)\} + E\{x(\tau, n+1), \underline{a}(n+1)|\delta_z(n+1|n)\} \quad \text{(C-7)}$$

We shall first discuss estimation of $x(\tau, n)$, $\tau \in [\tau_1, \tau_2]$. Define the matrix $A_\tau(n+1)$ (a matrix function of one variable):

$$A_\tau(n+1) = E\left\{\begin{bmatrix} x(\tau, n+1) \\ \delta_z(n+1|n) \end{bmatrix}\begin{bmatrix} x(\tau, n+1) \\ \delta_z(n+1|n) \end{bmatrix}'\right\} \quad \text{(C-8)}$$

We then have[12]:

$$E\{x(\tau, n+1)|\delta_z(n+1|n)\} = A_{12,\tau}(n+1)A^{-1}_{22,\tau}(n+1)\delta_z(n+1|n) \; ; \quad (\tau_1 \le \tau \le \tau_2) \quad \text{(C-9)}$$

We now obtain $A_{22,\tau}$ and $A_{12,\tau}$. Define the following quantities:

$$\delta_{x_\tau}(n|n) = x(\tau, n) - \hat{x}(\tau, n|n) \quad \text{(C-10)}$$

$$\delta_{\underline{a}}(n|n) = \underline{a}(n) - \underline{\hat{a}}(n|n) \quad \text{(C-11)}$$

$$\delta_{x_\tau}(n+1|n) = x(\tau, n+1) - \hat{x}(\tau, n+1|n) \quad \text{(C-12)}$$

$$\delta_{\underline{a}}(n+1|n) = \underline{a}(n+1) - \underline{\hat{a}}(n+1|n) \quad \text{(C-13)}$$

Because, by Eq. (41):

$$z(n+1) = \int_{\tau_1}^{\tau_2} p_d(\tau)x(\tau, n+1)d\tau + \underline{H}(n+1)\underline{a}(n+1) + v(n+1) \quad \text{(C-14)}$$

and, by Eq. (35) and Eq. (42):

$$x(\tau, n+1) = \Phi_\tau x(\tau, n) + w(\tau, n) \quad \text{(C-15)}$$

$$\underline{a}(n+1) = \underline{\Phi}(n)\underline{a}(n) + \underline{B}(n)\underline{u}(n) \quad \text{(C-16)}$$

we have that:

$$z(n+1) = \int_{\tau_1}^{\tau_2} p(\tau)\Phi_\tau x(\tau, n)d\tau + \int_{\tau_1}^{\tau_2} p(\tau)w(\tau, n)d\tau$$

$$+ \underline{H}(n+1)\underline{\Phi}(n)\underline{a}(n) + \underline{H}(n+1)\underline{B}(n)\underline{u}(n) + v(n+1)$$

Thus:

$$E(z(n+1)|n) = \int_{\tau_1}^{\tau_2} p(\tau)\Phi_\tau \hat{x}(\tau, n|n)d\tau + \int_{\tau_1}^{\tau_2} p(\tau)\hat{w}(\tau, n|n)d\tau$$

$$+ \underline{H}(n+1)\underline{\Phi}(n)\underline{\hat{a}}(n|n) + \underline{H}(n+1)\underline{B}(n)\underline{\hat{u}}(n|n) + \hat{v}(n+1|n)$$

But $(z(1), \cdots, z(n))$ have no information about white process samples $w(\tau, n)$, $\underline{u}(n)$, or $v(n+1)$. Thus:

$$\underline{\hat{u}}(n|n) = \underline{0} \;;\quad \hat{v}(n+1|n) = 0 \;;\quad \hat{w}(\tau, n|n) = 0 \;;\quad (\forall\tau \in [\tau_1, \tau_2]) \tag{C-17}$$

Thus:

$$E(z(n+1)|n) = \int_{\tau_1}^{\tau_2} p(\tau)\Phi_\tau \hat{x}(\tau, n|n)d\tau + \underline{H}(n+1)\underline{\Phi}(n)\underline{\hat{a}}(n|n) \tag{C-18}$$

Then, by Eq. (C-3) we have that:

$$\delta_z(n+1|n) = z(n+1) - \int_{\tau_1}^{\tau_2} p(\tau)\Phi_\tau \hat{x}(\tau, n|n)d\tau - \underline{H}(n+1)\underline{\Phi}(n)\underline{\hat{a}}(n|n) \tag{C-19}$$

Also, Eq. (C-17) implies that:

$$\hat{x}(\tau, n+1|n) = \Phi_\tau \hat{x}(\tau, n|n) \tag{C-20}$$

$$\underline{\hat{a}}(n+1|n) = \underline{\Phi}(n)\underline{\hat{a}}(n|n) \tag{C-21}$$

Thus, using Eq. (C-12) through (C-14) and Eq. (C-19) through (C-21):

$$\delta_z(n+1|n) = \int_{\tau_1}^{\tau_2} p(\tau)\delta_{x_\tau}(n+1|n)d\tau + \underline{H}(n+1)\delta_{\underline{a}}(n+1|n) + v(n+1) \tag{C-22}$$

Observation of Eq. (C-8) shows that:

$$A_{22,\tau}(n+1) = E\{\delta_z(n+1|n)\delta_z'(n+1|n)\} \tag{C-23}$$

Thus, from Eq. (C-22):

$$
A_{22,\tau}(n+1) = \begin{bmatrix} r(n+1) + \underline{H}(n+1)\Sigma_{\underline{a},\underline{a}}(n+1|n)\underline{H}'(n+1) \\ + \int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2} p(\gamma)p(\beta)\Sigma_{\gamma,\beta}(n+1|n)d\gamma d\beta + \\ + \int_{\tau_1}^{\tau_2} p(\tau)\Sigma_{\tau,\underline{a}}(n+1|n)\underline{H}'(n+1)d\tau \\ + \int_{\tau_1}^{\tau_2} \underline{H}(n+1)\Sigma_{\underline{a},\tau}(n+1|n)p(\tau)d\tau \end{bmatrix} \tag{C-24}
$$

(where definitions Eq. (46) through (53) are used.) We will now obtain $A_{12,\tau}(n+1)$. By Eq. (C-10) and (C-15) we have:

$$
x(\tau, n+1) = \Phi_\tau[\hat{x}(\tau, n|n) + \delta_{x_\tau}(n|n)] + w(\tau, n) \tag{C-25}
$$

But by Eq. (C-12) and Eq. (C-20) we have:

$$
\delta_{x_\tau}(n+1|n) = x(\tau, n+1) - \Phi_\tau\hat{x}(\tau, n|n) \tag{C-26}
$$

Then, by Eq. (C-15) and Eq. (C-10):

$$
\delta_{x_\tau}(n+1|n) = \Phi_\tau\delta_{x_\tau}(n|n) + w(\tau, n) \tag{C-27}
$$

Thus, in Eq. (C-25):

$$
x(\tau, n+1) = \Phi_\tau\hat{x}(\tau, n|n) + \delta_{x_\tau}(n+1|n) \tag{C-28}
$$

By Eq. (C-8):

$$
A_{12,\tau}(n+1) = E\{x(\tau, n+1)\delta_z(n+1|n)\}
$$

By Eq. (C-22) and (C-28):

$$
A_{12,\tau}(n+1) = E\{[\Phi_\tau\hat{x}(\tau, n|n) + \delta_{x_\tau}(n+1|n)][\int_{\tau_1}^{\tau_2} p(\tau)\delta_{x_\tau}(n+1|n)d\tau + \underline{H}(n+1)\delta_{\underline{a}}(n+1|n) + v(n+1)]'\}
$$

Four of the six terms in the above product are zero by orthogonality or independence, leaving:

$$
A_{12,\tau}(n+1) = [\int_{\tau_1}^{\tau_2} p(\gamma)\Sigma_{\tau,\gamma}(n+1|n)d\gamma + \Sigma_{\tau,\underline{a}}(n+1|n)\underline{H}'(n+1)] \tag{C-29}
$$

Thus, by Eq. (C-7), (C-9), (C-19), and (C-20) we have that:

$$\hat{x}(\tau, n+1|n+1) = \Phi_\tau \hat{x}(\tau, n|n) + A_{12,\tau}(n+1)\, A_{22,\tau}^{-1}(n+1)[z(n+1) - \int_{\tau_1}^{\tau_2} p(\tau)\Phi_\tau \hat{x}(\tau, n|n)d\tau - \underline{H}(n+1)\underline{\Phi}(n)\hat{\underline{a}}(n|n)] \quad \text{(C-30)}$$

We shall now discuss estimation of $\underline{a}(n)$. Analogously to Eq. (C-8), we define $A_{\underline{a}}(n+1)$:

$$A_{\underline{a}}(n+1) = E\left\{\begin{bmatrix} \underline{a}(n+1) \\ \delta_z(n+1|n) \end{bmatrix} \begin{bmatrix} \underline{a}(n+1) \\ \delta_z(n+1|n) \end{bmatrix}'\right\} \quad \text{(C-31)}$$

Then as in Eq. (C-9) we have:

$$E\{\underline{a}(n+1)|\delta_z(n+1|n)\} = A_{12,\underline{a}}(n+1)A_{22}^{-1}(n+1)\delta_z(n+1|n) \quad \text{(C-32)}$$

It is clear that (see Eq. (C-8)):

$$A_{22,\underline{a}}(n+1) = E\{\delta_z(n+1|n)\delta_z(n+1|n)\} = A_{22,\tau}(n+1) \quad \text{(C-33)}$$

Thus we may define:

$$A_{22}(n+1) \triangleq A_{22,\underline{a}}(n+1) = A_{22,\tau}(n+1) \text{ (in Eq. (C-24))} \quad \text{(C-34)}$$

As in Eq. (C-25) we have, from Eq. (C-11) and (C-16):

$$\underline{a}(n+1) = \underline{\Phi}(n)[\hat{\underline{a}}(n|n) + \delta_{\underline{a}}(n|n)] + \underline{B}(n)\underline{u}(n) \quad \text{(C-35)}$$

which reduces, as in Eq. (C-28), to:

$$\underline{a}(n+1) = \underline{\Phi}(n)\hat{\underline{a}}(n|n) + \delta_{\underline{a}}(n+1|n) \quad \text{(C-36)}$$

We thus have, using Eq. (C-22):

$$A_{12,\underline{a}}(n+1) = E\{\underline{a}(n+1)\delta_z(n+1|n)\} = E\{[\underline{\Phi}(n)\hat{\underline{a}}(n|n) + \delta_{\underline{a}}(n+1|n)] \times$$

$$\times [\int_{\tau_1}^{\tau_2} p(\tau)\delta_{x_\tau}(n+1|n)d\tau + \underline{H}(n+1)\delta_{\underline{a}}(n+1|n) + v(n+1)]'\}$$

Four terms in the above product cancel by orthogonality or independence, leaving:

$$A_{12,\underline{a}}(n+1) = [\int_{\tau_1}^{\tau_2} p(\tau)\Sigma_{\underline{a},\tau}(n+1|n)d\tau + \Sigma_{\underline{a},\underline{a}}(n+1|n)\underline{H}'(n+1)] \quad \text{(C-37)}$$

which is a column vector. Then by Eq. (C-7), (C-19), (C-21), and (C-32) we have:

$$\underline{\hat{a}}(n+1|n+1) = \underline{\Phi}(n)\underline{\hat{a}}(n|n) + A_{12,\underline{a}}(n+1)A_{22}^{-1}(n+1)[z(n+1)$$

$$- \int_{\tau_1}^{\tau_2} p(\tau)\Phi_\tau \hat{x}(\tau, n|n)d\tau - \underline{H}(n+1)\underline{\Phi}(n)\underline{\hat{a}}(n|n)] \qquad \text{(C-38)}$$

In summary, Eq. (C-30) and Eq. (C-38) represent the conditional expectation filter for the states $x(\tau, n)$, $\tau \in [\tau_1, \tau_2]$, and $\underline{a}(n)$. Because the signal and gyro noise are both zero mean, we use as boundary conditions for Eq. (C-30) and (C-38):

$$\hat{x}(\tau, 0|0) = 0 \; ; \quad \forall\tau \in [\tau_1, \tau_2] \qquad \text{(C-39a)}$$

$$\underline{\hat{a}}(0|0) = \underline{0} \qquad \text{(C-39b)}$$

We next discuss the covariance equations of this filter.

Derivation of Covariance Equations:

By Eq. (C-16) we have that:

$$\underline{a}(n+1) = \underline{\Phi}(n)\underline{a}(n) + \underline{B}(n)\underline{u}(n) \qquad \text{(C-40)}$$

and by Eq. (C-21):

$$\underline{\hat{a}}(n+1|n) = \underline{\Phi}(n)\underline{\hat{a}}(n|n) \qquad \text{(C-41)}$$

Subtract Eq. (C-41) from Eq. (C-40), using Eq. (C-11) and (C-13) to obtain:

$$\delta_{\underline{a}}(n+1|n) = \underline{\Phi}(n)\delta_{\underline{a}}(n|n) + \underline{B}(n)\underline{u}(n) \qquad \text{(C-42)}$$

Multiply Eq. (C-42) by its transpose, and take an expectation of the result. Because $\underline{z}_n$ is independent of $\underline{u}(n)$, we obtain, using Eq. (59) and Eq. (C-13):

$$\Sigma_{\underline{a},\underline{a}}(n+1|n) = \underline{\Phi}(n)\Sigma_{\underline{a},\underline{a}}(n|n)\underline{\Phi}'(n) + \underline{B}(n)\underline{Q}_1(n)\underline{B}'(n) \qquad \text{(C-43)}$$

Similarly, by Eq. (C-15):

$$x(\tau, n+1) = \Phi_\tau x(\tau, n) + w(\tau, n) \qquad \text{(C-44)}$$

and by Eq. (C-20):

$$\hat{x}(\tau, n+1|n) = \Phi_\tau \hat{x}(\tau, n|n) \qquad \text{(C-45)}$$

Subtract, using Eq. (C-10) and (C-12) to obtain:

$$\delta_{x_\tau}(n+1|n) = \Phi_\tau \delta_{x_\tau}(n|n) + w(\tau, n) \tag{C-46}$$

By Eq. (C-12) and Eq. (53) we have:

$$\Sigma_{\gamma,\beta}(n+1|n) = E\{\delta_{x_\gamma}(n+1|n)\delta_{x_\beta}(n+1|n)\}$$

$$\Sigma_{\gamma,\beta}(n+1|n) = \Phi_\gamma \Sigma_{\gamma,\beta}(n|n)\Phi_\beta + E\{w(\gamma, n)w(\beta, n)\}$$

Then from Eq. (37) we have that:

$$\Sigma_{\gamma,\beta}(n+1|n) = \Phi_\gamma \Sigma_{\gamma,\beta}(n|n)\Phi_\beta + Q(\gamma)\delta(\gamma-\beta) \tag{C-47}$$

Using Eq. (C-42) and (C-46) we obtain:

$$\Sigma_{\tau,\underline{a}}(n+1|n) = \Phi_\tau \Sigma_{\tau,\underline{a}}(n|n)\underline{\Phi}'(n) \tag{C-48}$$

and:

$$\Sigma_{\underline{a},\tau}(n+1|n) = \underline{\Phi}(n)\Sigma_{\underline{a},\tau}(n|n)\Phi_\tau \tag{C-49}$$

Equations (C-43), (C-47), (C-48), and (C-49) form one set of covariance equations. The second set will now be formed.

We have, by Eq. (C-11) and (C-38) that:

$$\delta_{\underline{a}}(n+1|n+1) = \underline{a}(n+1) - \underline{\Phi}(n)\underline{\hat{a}}(n|n) - A_{12,\underline{a}}(n+1)A_{22}^{-1}(n+1)\delta_z(n+1|n) \tag{C-50}$$

Using Eq. (C-13) and (C-21) we obtain:

$$\delta_{\underline{a}}(n+1|n+1) = \delta_{\underline{a}}(n+1|n) - A_{12,\underline{a}}(n+1)A_{22}^{-1}(n+1)\delta_z(n+1|n) \tag{C-51}$$

Taking the transpose of Eq. (C-51), multiplying, and taking the expectation, we obtain:

$$\Sigma_{\underline{a},\underline{a}}(n+1|n+1) = \begin{bmatrix} \Sigma_{\underline{a},\underline{a}}(n+1|n) + A_{12,\underline{a}}(n+1)A_{22}^{-1}(n+1)A_{22}(n+1)A_{22}^{-1'}(n+1)A'_{12,\underline{a}}(n+1) + \\ - E\{\underline{a}(n+1)\delta_z'(n+1|n)A_{22}^{-1'}(n+1)A'_{12,\underline{a}}(n+1)\} + E\{\underline{\Phi}(n)\underline{\hat{a}}(n|n)\delta_z'(n+1|n) \times \\ \times A_{22}^{-1'}(n+1)A'_{12,\underline{a}}(n+1)\} + \\ - E\{A_{12,\underline{a}}(n+1)A_{22}^{-1}(n+1)\delta_z(n+1|n)[\underline{a}'(n+1) - \underline{\hat{a}}'(n|n)\underline{\Phi}'(n)]\} \end{bmatrix} \tag{C-52}$$

Note that z, hence $\delta_z(n+1|n)$ and $A_{22}(n+1)$, are scalars, and we leave the transposition symbols over them in Eq. (C-52) merely for clarity in the multiplication. By orthogonality the fourth term and the second half of the last term of Eq. (C-52) are both $\underline{0}$. The second and third terms of Eq. (C-52) cancel, because in Eq. (C-31):

$$E\{\underline{a}(n+1)\delta_z'(n+1|n)\} = A_{12,\underline{a}}(n+1) \qquad \text{(C-53)}$$

Thus only the first term and the first half of the last term remain:

$$\Sigma_{\underline{a},\underline{a}}(n+1|n+1) = \Sigma_{\underline{a},\underline{a}}(n+1|n) - A_{12,\underline{a}}(n+1)A_{22}^{-1}(n+1)A_{12,\underline{a}}'(n+1) \qquad \text{(C-54)}$$

By identical manipulations, we find that:

$$\Sigma_{\gamma,\beta}(n+1|n+1) = \Sigma_{\gamma,\beta}(n+1|n) - A_{12,\gamma}(n+1)A_{22}^{-1}(n+1)A_{12,\beta}'(n+1) \qquad \text{(C-55)}$$

(where the transpose in the last term is not important, as $A_{12,\beta}(n+1)$ is a scalar). Analogously to Eq. (C-51), we have:

$$\delta_{x_\tau}(n+1|n+1) = \delta_{x_\tau}(n+1|n) - A_{12,\tau}(n+1)A_{22}^{-1}(n+1)\delta_z(n+1|n) \qquad \text{(C-56)}$$

Multiply Eq. (C-51) by the transpose of Eq. (C-56), and take an expectation:

$$\Sigma_{\underline{a},\tau}(n+1|n+1) = \begin{bmatrix} \Sigma_{\underline{a},\tau}(n+1|n) + A_{12,\underline{a}}(n+1)A_{22}^{-1}(n+1)A_{22}(n+1)A_{22}^{-1\,\prime}(n+1)A_{12,\tau}'(n+1) + \\ - E\{\underline{a}(n+1)\delta_z'(n+1|n)A_{22}^{-1\,\prime}(n+1)A_{12,\tau}'(n+1)\} \\ + E\{\underline{\Phi}(n)\hat{\underline{a}}(n|n)\delta_z'(n+1|n)A_{22}^{-1\,\prime}(n+1)A_{12,\tau}'(n+1)\} \\ - E\{A_{12,\underline{a}}(n+1)A_{22}^{-1}(n+1)\delta_z(n+1|n)[x'(\tau,\ n+1) - \hat{x}'(\tau,\ n|n)\Phi_\tau']\} \end{bmatrix}$$

(C-57)

By orthogonality the fourth term and the second half of the last term of Eq. (C-57) are both $\underline{0}$. The second and third terms cancel, because of Eq. (C-53). From Eq. (C-8) we have:

$$E\{x(\tau,\ n+1)\delta_z'(n+1|n)\} = A_{12,\tau}(n+1) \qquad \text{(C-58)}$$

Thus the remaining terms of Eq. (C-57) are:

$$\Sigma_{\underline{a},\tau}(n+1|n+1) = \Sigma_{\underline{a},\tau}(n+1|n) - A_{12,\underline{a}}(n+1)A_{22}^{-1}(n+1)A_{12,\tau}'(n+1) \qquad \text{(C-59)}$$

By reversing the order of the multiplication of Eq. (C-51) and Eq. (C-56), we obtain:

$$\Sigma_{\tau,\underline{a}}(n+1|n+1) = \Sigma_{\tau,\underline{a}}(n+1|n) - A_{12,\tau}(n+1)A_{22}^{-1}(n+1)A_{12,\underline{a}}'(n+1) \qquad \text{(C-60)}$$

We thus have the second set of covariance equations (Eq. (C-54), (C-55), (C-59), and (C-60)). The model boundary conditions (Eq. (40) and (45)) and the independence of the signal and gyro noise determine the boundary conditions of the covariance equations:

$$\Sigma_{\gamma,\beta}(0|0) = \frac{W\gamma}{2} \cdot \delta(\gamma-\beta) \quad \text{(C-61)}$$

$$\Sigma_{\tau,\underline{a}}(0|0) = \underline{0} \quad \text{(C-62)}$$

$$\Sigma_{\underline{a},\tau}(0|0) = \underline{0} \quad \text{(C-63)}$$

$$\Sigma_{\underline{a},\underline{a}}(0|0) = \underline{P} \quad \text{(C-64)}$$

The filter derived in this appendix, the covariance equations, and the boundary conditions are repeated in Eq. (60) through (78) of the main text.

APPENDIX D

A SOLUTION OF THE STEADY-STATE RICCATI OPERATOR EQUATION

As was shown in Eq. (202), a bounded self-adjoint linear operator $\hat{K}$ exists such that (for fixed i):

$$K(i, (0,T)) \overset{W}{\to} \hat{K} \text{ as } T \to \infty, \text{ monotonically} \tag{D-1}$$

In this appendix we shall show that $\hat{K}$ is in fact a solution of the steady-state Riccati equation:

$$\hat{K} = \Phi_s(\hat{K} - \hat{K}H_s^*[r + H_s\hat{K}H_s^*]^{-1} H_s\hat{K})\Phi_s^* + D \tag{D-2}$$

We first establish that:

$$y'(i)\hat{K}y(i) = \inf_{\{u(j)\}} \sum_{j=i}^{\infty} (y'(j)Dy(j) + u'(j)ru(j)) \tag{D-3}$$

From Eq. (198) we have:

$$y'(i)K(i, (0,T))y(i) = \inf_{\{u(j)\}} \sum_{j=i}^{T-1} (y'(j)Dy(j) + u'(j)ru(j)) \tag{D-4}$$

By the nonnegativity of the summand:

$$\inf_{\{u(j)\}} \sum_{j=i}^{T-1} (y'(j)Dy(j) + u'(j)ru(j)) \le \inf_{\{u(j)\}} \sum_{j=i}^{\infty} (y'(j)Dy(j) + u'(j)ru(j)) \; ; \quad (\forall T) \tag{D-5}$$

Thus we have that:

$$\lim_{T\to\infty} (\inf_{\{u(j)\}} \sum_{j=i}^{T-1} (y'(j)Dy(j) + u'(j)ru(j))) \le \inf_{\{u(j)\}} \sum_{j=i}^{\infty} (y'(j)Dy(j) + u'(j)ru(j)) \tag{D-6}$$

From Eq. (D-1), (D-4), and (D-6) we obtain:

$$y'(i)\hat{K}y(i) \le \inf_{\{u(j)\}} \sum_{j=i}^{\infty} (y'(j)Dy(j) + u'(j)ru(j)) \tag{D-7}$$

Equation (D-7) is the first half of the desired result (Eq. (D-3)). By the monotone nature of the sequence:

$$\{0 \le K(T-1,\ (0,T)) \le K(T-2,\ (0,T)) \le \cdots \le \hat{K}\} \tag{D-8}$$

we know that:

$$y'(i)\hat{K}y(i) \ge y'(i)K(i,\ (0,T))y(i) = \inf_{\{u(j)\}} \sum_{j=i}^{T-1} (y'(j)Dy(j) + u'(j)ru(j)) \ ; \quad (\forall T) \tag{D-9}$$

We see from Lemma 3 (Section 4) that the optimal control, starting at fixed time (i), for the dual control problem yields (under the two assumed conditions of that lemma, which we assume here):

$$\left(\frac{\|y(T)\|}{\|y(i)\|}\right) \to 0 \ ; \quad \text{as } T \to \infty \tag{D-10}$$

Pick any $\varepsilon > 0$. Then we can choose T_ε such that:

$$\|y(T_\varepsilon)\| \le \varepsilon\|y(i)\| \tag{D-11}$$

If we now apply the "stabilizing" controls of Lemma 1 (Section 4) beginning at time T_ε, we deduce that the cost-to-go, starting at time T_ε and continuing for infinite time, satisfies:

$$J_{\inf}(\{u(j)\},\infty,T_\varepsilon) \triangleq \frac{1}{2}\inf_{\{u(j)\}} \sum_{j=T_\varepsilon}^{\infty} (y'(j)Dy(j)+u'(j)ru(j)) \le \frac{1}{2}F_8\|y(T_\varepsilon)\|^2 \tag{D-12}$$

for some constant F_8. Hence by Eq. (D-11):

$$\inf_{\{u(j)\}} \sum_{j=T_\varepsilon}^{\infty} (y'(j)Dy(j) + u'(j)ru(j)) \le F_8\|y(T_\varepsilon)\|^2 \le F_8\varepsilon^2\|y(i)\| \tag{D-13}$$

We observe, by optimality, that:

$$\inf_{\{u(j)\}} \sum_{j=i}^{\infty} (y'(j)Dy(j) + u'(j)ru(j)) \le \inf_{\{u(j)\}} \left(\sum_{j=i}^{T_\varepsilon-1} (y'(j)Dy(j) + u'(j)ru(j))\right) + \inf_{\{u(j)\}} \left(\sum_{j=T_\varepsilon}^{\infty} (y'(j)Dy(j) + u'(j)ru(j))\right) \tag{D-14}$$

where the initial state $(y(T_\varepsilon))$ for the second summation is found through the solution of the minimization problem (optimal control problem) in the first summation (as solved in Eq. (122)). Substitute Eq. (D-13) into Eq. (D-14) and transpose terms to obtain:

$$\inf_{\{u(j)\}} \sum_{j=i}^{T_\varepsilon - 1} (y'(j)Dy(j) + u'(j)ru(j)) \geq \inf_{\{u(j)\}} \left(\sum_{j=i}^{\infty} (y'(j)Dy(j) + u'(j)ru(j))\right) + - F_8 \varepsilon^2 \|y(i)\|^2 \tag{D-15}$$

By Eq. (D-9) and (D-15) we see that:

$$y'(i)\hat{K}y(i) \geq \inf_{\{u(j)\}} \left(\sum_{j=i}^{\infty} (y'(j)Dy(j) + u'(j)ru(j))\right) - F_8 \varepsilon^2 \|y(i)\|^2 \tag{D-16}$$

Because ε was arbitrary, we thus have:

$$y'(i)\hat{K}y(i) \geq \inf_{\{u(j)\}} \sum_{j=i}^{\infty} (y'(j)Dy(j) + u'(j)ru(j)) \tag{D-17}$$

Equations (D-7) and (D-17) together imply Eq. (D-3):

$$J_{inf}(\{u(j)\},\infty,i) \triangleq \frac{1}{2} \inf_{\{u(j)\}} \sum_{j=i}^{\infty} (y'(j)Dy(j) + u'(j)ru(j)) = \frac{1}{2} y'(i)\hat{K}y(i) \tag{D-18}$$

We now use Eq. (D-18) to show that $\hat{K}$ satisfies the steady-state operator Riccati equation:

$$\hat{K} = \Phi_s(\hat{K} - \hat{K}H_s^*[r + H_s\hat{K}H_s^*]^{-1} H_s\hat{K})\Phi_s^* + D \tag{D-19}$$

Using dynamic programming (precisely as in the solution of the dual optimal control problem (Eq. (191) through (196)) on Eq. (D-18) to determine the optimal control at time (i - 1), we find that the cost-to-go at time (i - 1), from any given state $y \in V_s$, is given by:

$$J_{inf}(\{u(j)\},\infty,i-1) = \frac{1}{2} y'[\Phi(\hat{K} - \hat{K}H_s^*[r + H_s\hat{K}H_s^*]^{-1} H_s\hat{K})\Phi_s^* + D]y \tag{D-20}$$

However, it is clear from the time invariance of the operators in the summand of Eq. (D-18) that:

$$J_{inf}(\{u(j)\},\infty,i-1,y) = J_{inf}(\{u(j)\},\infty,i,y) \tag{D-21}$$

where the left-hand-side of Eq. (D-21) means the cost-to-go starting at state (y) at time (i - 1). Thus by Eq. (D-18), (D-20) and (D-21) we have that:

$$y'\hat{K}y = 2J_{inf}(\{u(j)\},\infty,i,y) = 2J_{inf}(\{u(j)\},\infty,i-1,y)$$

$$= y'[\Phi_s(\hat{K} - \hat{K}H_s^*[r + H_s\hat{K}H_s^*]^{-1} H_s\hat{K})\Phi_s^* + D]y \qquad \text{(D-22)}$$

Because Eq. (D-22) holds for all $y \varepsilon V_s$, we see that $\hat{K}$ satisfies:

$$\hat{K} = \Phi(\hat{K} - \hat{K}H_s^*[r + H_s\hat{K}H_s^*]^{-1} H_s\hat{K})\Phi_s^* + D \qquad \text{(D-23)}$$

Thus we have shown that $\hat{K}$ satisfies the steady-state Riccati operator equation.

APPENDIX E

A SOLUTION OF THE DUAL CONTROL PROBLEM ON AN INFINITE INTERVAL

In this appendix we show that the steady-state controls $\{u_s(n)\}$ given by:

$$u_s(i) = -[r + H_s\hat{K}H_s^*]^{-1} H_s\hat{K}\Phi_s^* y(i) \quad \text{(E-1)}$$

solve the dual control problem with penalty function:

$$J(\{u(n)\},\infty,i) = \frac{1}{2}(\sum_{j=i}^{\infty} (y'(j)Dy(j) + u'(j)ru(j))) \quad \text{(E-2)}$$

If we use Eq. (231) in Eq. (228), and use the fact that $\hat{K}$ is bounded (Section 4, Lemma 5), so $y'(M+1)\hat{K}y(M+1) \to 0$ as $M \to \infty$, we find that:

$$y'(i)\hat{K}y(i) = \sum_{j=i}^{\infty} (y'(j)Dy(j) + u_s'(j)ru_s(j)) \quad \text{(E-3)}$$

By Eq. (219) and (199) we then have that:

$$\begin{aligned}\lim_{T\to\infty}[J_{inf}(\{u(n)\}, (\Sigma_0,T), i)] &= \lim_{T\to\infty}[(\tfrac{1}{2})y'(i)K(i, (\Sigma_0,T))y(i)] = \\ &= (\tfrac{1}{2})y'(i)\hat{K}y(i) \\ &= \frac{1}{2}(\sum_{j=i}^{\infty} (y'(j)Dy(j) + u_s'(j)ru_s(j)));\end{aligned}$$

$$(\forall\Sigma_0 \geq 0) \quad \text{(E-4)}$$

By the definition of the cost on the left of Eq. (E-4)(take $\Sigma_0 = 0$):

$$\lim_{T\to\infty}(\inf_{\{u(j)\}} (\tfrac{1}{2}) \sum_{j=i}^{T-1} (y'(j)Dy(j) + u'(j)ru(j))) = (\tfrac{1}{2}) \sum_{j=i}^{\infty} (y'(j)Dy(j) + u_s'(j)ru_s(j))$$

$$\text{(E-5)}$$

By the positive semidefiniteness of D, and the positive definiteness of r, we have:

$$\inf_{\{u(j)\}} \left[\sum_{j=i}^{T-1} (y'(j)Dy(j) + u'(j)ru(j)) \right] \le \inf_{\{u(j)\}} \left[\sum_{j=i}^{\infty} (y'(j)Dy(j) + u'(j)ru(j)) \right] ; \quad (\forall T) \tag{E-6}$$

Thus:

$$\lim_{T\to\infty} \left(\inf_{\{u(j)\}} \left[\sum_{j=i}^{T-1} (y'(j)Dy(j) + u'(j)ru(j)) \right] \right) \le \inf_{\{u(j)\}} \left[\sum_{j=i}^{\infty} (y'(j)Dy(j) + u'(j)ru(j)) \right] \tag{E-7}$$

Now, we substitute Eq. (E-5) into Eq. (E-7) to obtain:

$$\left(\frac{1}{2}\right) \sum_{j=i}^{\infty} (y'(j)Dy(j) + u_s'(j)ru_s(j)) \le \inf_{\{u(j)\}} \left[\left(\frac{1}{2}\right) \sum_{j=i}^{\infty} (y'(j)Dy(j) + u'(j)ru(j)) \right] \tag{E-8}$$

By optimality, Eq. (E-8) must in fact be satisfied with equality. Thus:

$$\inf_{\{u(j)\}} \left[\left(\frac{1}{2}\right) \sum_{j=i}^{\infty} (y'(j)Dy(j) + u'(j)ru(j)) \right] = \left(\frac{1}{2}\right) \sum_{j=i}^{\infty} (y'(j)Dy(j) + u_s'(j)ru_s(j)) \tag{E-9}$$

Hence $\{u_s(j)\}$ solves the optimal control problem over infinite interval, with cost criterion:

$$J(\{u(j)\},\infty,i) = \left(\frac{1}{2}\right) \sum_{j=i}^{\infty} (y'(j)Dy(j) + u'(j)ru(j)) \tag{E-10}$$

and, from Eq. (E-3) and (E-9):

$$J(\{u(j)\},\infty,i) = \frac{1}{2}y'(i)\hat{K}y(i) \tag{E-11}$$

APPENDIX F

THE EQUIVALENCE OF TWO CONDITIONS FOR FINITE-DIMENSIONAL LINEAR SYSTEMS

In this appendix we show the equivalence of two conditions on finite-dimensional linear systems which arise in Section 4.3 of the text.

(1") $(\underline{H}, \underline{\Phi})$ is detectable, i.e. there exists a matrix $\underline{L}$ such that $\|(\underline{\Phi} - \underline{LH})^{M_1}\| \leq q_1$; for some $0 \leq q_1 < 1$, and some M_1 a positive integer.

(1"') There exists a matrix $\underline{L}$ such that

$$\text{spectral radius } (\underline{\Phi}-\underline{LH}) \triangleq \max\{|\lambda_i| \,\big|\, (\underline{\Phi}-\underline{LH})\underline{x} = \lambda_i\underline{x};\ (\forall \underline{x}\varepsilon R^n)\} = \rho < 1$$

Proof: (1") → (1"'): If for any given $\underline{L}$, the spectral radius of $(\underline{\Phi}-\underline{LH})$ is equal to $(\gamma_{\underline{L}} > 1)$, then pick $\underline{x}$ such that $\|\underline{x}\| = 1$, and such that

$$(\underline{\Phi} - \underline{LH})\underline{x} = \gamma_{\underline{L}}\underline{x} \tag{F-1}$$

Then:

$$\|(\underline{\Phi} - \underline{LH})^{M_1}\underline{x}\| = (\gamma_{\underline{L}})^{M_1}\|\underline{x}\| = (\gamma_{\underline{L}})^{M_1} > 1\ ; \quad (\forall M_1\ \varepsilon\ Z^+) \tag{F-2}$$

So that:

$$\|(\underline{\Phi} - \underline{LH})^{M_1}\| > 1\ ; \quad (\forall M_1\ \varepsilon\ Z^+) \tag{F-3}$$

which implies that $(\underline{H}, \underline{\Phi})$ is not detectable.

(1"') → (1"):

We assume that there exists a matrix $\underline{L}$ such that the spectral radius of $(\underline{\Phi} - \underline{LH})$ is equal to $(\rho < 1)$. There exists an invertible matrix P such that:

$$(\underline{\Phi} - \underline{LH})_J = \underline{P}^{-1}(\underline{\Phi} - \underline{LH})\underline{P} \tag{F-4}$$

where $(\underline{\Phi} - \underline{LH})_J$ denotes the Jordan canonical form of $(\underline{\Phi} - \underline{LH})$ [31]. The typical Jordan block of $(\underline{\Phi} - \underline{LH})$ is of form:

$$[(\underline{\Phi} - \underline{LH})_J]_j = \begin{bmatrix} \lambda_i & 1 & 0 & 0 \cdots 0 \\ & \lambda_i & 1 & 0 \cdots 0 \\ & & \lambda_i & \ddots \quad \vdots \\ & & & \ddots \quad 1 \\ & \underline{0} & & \lambda_i \end{bmatrix} \tag{F-5}$$

where (λ_i) is an eigenvalue of $(\underline{\Phi} - \underline{LH})$. For any given constant $(\varepsilon > 0)$ it is easy to see that for M_1 large enough we have that:

$$\| [(\underline{\Phi} - \underline{LH})_J]^{M_1} \| < \varepsilon \tag{F-6}$$

Now let ε be chosen such that $\left[\varepsilon < \left(\frac{1}{(\|\underline{P}\|)(\|\underline{P}^{-1}\|)}\right)\right]$. Then:

$$\| (\underline{\Phi} - \underline{LH})^{M_1} \| = \| [\underline{P}(\underline{\Phi} - \underline{LH})_J \underline{P}^{-1}]^{M_1} \| = \| \underline{P}[(\underline{\Phi} - \underline{LH})_J]^{M_1} \underline{P}^{-1} \|$$

$$\leq (\|\underline{P}\|)(\| [(\underline{\Phi} - \underline{LH})_J]^{M_1} \|)(\|\underline{P}^{-1}\|)$$

$$\leq (\|\underline{P}^{-1}\|)(\varepsilon)(\|P\|) < 1 \tag{F-7}$$

Thus we have:

$$\| (\underline{\Phi} - \underline{LH})^{M_1} \| < 1 \tag{F-8}$$

which shows that $(\underline{H}, \underline{\Phi})$ is detectable. QED

BIOGRAPHICAL NOTE

Larry L. Horowitz was born in Flushing, New York, on April 17, 1949. He received the S.B. and S.M. degrees from the Massachusetts Institute of Technology in 1972. Mr. Horowitz's doctoral work has been supported by a National Science Foundation Graduate Fellowship. He has been working in association with The Charles Stark Draper Laboratory, Inc., Cambridge, Massachusetts.

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