ON THE DECOUPLING OF LINEAR MULTIVARIABLE SYSTEMS

by

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ABSTRACT

The problem of decoupling a linear, time invariant, multivariable system by a state feedback control law is considered. The original results of Falb and Wolovich for decoupling an m-input, m-output system into m scalar input, scalar output subsystems is presented and extended to the case where the number of inputs exceeds the number of outputs.

The geometric problem formulation of Wonham and Morse is presented and a partial decomposition for general (A,B) invariant subspaces is given. Results on the generic solvability of a class of decoupling problems are examined and it is shown that almost all systems of the type considered by Falb and Wolovich are decoupleable.

Connections between the Wonham-Morse and Falb-Wolovich approaches to the decoupling problem are explored and the direct equivalence is demonstrated for a special case. A result of the former is then shown to imply a strong necessary condition for decoupling linear systems into single input, multiple output subsystems.

Finally, the controllability subspaces of a matrix pair (A,B), instrumental in the geometric formulation of the decoupling problem are shown to have a natural analog in terms of the kernel of the singular pencil of matrices $(\lambda I-A;-B)$. The possible dimensions of controllability subspaces are proved to be completely determined by a set of invariants of this pencil of matrices. The minimal dimension of controllability subspaces which contain arbitrary subspaces of the image of the input gain B is ascertained, and a construction for such subspaces given.

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Dedicated to

a little red-haired boy and his Mother;

when they smile the sun shines.

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CHAPTER 1

INTRODUCTION

1.1 Problem Statement

Real life systems tend to be complicated objects. Inputs interact with and counteract each other rendering it difficult to determine how they effect outputs. Mathematical system models built to help understand the system they represent, frequently fall victim to this malaise. Often composed of scores of coupled differential equations, such system models are only slightly more tractable than the system they attempt to define. The abundance of misunderstood econometric models with hundreds, even thousands of state variables gives ample evidence of this plight.

In many such systems, the culprit which tends to complicate our understanding is not so much size, but rather a lack of modularity. It is the interaction which causes every input to affect every output and destroys any attempt to analyze the system on an organized piece by piece basis. Intuitively, it would be so much more pleasing if systems could be partitioned into isolated subsystems of reasonable size and complexity. These subsystems could be analyzed independently and then the overall system dynamics determined from the subsystems.

For approximately the past decade, considerable literature has been generated about partitioning or decoupling finite dimensional, linear, time-invariant systems into smaller order subsystems through the use of state feedback control laws. For the general, linear, time-invariant, multivariable system the decoupling problem can be stated as follows: having partitioned the outputs into a number of subsets, is there a feedback control law which transforms the given system into one where each subset of outputs is completely controlled by a corresponding subset of the inputs which affects no other output subsets? In its most general form, this problem remains unanswered.

Most previous research in this area has been centered on two different approaches. At first considerable emphasis was given toward systems with an equal number of inputs and outputs. Decoupling meant a decomposition into single input, single output subsystems, and became equivalent to the nonsingular diagonalization of the system transfer function matrix by a state feedback control law. In 1967 Falb and Wolovich [F2] completely solved this problem, giving a constructive solution and showing the possibility of simultaneous pole allocation while decoupling. Their method, which in this paper we shall designate the input-output method due to its exclusive use of input to output mappings, made extensive use of matrix operations to secure necessary and sufficient conditions for solvability.

Beginning in early 1970, Wonham and Morse [W7] introduced a theory of decoupling applicable to a more general class of linear systems. Having as its central theme a decomposition of the system state space into specialized subspaces, this geometric approach, as it was called, led to significant advances in our knowledge about the decoupling of linear systems. Complete solutions for several important special cases were determined, and the problem of decoupling via more general feedback compensators was formulated and solved [M7].

The research reported on herein is concerned with the following issues:

i) to elucidate the input-output and geometric theories of decoupling, and where appropriate to rederive or interpret elements of these theories;

ii) to explore the connections between these apparently disparate approaches to a common problem, and to extend the results of the input-output method to the more general problem formulations solvable via the geometric method;

iii) to develop a characterization of the key elements of the geometric theory and relate them to basic system invariants.

Through i) and ii) we hope to expand our general knowledge about the decoupling of linear systems. Few attempts have been made thus far to unify these differing approaches to decoupling and to evolve a common, more easily implemented theory. By iii) we hope to add some concrete structure to the specialized subspaces which play a vital role in the geometric formulation of the decoupling problem, structure which has so far been lacking.

1.2 Brief Historical Background

Prior to 1964, the status of decoupling theory left much to be desired. Results were few and generally limited to involved frequency domain manipulations. Morgan [M3] in 1964 formulated the problem of decoupling an m-input, m-output linear time invariant system into single input, single output subsystems, in terms of state space techniques. With this approach he was able to find a sufficient condition for decoupling.

In 1967, Falb and Wolovich [F3] completely solved this problem, showing that the existence of a decoupling control law was equivalent to the nonsingularity of an m x m matrix easily constructed from the system parameters. Falb and Wolovich further gave a synthesis procedure for determining decoupling control laws and showed that one could decouple and achieve at least partial pole assignment at the same time.

Approximately a year later, Gilbert [G2] refined and put into perspective the work of Falb and Wolovich. Gilbert pointed out certain feedback inavriants used in the construction of decoupling control laws and gave a canonical form for decoupled systems. In addition, he showed the detailed structure of the decoupled subsystems indicating the possibility of pole zero cancellations and hence the loss of observability of the decoupled system.

Numerous other papers ([C1],[F5],[L1],[M2],[M9],[M10],[N1],[P2],[S1],[S2],[S3],[S4],[S5],[T1],[W3]) have been published presenting extensions and/or variations of the original results of Falb and Wolovich. Questions about the decoupling of time varying linear systems, synthesis of inverse systems, and decoupling of special classes of nonlinear systems have been considered in the papers listed, all basically following from [F2].

Beginning early 1970, Wonham and Morse published the first of their papers [W7] proposing a different approach to the decoupling of linear systems. Using a geometric method, they were able to formulate the problem of decoupling a general linear systems into arbitrary size subsystems, achieving complete solutions for a number of important special cases. At about the same time, Basile and Marro [B1],[B2] independently derived some of the results on invariant subspaces which play such a key role in the geometric approach to decoupling.

In a subsequent paper [M7], Morse and Wonham extended their results to the decoupling of linear systems by dynamic compensation and completely determined when a solution exists. Additionally, they have shown [W8] that the subspaces generated in their decoupling theory may be used to rederive a canonical form for the input-state dynamics of linear time invariant systems ([B7],[K2],[P1],[R1],[W4]). The survey paper [M8] presents an extensive overview of their geometric method.

More recently Fabian and Wonham [F1] explored the question of generic solvability of decoupling problems. They were able to show that with minor constraints on the dimensions of the system and the decoupled subsystems, almost all linear systems are decoupleable by the use of dynamic compensation.

1.3 Thesis Outline

The remainder of this introduction will include an outline of the body of the thesis followed by a section introducing the notational conventions we shall employ throughout the text.

In Chapter 2 we shall present the input-output approach to the decoupling problem. We will begin with the development of the results of Falb and Wolovich and then give the canonical form for decoupled systems of Gilbert. As an extension we will consider the problem of decoupling systems with more inputs than outputs, and show that its solution may be related to the original Falb and Wolovich conditions.

The methodology and results of Wonham and Morse are presented in Chapter 3. We will review the development of (A,B) invariant subspaces and controllability subspaces, and show how the decoupling problem may be formulated in terms of a set of controllability subspaces. Solutions for important special cases will be given.

Continuing, we shall extend an important lemma from [M8] to show how general (A,B) invariant subspaces may be decomposed into simpler invariant subspaces. We then briefly present the work of Morse and Wonham on decoupling linear systems by dynamic compensation. The results on the generic solvability of this problem by Fabian and Wonham are given and used to show that the problem considered by Falb and Wolovich is generically solvable.

In Chapter 4 we try to illuminate the connections between the input-output and geometric approaches to decoupling. We begin by demonstrating the direct equivalence of the Falb and Wolovich result and a geometric condition of Wonham and Morse. Then we return to the canonical form of Gilbert and re-examine it in light of previous results on invariant subspaces. Finally we make use of a necessary and sufficient geometric condition for decoupling a system into single input, multiple output subsystems to derive a strong necessary condition in the form of the original Falb and Wolovich result.

We switch directions somewhat in Chapter 5 and develop structures for (A,B) invariant and controllability subspaces. We show first that the invariant subspaces of matrix pairs related by common transformations, are themselves simply related. This allows us to restrict our consideration to matrix pairs (A,B) in a canonical form.

Then we derive a characterization for controllability subspaces in terms of the kernel of a particular polynomial matrix. This characterization leads directly to a well-known canonical form of Brunovsky [B7]. The possible dimensions of controllability subspaces are then shown to be completely determined by a set of system invariants. Finally a method to construct minimal dimension controllability subspaces which contain certain other subspaces is given.

In Chapter 7, some conclusions about this research are drawn and some avenues of future research discussed.

1.4 Notation Conventions

Due to the large and varied amount of symbols used in the following text, it will be to our advantage to arrive at some specific notational conventions at the start. Whenever and wherever possible we will try to adhere to the guidelines set herein. Deviations, if any, will be minor and will be clear from context and/or explanation. To be consistent with established literature, we will basically follow the notation of [G2] and [M8].

For concreteness, we shall assume that the vector spaces encountered are real, Euclidean spaces, although most everything we develop will hold for vector spaces over an arbitrary infinite field. We let R^n designate the real, Euclidean n-space. Upper case block letters, A,B,C, etc. will indicate linear transformations between vector spaces or their associated matrices. Transposes will be demonstrated by a superscript T, e.g., A^T. The lower case block letters a thru h and u thru z will usually refer to vectors, while the letters i thru t will generally denote positive integers.

Script letters will be used to indicate vector spaces or subspaces, with lower case script letters denoting one dimensional subspaces. In particular, if B is a specified linear map, $\mathcal{B}$ will represent its image, similarly $\mathcal{B}^T$ will designate the image of B^T. If x is a vector, we shall indicate the subspace spanned by x either by $\mathcal{X}$ or Span{x}. In the same vein, the subspace spanned by a set of vectors $\{x_1,\ldots,x_k\}$ will be denoted Span$\{x_1,\ldots,x_k\}$. The orthogonal complement of a subspace $\mathcal{M}$, that is the subspace of all vectors orthogonal to $\mathcal{M}$, is given by $\mathcal{M}^{\perp}$. Further the kernel or nullspace of a map C is indicated Ker C. We note that $\mathcal{C}^{\perp} = \ker C^T$. If A is a linear map and $\mathcal{R}$ a subspace of the range of A, the inverse image of $\mathcal{R}$ under A, that is, the subspace $\{x \mid Ax \in \mathcal{R}\}$, is denoted by $A^{-1}\mathcal{R}$.

For any positive integer k, the set of integers $\{1,2,\ldots,k\}$ will be denoted $\underline{k}$. For any set of integers $\{k_1,\ldots,k_m\}$ we define

$$k_i^* = \sum_{\substack{j \neq i \\ j \in \underline{m}}} k_j, \qquad i \in \underline{m}$$

and for any ordered set $\{k_1,\ldots,k_m\}$

$$k_i^+ = \sum_{\substack{j \leq i \\ j \in \underline{m}}} k_j, \qquad i \in \underline{m}.$$

If $\{R_1,\ldots,R_k\}$ is a set of subspaces we define

$$R_i^* = \sum_{\substack{j\neq i\\ j\in \underline{k}}} R_i, \qquad i \in \underline{k}$$

and

$$R^* = \bigcap_{i\in\underline{k}} R_i^*$$

We call the subspace R^* the radical of the set $\{R_1,\ldots,R_k\}$.

The dimension of a subspace R is given by dim R. Two subspaces R_1 and R_2 are said to be independent if their intersection is the zero subspace, which we indicate either 0 or $\{0\}$. A set of subspaces $\{R_1,\ldots,R_k\}$ is independent if $R_i \cap R_i^* = 0$, for all $i \in \underline{k}$. It follows that the radical of a set of independent subspaces is zero. When we wish to emphasize the independence of two subspaces, we will write their sum as a direct sum $R_1 \oplus R_2$. Further, if $R_1 \subset R_2$, then we indicate the vector space of cosets of R_2 modulo R_1 by R_2/R_1.

If A is a linear map from R^n to R^n, $A: R^n \to R^n$, and R is an A invariant subspace, i.e. $AR \subset R$, then we write $A|R$ to represent the restriction of A to the sub-domain R. If $A : R^n \to R^n$, and $B \subset R^n$, then

$$\{A|B\} = B + AB + \ldots + A^{n-1}B,$$

and (A,B) is called a controllable pair if $\{A|B\} = R^n$.

Finally, the degree of a polynomial $\psi(\lambda)$ is indicated deg $\psi(\lambda)$, while the space of polynomials with coefficients in R^n is given by

$R^n[\lambda]$. We denote the j^{th} standard unit vector of R^n (1 in the j^{th} component, zeros elsewhere) by e_j, and let lower case Greek letters refer to field scalars. The phrases invariant subspace and controllability subspace will often be abbreviated i.s. and c.s. respectively. The terminus of a proof will be indicated by $\square$.

We will be interested solely in linear, time-invariant, multivariable systems described by the differential equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) \tag{1.4-1}$$

or the difference equation

$$x(j+1) = Ax(j) + Bu(j), \quad y(j) = Cx(j). \tag{1.4-2}$$

The algebraic structure which we will use applies equally well to either system formulation. We shall often refer to (1.4-1) or (1.4-2) simply by the matrix triple (A,B,C). It is assumed that x,u, and y (the state, input, and output) are real n,m, and q vectors respectively, with the matrices A,B,C appropriately dimensioned. When we consider a system to be decoupled, an implied partition of the outputs into k suboutputs, $y_i = C_i x$, with y_i a q_i vector for $i \in \underline{k}$ is always assumed. Further, unless otherwise noted the pair (A,B) is always presumed controllable, with B having full column rank, and the output submatrices C_i, $i \in \underline{k}$ are always assumed nontrivial.

CHAPTER 2

THE INPUT-OUTPUT APPROACH TO DECOUPLING

2.1 Introduction

Prior to the work by Wonham and Morse, the significant results in decoupling theory were achieved by Falb and Wolovich [F2] who considered the problem of decoupling m-input, m-output linear systems into m scalar input, scalar output subsystems. Such a system is decoupled if its transfer function matrix is diagonal with nonzero diagonal elements. Falb and Wolovich determined a condition for the existence of a feedback control law which accomplishes this in terms of certain feedback invariants of this transfer function matrix. To differentiate it from the procedures of Wonham and Morse, we designate this latter method the input-output approach to decoupling.

In this chapter we will review the input-output method, primarily following Gilbert [G2], who has extended and put in perspective the results of Falb and Wolovich. These authors consider m-input, m-output linear systems governed by equations of the form (1.4-1). It is further assumed that the k output subvectors are all scalars, that is $k = q = m$, and $q_i = 1$, $i \in \underline{m}$. This latter presumption holds throughout this chapter. In this development, the discrete time system formulation (1.4-2) will be used. As has been previously indicated, the algebraic structure we will employ applies both to continuous or discrete time linear systems.

The paper by Gilbert to which we refer is an easily readable reformulation and clarification of the original work by Falb and

Wolovich. In addition, Gilbert has defined both transfer function and state space representations for what he calls canonically decoupled systems. With these formulations, the structure of decoupled systems become easily visible.

We shall present the significant results of the input-output method without proof, referring the reader to the appropriate literature as needed. Where adviseable, however, we will give various interpretations to elements of this approach, particularly when the link to discrete time systems differs from that to continuous time systems.

Finally we will show how the basic input-output approach is easily extended to the case where the number of inputs may exceed the number of outputs, but where decoupling into single output subsystems is still desired. We are able to determine necessary and sufficient conditions for decoupling of this more general problem which are very similar to those given by Falb and Wolovich.

2.2 Basic Results

Let us consider an m-input, m-output, n-th order system described by (1.4-2) where A,B, and C are real n x n, n x m, and m x n matrices respectively, with $m \leq n$. Associated with (1.4-2) is the transfer function

$$H(\lambda) = C(I\lambda - A)^{-1}B \qquad (2.2\text{-}1)$$

which may be viewed as a mapping from the space of finite length input sequences (i.e. $u^*(\lambda) = \lambda^j u(-j) + \ldots + \lambda\, u(-1) + u(0)$) to the space of infinite length output sequences (i.e. $y^*(\lambda) = \lambda^{-1}y(1) + \lambda^{-2}y(2) + \ldots$). (See [K3],Ch. 10 for a more detailed description of this module based linear system theory.) The feedback control law denoted by the matrix pair (F,G),

$$u(j) = Fx(j) + Gv(j) \tag{2.2-2}$$

where v is a new set of m inputs, transforms the external description of the system (2.2-1) to

$$H(\lambda;F,G) = C(I\lambda-A-BF)^{-1}BG \tag{2.2-3}$$

where we have explicitly shown the dependence of $H(\lambda)$ on the control law (F,G). We shall say that the system (1.4-2) is decoupleable if there exists a feedback control law (2.2-2) such that (2.2-3) is diagonal and nonsingular. (A matrix of rational functions is nonsingular if its determinant is not identically zero.)

Letting C_i denote the i^{th} row of C, $i \varepsilon \underline{m}$ we define a set of integers $\{d_i, i \varepsilon \underline{m}\}$ and a set of linear forms (row vectors) $\{D_i, i \varepsilon m\}$.

(2.2.1) <u>Definition</u>: (Falb and Wolovich [F2]). Given a system (A,B,C) for each $i \varepsilon \underline{m}$ let

$$d_i = \min\{j \mid C_iA^jB \neq 0, \quad j = 0,1,\ldots,n-1\}$$

or (2.2-4)

$$d_i = n - 1 \quad \text{if} \quad C_iA^jB = 0 \text{ for all } j \geq 0$$

and

$$D_i = C_iA^{di}B \ . \tag{2.2-5}$$

For a controllable system described by (1.4-2) d_i+1 represents the minimum time delay for the effect of any input to be visible at output i, and D_i represents the first non-trivial pointwise mapping from inputs to output i. That is given an input $u(0) \neq 0$, the outputs $y_i(1)$ through $y_i(d_i)$ will be identically zero with $y_i(d_i + 1) = C_i A^{d_i} Bu(0)$ possibly nonzero for each $i \in \underline{m}$.

(2.2.2) Remark: We may alternatively define the quantities d_i and D_i, $i \in \underline{m}$ directly from the transfer function $H(\lambda)$. Letting $H_i(\lambda)$ denote the i^{th} row of $H(\lambda)$ it is readily established (see[G2]) that if $H_i(\lambda) \neq 0$, then

$$d_i = j \text{ such that } D_i^j = \lim_{\lambda\to\infty} \lambda^{j+1} H_i(\lambda) \text{ is nonzero and finite}$$

and

$$D_i = D_i^{d_i}$$

while if $H_i(\lambda) = 0$, $d_i = n-1$ and $D_i = 0$. In the continuous time state variable formulation (1.4-1), these quantities represent high frequency system properties.

Consider a discrete system (A,B,C) of the form (1.4-2), with feedback law (F,G). We may calculate the quantities d_i and D_i, $i \in \underline{m}$ for the closed loop system (A + BF, BG, C) via Def. 2.2.1. To indicate the explicit dependence of these quantities on the control law (F,G) we will write them as $d_i(F,G)$ and $D_i(F,G)$ respectively. That is $d_i \equiv d_i(0,I_m)$, $D_i = D_i(0,I_m)$, $i \in \underline{m}$.

Consider now the effect of the control law (F,G) on the integers $d_i(F,G)$ $i \in \underline{m}$. Since feedback in the discrete system (1.4-2) acts as an input delayed one unit of time, it is clear that feedback cannot affect the minimum time delay from inputs to any particular output.

Hence $d_i(F,G)$ is independent of F, $i \in \underline{m}$, and is thus a feedback invariant. Further if G is nonsingular, it is easily seen that $d_i(F,G) = d_i$ while $D_i(F,G) = D_iG$, $i \in \underline{m}$ for any input gain matrix G. These equalities may be easily checked by expanding $C_i(A + BF)^jBG$ and using Def. 2.2.1.

Now we form the m x m matrix D from the row vectors D_i, $i \in \underline{m}$,

$$D = [D_1^T;\dots;D_m^T]^T .$$

This matrix contains all the information necessary to solve our problem.

(2.2.3) Theorem: (Falb and Wolovich [F2]) The system (A,B,C) may be decoupled if and only if D is nonsingular. If the feedback law (F, G) decouples (A,B,C) then $G = D^{-1}\Gamma$ for some nonsingular diagonal matrix Γ.

Gilbert [G2] demonstrates this result in an easily understood exposition.

Falb and Wolovich show that the feedback law

$$F = - D^{-1}A^*, \quad G = D^{-1} \tag{2.2-7}$$

where A^* is an m x n matrix whose i^{th} row is given by $C_iA^{d_i+1}$, $i \in \underline{m}$, will always decouple a system (A,B,C) provided, of course, that D^{-1} exists. With this feedback law it is easily demonstrated that $C_i(A + BF)^{d_i+1} = 0$ for all $i \in \underline{m}$. Indeed the feedback law (2.2-7) yields a closed loop system with a most simple input-output structure. The diagonal elements of the transfer function matrix are given by

$$h_i(\lambda;F,G) = \lambda^{-d_i-1} \qquad i \in \underline{m} , \tag{2.2-8}$$

and thus the i^{th} decoupled subsystem is just a $d_i + 1$ order delay. In the continuous time analog, the transfer function elements (2.2-8) would denote $d_i + 1$ order integrators, prompting Gilbert to call such a system "integrator decoupled" (i.d.).

Falb and Wolovich further show that it is possible to specify at least $m + \sum_{i \in \underline{m}} d_i$ of the closed loop system poles while still achieving decoupling. Indeed let $d = \max\{d_i | i \in \underline{m}\}$, and consider the m x m matrices.

$$M_j = \text{diagonal}[\mu_{1j},\ldots,\mu_{mj}] \quad j = 0,1,\ldots d$$

where the μ_{ij}'s are scalars with $\mu_{ij} = 0$ for $j > d_i$, $i \in \underline{m}$. Then given (A,B,C) decoupleable and the feedback law

$$F = -D^{-1}\left(\sum_{j=0}^{d} M_j CA^j - A^*\right), \quad G = D^{-1} \tag{2.2-9}$$

it follows that the input-output dynamics of the closed loop system (A + BF, BG, C) are given by

$$y_i(d_i+1+j) = \mu_{i0}y_i(j) + \mu_{i1}y_i(j+1)+\ldots+ \mu_{id_i}y_i(d_i+j) + u_i(j) \qquad i \in \underline{m}. \tag{2.2-10}$$

By varying the μ_{ij}, $j = 0,\ldots,d_i$, $i \in \underline{m}$, and hence changing the feedback of (2.2-9), the dynamics of (2.2-10) may be arbitrarily altered without sacrificing the decoupled structure of the system.

Gilbert has defined a particularly simple form for integrator decoupled systems, and has shown that every i.d. system may be represented in such a form by an appropriate coordinate transformation. We present this form here as it graphically illustrates the structure of a decoupled m-input, m-output system, and will re-examine it at greater length in Chapter 4 after some of the invariant subspace machinery of the geometric approach is developed.

(2.2.4) Definition: (Gilbert [G2]) An m-input, m-output linear system (A,B,C) is canonically decoupled (c.d.) if

1) The matrices A,B, and C have the partitioned form:

$$A = \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 & 0 & A_1^u \\ 0 & A_2 & 0 & \cdots & 0 & 0 & A_2^u \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_m & 0 & A_m^u \\ A_1^c & A_2^c & A_3^c & \cdots & A_m^c & A_{m+1} & A_{m+1}^u \\ 0 & 0 & 0 & \cdots & 0 & 0 & A_{m+2} \end{bmatrix} \quad \begin{array}{l} A_i \text{ is } p_i \times p_i, \\ A_i^c \text{ is } p_{m+1} \times p_i, \\ A_i^u \text{ is } p_i \times p_{m+2}, \end{array} \tag{2.2-11}$$

$$B = \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & b_m \\ b_1^c & b_2^c & \cdots & b_m^c \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad \begin{array}{l} b_i \text{ is } p_i \times 1, \\ b_i^c \text{ is } p_{m+1} \times 1, \end{array} \tag{2.2-12}$$

$$C = \begin{bmatrix} c_1 & 0 & \cdots & 0 & 0 & c_i^u \\ 0 & c_2 & \cdots & 0 & 0 & c_2^u \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & c_m & 0 & c_m^u \end{bmatrix} \quad \begin{array}{l} c_i \text{ is } 1 \times p_i, \\ c_i^u \text{ is } 1 \times p_{m+2}, \end{array} \tag{2.2-13}$$

where $p_i \geq d_i + 1$, $i \in \underline{m}$;

2) the submatrices A_i, b_i, and c_i, $i \in \underline{m}$ have the form:

$$A_i = \begin{bmatrix} \begin{bmatrix} 0 & I_{d_i} \\ 0 & 0 \end{bmatrix} & 0 \\ T_i & \Phi_i \end{bmatrix}, \qquad \begin{array}{l} T_i \text{ is } r_i \times (d_i + 1), \\ \Phi_i \text{ is } r_i \times r_i, \end{array}$$

$$b_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \gamma_i \\ \beta_i \end{bmatrix}, \qquad \beta_i = \begin{bmatrix} \beta_{i1} \\ \beta_{i2} \\ \vdots \\ \beta_{ir_i} \end{bmatrix}, \qquad (2.2\text{-}14)$$

$$c_i = [1 \quad 0 \quad \cdots \quad 0],$$

where $r_i = p_i - 1 - d_i$;

3) the p_i vectors, b_i, $Ab_i, \ldots, A^{p_i-1} b_i$ are linearly independent;

4) if p_{m+1} is nonzero and the row vector $\eta = [\eta_1, \ldots, \eta_n]$ is such that $\eta_{p+1}, \ldots, \eta_{p+p_{m+1}}$ are not all zero, where $p = \sum_{i \in \underline{m}} p_i$, then the row vector $\eta(I\lambda - A)^{-1}B$ has at least two nonzero elements.

We note immediately from the form of the matrices in 1) that the transfer function matrix of a c.d. system, $H(\lambda)$, is diagonal with diagonal elements $h_i(\lambda) = \lambda^{-d_i-1}$, $i \in \underline{m}$. Further for $p_{m+2} > 0$ the pair (A_1B) is not completely controllable, nor in general is the pair (A,C) completely observable. Indeed from 2) we see that the subpairs (A_i,c_i) need not be completely observable, however the subpairs (A_i,b_i) are controllable from 3). Property 4) tells us that there is a segment of the state space which is always driven by at least two inputs. Since this subspace cannot be controlled via any one input, it must be unobservable as is seen from 1).

Finally Gilbert demonstrates that those feedback laws (F,G) which do not destroy the decoupled structure of a canonically decoupled system have a relatively simple form.

(2.2.5) <u>Proposition</u>: (Gilbert [G2]) Given a c.d. system (A,B,C) the control law (F,G) will render the closed loop system (A + BF, BG, C) decoupled if and only if

$$F = \begin{bmatrix} \theta_1 & 0 & 0 & \cdots & 0 & 0 & \theta_1^u \\ 0 & \theta_2 & 0 & \cdots & 0 & 0 & \theta_2^u \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \theta_m & 0 & \theta_m^u \end{bmatrix}, \qquad (2.2\text{-}15)$$

where θ_i is $1 \times p_i$ and θ_i^u is $1 \times p_{m+2}$, and

$$G = \mathrm{diag}(\delta_1,\ldots,\delta_m) \qquad \delta_i \neq 0 \qquad i \in \underline{m} \ . \qquad (2.2\text{-}16)$$

2.3 Decoupling with an Excess of Inputs

We now consider an extension of the results presented in the previous section to the case where the linear system may have more inputs than outputs. That is, we are still concerned with systems described by (1.4-2), but now u(j) and y(j) are m and q vectors respectively, with $m \geq q$, $q = k$ and $q_i = 1$ for $i \in \underline{k}$. It is clear that we may relate this problem to that solved by Falb and Wolovich if we ignore $m - q$ inputs and consider the system resulting from the remaining q inputs and the q outputs. In what follows we shall make the correspondence in this manner, and then derive necessary and sufficient conditions for decoupling of this more general problem closely paralleling those of Falb and Wolovich.

First we shall expand our view of a decoupled system.

(2.3.1) Definition: A system (A,B,C) with A $n \times n$, B $n \times m$, and C $q \times n$ real matricies $m \geq q$ is effectively decoupled (e.d.) if the associated transfer function matrix $H(\lambda)$ contains a nonsingular $q \times q$ diagonal submatrix.

We say that such a system is effectively decoupled for the reason that although the transfer function may exhibit interaction, a wise choice of those inputs which are to be inactivated will yield a strictly decoupled input-output structure. Note that this definition permits a broad view of a decoupled system and specifically allows redundancy in the system. That is, it is perfectly permissible for several inputs to control a given output, an often desirable design criterion. Systems with transfer function matrices such as

$$H(\lambda) = \begin{bmatrix} h_1(\lambda) & h_2(\lambda) & 0 & 0 \\ 0 & h_3(\lambda) & h_4(\lambda) & 0 \\ 0 & 0 & 0 & h_5(\lambda) \end{bmatrix} \qquad (2.3\text{-}1a)$$

or

$$H(\lambda) = \begin{bmatrix} h_1(\lambda) & 0 & 0 & 0 \\ 0 & h_2(\lambda) & h_3(\lambda) & 0 \\ 0 & 0 & 0 & h_4(\lambda) \end{bmatrix} \qquad (2.3\text{-}1b)$$

where the $h_i(\lambda)$ are nonzero rational functions are e.d. by Def. 2.3.1.

The transfer function matrix of any e.d. system may be transformed into

$$H(\lambda) = [\bar{H}(\lambda) \ \vdots \ 0] \qquad (2.3\text{-}2)$$

where $\bar{H}_1$ is q x q, diagonal and nonsingular, by the application of an appropriate control law (0,G). If the nonsingular diagonal submatrix of $H(\lambda)$ for an e.d. system consists of the columns $i_1 < \ldots < i_q$, then for G of the form $[G_1 \ \vdots \ 0]$ with G_1 m x q and the j^{th} column of G_1 given by $\bar{e}_{i_j}$ (the i_jth standard unit vector in R^m it follows that $H(\lambda)G$ is of the form (2.3-2).

Now define the quantities d_i and D_i $i \in \underline{m}$ as per (2.2-4) and (2.2-5). It is clear that these are again feedback invariants, $D_i(F,G) = D_iG$, and for G nonsingular $d_i(F,G) = d_i$, $i \in \underline{q}$. However this last requirement is somewhat restrictive and not necessary. Indeed choose if we can a subset of q columns of B (without loss of generality assume $B_1,\ldots,B_q$ and then write $B = [\bar{B}_1 \ \vdots \ \bar{B}_2]$ with $\bar{B}_1$ n x q) such that $C_iA^{d_i}\bar{B}_1 \neq 0$, $i \in \underline{q}$. Then for G of the form

$$G = \begin{pmatrix} G_1 & \vdots & G_2 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & G_3 \end{pmatrix} \qquad (2.3\text{-}3)$$

with G_1 q x q and nonsingular, G_2, and G_3 conformably dimensioned, we have

$$BG = [\bar{B}_1 G_1 \vdots \bar{B}_1 G_2 + \bar{B}_2 G_3]$$

and it follows that $d_i(F,G) = d_i$; $i \in \underline{q}$. Define the q x m matrix $D = [D_1^T; \ldots; D_q^T]^T$.

Borrowing some terminology from Gilbert via Section 2.2, we shall say a system is integrator decoupled (i.d.) if its transfer function matrix has the form of (2.3-2) with the diagonal elements of $\bar{H}(\lambda)$ given by $h_i(\lambda) = \gamma_i \lambda^{-d_i-1}$, $i \in \underline{q}$.

(2.3.2) Lemma: The system (A,B,C) is integrator decoupled if $D = [\Gamma \vdots 0]$ where Γ is q x q diagonal and nonsingular and $C_i A^{d_i+1} = 0$, $i \in \underline{q}$.

Proof: From the definition of D_i and since $C_i A^{d_i+1} = 0$, $i \in \underline{q}$ it follows that the i^{th} row of the transfer function matrix of an i.d. system, $H_i(\lambda)$, is given by (see[G2, Sect. 4])

$$H_i(\lambda) = \psi^{-1}(\lambda)(\lambda^{n-1-d_i} - \psi_1 \lambda^{n-2-d_i} - \ldots - \psi_{n-1-d_i})[\gamma_i \hat{e}_i^T \vdots 0],$$

where $\psi(\lambda)$ is the characteristic equation of A, $\psi(\lambda) = \det(\lambda I - A) = \lambda^n - \psi_1 \lambda^{n-1} - \ldots - \psi_n$, and $\hat{e}_i$ is the i^{th} unit vector of R^q. Using the Cayley-Hamilton theorem together with $C_i A^j B = 0$ for $j \neq d_i$, it follows easily that $H_i(\lambda) = \gamma_i \lambda^{-d_i-1} (\hat{e}_i^T \vdots 0)$.

Now we shall show that given a system (A,B,C), if D contains a q x q nonsingular submatrix R, then there exists a control law (F,G) such that (A + BF, BG, C) is integrator decoupled. To avoid undue notational difficulties we assume that R consists of the first q columns of D. (This condition does not involve a loss of generality as it presumes only a renumbering of the inputs.) Partition B correspondingly into an n x q and an n x m-q block $(\bar{B}_1 \,\vdots\, \bar{B}_2)$.

(2.3.3) Lemma: Consider the system (A,B,C) with $D = (R \,\vdots\, Q)$, R q x q and nonsingular. Let A* denote the q x n matrix whose i^{th} row is given by $C_i A^{d_i+1}$, $i \in \underline{m}$. Then for the control law (F,G) where

$$F = \begin{pmatrix} -R^{-1}A^* \\ 0 \end{pmatrix} \qquad\qquad G = \begin{pmatrix} R^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

the system (A + BF, BG, C) is integrator decoupled.

Proof. From the form of G, it follows from the preceding discussion that $d_i(F,G) = d_i$, $i \in \underline{m}$. Further $D_i(F,G) = D_iG$ which equals

$$(C_iA^{d_i}\bar{B}_1 \,\vdots\, C_iA^{d_i}\bar{B}_2)\begin{pmatrix} R^{-1} & 0 \\ 0 & 0 \end{pmatrix} = (\hat{e}_i^{\,T} \,\vdots\, 0)$$

since $C_iA^{d_i}\bar{B}_1$ is the i^{th} row of R. Thus we have $D(F,G) = (I_q \,\vdots\, 0)$. Continuing,

$$C_i(A+BF)^{d_i+1} = C_iA^{d_i+1} + C_iA^{d_i}BF + \text{terms of the form}$$

$$C_iA^jBF(\cdot) \text{ for } 0 \leq j < d_i \text{ if } d_i > 0.$$

If present, these last terms are clearly zero, whence

$$C_i(A+BF)^{d_i+1} = C_iA^{d_i+1} + C_iA^{d_i}(\bar{B}_1 \vdots \bar{B}_2)\begin{pmatrix} -R^{-1}A^* \\ \\ 0 \end{pmatrix}$$

$$= C_iA^{d_i+1} - C_iA^{d_i}\bar{B}_1R^{-1}A^*$$

$$= C_iA^{d_i+1} - \hat{e}_i^TA^* = 0.$$

Then by the preceding lemma the result follows. ■

(2.3.4) Remark: If we don't assume that the nonsingular submatrix R consists of the first q columns of D, but rather columns $i_1,\ldots,i_q$, then the appropriate control law (F,G) has the following form. Let s_j be the j^{th} row of $-R^{-1}A^*$ and f_j be the j^{th} row of F; then $f_{i_j} = s_j$ $j \in \underline{q}$, zero otherwise. Let r_j be the j^{th} row of R^{-1} and g_j be the j^{th} row of G; then $g_{i_j} = (r_j \vdots 0)$ $j \in \underline{q}$, zero otherwise.

(2.3.5) Theorem: The system represented by the matrix triple (A,B,C) may be effectively decoupled if and only if D contains a nonsingular q x q submatrix.

Proof: (Necessity): Since the transfer function matrix of any e.d. system will be of the form (2.3-2) for some appropriate control law (0,G), we may assume without loss of generality that the i^{th} row of $H(\lambda;F,G)$ is given by

$$H_i(\lambda;F,G) = h_i(\lambda;F,G)(\hat{e}_i^T \vdots 0) \qquad i \in \underline{q}.$$

It then follows from Remark 2.2.2 that $D_i(F,G) = D_iG = (\gamma_i\hat{e}_i^T \vdots 0)$.

Now $\gamma_i \neq 0$ for all $i \in \underline{q}$, otherwise we have $D_j(F, G) = 0$ for some $j \in \underline{q}$ implying that $H_j(\lambda;F,G) = 0$ and contradicting the nonsingularity of $\overline{H}(\lambda;F,G)$. Then $D(F,G) = DG = (\Gamma \vdots 0)$ where Γ is q x q, diagonal and nonsingular, implying that rank D = q, and therefore that D contains a q x q nonsingular submatrix.

(Sufficiency): Follows from Lemma 2.3.3 and Remark 2.3.4. ∎

CHAPTER 3

THE GEOMETRIC APPROACH TO DECOUPLING

3.1 Introduction

The method of attack of Wonham and Morse presents a striking departure from that of Falb and Wolovich, and Gilbert. There is no mention of diagonalizing transfer function matrices, nor do we find conditions based upon the ranks of certain matrices. Instead, in their works ([M6], [M7], [M8], [W7], [W8]), Wonham and Morse have adopted a geometric setting for their decoupling theory, and have changed the question of whether there exists a control law (F,G) that decouples (A,B,C) to whether there exists a suitable set of subspaces dependent upon (A,B,C).

Using this coordinate free approach to decoupling, Wonham and Morse consider a more general problem than that solved by Falb and Wolovich. In the geometric formulation one may consider decomposing a linear system into multi-input, multi-output subsystems rather than just simply into scalar input, scalar output subsystems. That is, we specifically allow $q_i > 1$ for all $i \in \underline{k}$. The decoupling problem considered in Section 2.2 then becomes a solvable special case of this broader formulation.

The geometric method has led quite naturally to a vector subspace interpretation of the problem of decoupling using dynamic compensation. Gilbert [G2] first discussed the possibility of employing additional integrators to decouple systems which could not be decoupled by memoryless state feedback alone. In [M7], Morse and

Wonham consider this issue as an extension of the static feedback decoupling problem and offer a nonconstructive proof.

Fabian and Wonham [F1] have explored the question of generic solvability of the decoupling problem. Given certain conditions on the number of inputs and the number of outputs in each output block, they have established that almost every system (A,B,C) may be decoupled by dynamic compensation. Building upon their work, we will show that the scalar input, scalar output decoupling problem solved by Falb and Wolovich is also generically solvable.

In this chapter we will briefly present the geometric theory of Wonham and Morse, including the results for the solvability of special cases. Making use of a closed form expression for certain (A,B) invariant subspaces, we are then able to show how general (A,B) i.s. may be determined with respect to simpler subspaces. As (A,B) i.s. play a key role in the Wonham-Morse geometric formulation, this result provides us with some additional structure for these elements, and is useful in tying together the geometric and input-output approaches. For completeness, the results on decoupling via dynamic compensation and generic solvability of decoupling are presented. Finally we show an extension of the latter conditions and prove that the problem considered by Falb and Wolovich is generically solvable.

3.2 Decoupling with Memoryless State Feedback

In this section we will exhibit the geometric formulation of the decoupling problem following primarily [M8] and [W7]. We start by introducing the notions of (A,B) invariant and controllability subspaces (c.s.) and then show how the solution to a decoupling problem may be stated in terms of a set of c.s. Lastly, solutions for several cases are given.

Throughout this description of the methodology of Wonham and Morse we should keep in mind their primary thrust — noninteraction. The constructions which will be shown are all intended to generate maximal (with respect to inclusion) subspaces which create this noninteracting structure.

Let us consider a discrete time system represented by the difference equation (1.4-2). (As in the input-output approach we could equivalently consider the continuous time system represented by (1.4-1).) We no longer restrict our attention to cases where $k = q = m$. Given a control law of the form (2.2-2) the resultant system is described by

$$x(j+1) = (A+BF)x(j) + BGv(j), \qquad y(j) = Cx(j). \qquad (3.2\text{-}1)$$

In light of our quest for a noninteracting control law it is quite natural to examine the invariant subspaces of $A + BF$. Given the matrices A,B and F, this is an easy exercise consisting of finding the set of eigenvectors of the matrix $A + BF$. However given only A and B, it becomes a distinctly different problem to determine if there exists a feedback map F such that a given subspace is $(A + BF)$ invariant.

(3.2.1) <u>Definition</u>: A subspace $\mathcal{V}$ is (A,B) invariant if there exists some F such that $(A + BF)\mathcal{V} \subset \mathcal{V}$.

We will designate the space of (A,B) invariant subspaces (i.s.) by

$\mathcal{I}_{A,B}$. It is easily shown [W7, Lemma 3.2], [M8, Lemma 2] that $\mathcal{V}$ is (A,B) invariant if and only if

$$A\mathcal{V} \subset \mathcal{V} + \mathcal{B}.$$

A set of (A,B) i.s. $\{\mathcal{V}_i,\ i \in \underline{k}\}$ is said to be compatible if there exists a single F such that $(A + BF)\mathcal{V}_i \subset \mathcal{V}_i$, $i \in \underline{k}$. Given a compatible set $\{\mathcal{V}_i,\ i \in \underline{k}\}$, it is easily seen that $\mathcal{V}^*$, the space spanned by elements of $\mathcal{V}_i \cap \mathcal{V}_i^*$, for $i \in \underline{k}$, is (A,B) invariant, where $\mathcal{V}_i^* = \sum_{\substack{j \neq i \\ j \in \underline{k}}} \mathcal{V}_j$.

For it follows that

$$(A+BF)(\mathcal{V}_i \cap \mathcal{V}_j) = (A+BF)\mathcal{V}_i \cap (A+BF)\mathcal{V}_j \subset \mathcal{V}_i \cap \mathcal{V}_j$$

for any $i,j \in \underline{k}$ and hence $(A+BF)\mathcal{V}^* \subset \mathcal{V}^*$. However it is not generally true that given $\mathcal{V}^* \in \mathcal{I}_{A,B}$, the set $\{\mathcal{V}_i,\ i \in \underline{k}\}$ is compatible. At most one can say that there exists an F such that $(A+BF)(\mathcal{V}_i + \mathcal{V}^*) \subset \mathcal{V}_i + \mathcal{V}^*$ for all $i \in \underline{k}$. We note that if the $\mathcal{V}_i$'s are mutually independent, then the set $\{\mathcal{V}_i,\ i \in \underline{k}\}$ is trivially compatible.

Given any subspace $\mathcal{N}$, we turn our attention to the set of (A,B) invariant subspaces contained in $\mathcal{N}$. This set is nonempty (it always contains the zero subspace) and since $\mathcal{I}_{A,B}$ is clearly closed under addition it follows that this set contains a maximal element.

(3.2.2) <u>Proposition:</u> (Morse and Wonham [M8]) Given a subspace $\mathcal{N}$ of dimension s, define

$$\mathcal{V}_0 = \mathcal{N}, \quad \mathcal{V}_i = \mathcal{N} \cap A^{-1}(\mathcal{V}_{i-1} + \mathcal{B}), \quad i \in \underline{s}. \qquad (3.2\text{-}2)$$

Then $\mathcal{V}_s$ is the maximal (A,B) i.s. contained in $\mathcal{N}$.

If in the proposition above, dim N = n-1, the algorithm (3.2-2) takes on a very simple form. There exists a nonzero vector $z \in R^n$ such that $N = \mathrm{Ker}(z^T)$, where z^T is a linear form on R^n. Define d as the least nonnegative integer such that $z^T A^d B \neq 0$ (recall (2.2-4)). As (A,B) is assumed controllable, d is well defined. Letting Z denote the subspace spanned by z we have the following result.

(3.2.3) Corollary: (Morse and Wonham [M8]) If $N = \mathrm{Ker}(z^T)$, then the maximal (A,B) i.s. contained in N is given by

$$\overline{V} = (z + A^T z + \ldots + (A^T)^d z) \quad . \qquad (3.2\text{-}3)$$

In a subsequent section we will return to this corollary and show how it is instrumental in developing further structure of invariant subspaces and a connection between the input-output and geometric approaches.

Now given a subspace R, suppose we wish to find a feedback control law (F,G) such that R and no larger space is completely reachable. That is we want

$$R = \{A+BF|BG\} \triangleq BG + (A+BF)BG + \ldots + (A+BF)^{n-1}BG \quad . \qquad (3.2\text{-}4)$$

We note immediately that if (3.2-4) holds then R is (A,B) invariant. Furthermore it may be shown [W7, Lemma 4.1] that the explicit dependence of R on G may be eliminated; if there exists a pair (F,G) such that $R = \{A+BF|BG\}$, then

$$R = \{A+BF|B \cap R\} \qquad (3.2\text{-}5)$$

and conversely.

(3.3.4) <u>Definition</u>: A subspace $\mathcal{R}$ satisfying (3.2-5) is called a controllability subspace (c.s.).

We may further avoid the explicit dependence of $\mathcal{R}$ on the feedback map F indicated in (3.2-5) and develop a recursive construction for a c.s. First we take note of the following fact. If $\mathcal{R}$ is (A,B) invariant and $\mathcal{W} \subset \mathcal{R}$, then

$$(A+BF)\mathcal{W} + \mathcal{B} \cap \mathcal{R} = (A\mathcal{W} + \mathcal{B}) \cap \mathcal{R}$$

for any F such that $(A+BF)\mathcal{R} \subset \mathcal{R}$ [W7, Lemma 4.2]. This seemingly obscure identity allows us to transform the summative construction of (3.2-5) into a recursion independent of F. Indeed defining

$$\mathcal{R}^i = \mathcal{B} \cap \mathcal{R} + \ldots + (A+BF)^{i-1}\mathcal{B} \cap \mathcal{R} = (A+BF)\mathcal{R}^{i-1} + \mathcal{B} \cap \mathcal{R}, \quad i \in \underline{n}$$

and

$$\mathcal{S}^0 = 0, \quad \mathcal{S}_i = (A\mathcal{S}^{i-1} + \mathcal{B}) \cap \mathcal{R}, \quad i \in \underline{n} \qquad (3.2\text{-}6)$$

it follows that $\mathcal{S}^i = \mathcal{R}^i$, $i \in \underline{n}$ and

$$\mathcal{S}^n = \{A+BF|\mathcal{B} \cap \mathcal{R}\} \qquad (3.2\text{-}7)$$

for any F such that $(A+BF)\mathcal{R} \subset \mathcal{R}$. Clearly then $\mathcal{R}$ is a c.s. if and only if $\mathcal{R} = \mathcal{S}^n$. If $\mathcal{R}$ is not a c.s., then since $\mathcal{R} \in \mathcal{I}_{A,B}$, it follows from (3.2-7) that $\mathcal{S}^n$ is the largest c.s. contained in $\mathcal{R}$. This leads us to a characterization of maximal c.s. from [W7].

(3.2.5) <u>Proposition</u>: (Wonham and Morse) Let $\mathcal{N}$ be a given subspace and $\overline{\mathcal{V}}$ the maximal (A,B) i.s. contained in $\mathcal{N}$. Then the maximal c.s. contained in $\mathcal{N}$ is given by

$$\overline{\mathcal{R}} = \{A + BF | \mathcal{B} \cap \overline{\mathcal{V}}\} \qquad (3.2\text{-}8)$$

for any F such that $(A+BF)\overline{\mathcal{V}} \subset \overline{\mathcal{V}}$.

As noted in [M8], Prop. 3.2.5 provides a dual method of determining the maximal c.s. contained in a given subspace $\mathcal{N}$. We may compute $\overline{\mathcal{V}}$ from Prop. 3.2.2 and then use the algorithm (3.2-6). Alternatively, having $\overline{\mathcal{V}}$, we may determine an F such that $(A+BF)\overline{\mathcal{V}} \subset \overline{\mathcal{V}}$, and then use (3.2-8).

For linear systems represented by the matrix triple (A,B,C) it is well known [B6] that controllability of (A,B) implies arbitrary pole assignment via feedback. As a c.s. is a reachable subspace, we might expect some sort of pole assignment to hold on each c.s. Indeed this is the case.

(3.2.6) Proposition: (Wonham and Morse [W7, Theorem 4.2]) Let $\mathcal{R}$ be a c.s. and $\alpha(\lambda)$ a monic polynomial such that deg $\alpha(\lambda)$ = dim $\mathcal{R}$. There exists an F such that $(A+BF)\mathcal{R} \subset \mathcal{R}$ and the characteristic polynomial of $A + BF|\mathcal{R}$ is $\alpha(\lambda)$. Further, if $0 \neq b \in \mathcal{B} \cap \mathcal{R}$ is arbitrary, F may be chosen so that b generates $\mathcal{R}$, that is $\mathcal{R} = \{A+BF|b\}$.
An additional interpretation of c.s. in terms of open loop system properties is given in [M8]. In Chapter 5 a characterization in terms of polynomial matrices will be developed.

Now let us see how these constructions of (A,B) i.s. and c.s. relate to the decoupling problem. Assume that the ouput y(j) of the system (1.4-2) consists of k subvectors

$$y_i(j) = C_i x(j), \quad i \in \underline{k}$$

where y_i is a q_i vector and the q x n matrix C is partitioned $C = (C_1^T;\dots;C_k^T)^T$, with all the C_i's nonzero. A feedback control law of the form

$$u(j) = Fx(j) + \sum_{i\varepsilon \underline{k}} G_i v_i(j) \qquad (3.2\text{-}9)$$

(which is equivalent to (2.2-2) by letting $G = (G_i;\dots;G_k)$) where the v_i's are a set of external input subvectors, results in the closed loop linear system

$$x(j+1) = (A+BF)x(j) + B\sum_{i\varepsilon \underline{k}} G_i v_i(j), \quad y_i(j) = C_i x(j),\ i \varepsilon \underline{k}\ . \qquad (3.2\text{-}10)$$

The control law (3.2-9) decouples the system (3.2-10) if the input v_i controls output y_i and affects no other outputs for all $i \varepsilon \underline{k}$. More formally let $\mathcal{R}_i$ be the c.s. consisting of the states reachable from v_i,

$$\mathcal{R}_i = \{A+BF|B\mathcal{G}_i\}\ , \qquad i \varepsilon \underline{k}\ . \qquad (3.2\text{-}11)$$

Since y_i is to be controlled via v_i, we must have

$$C_i\mathcal{R}_i = \mathcal{C}_i\ , \qquad i \varepsilon \underline{k} \qquad (3.2\text{-}12)$$

and for noninteraction

$$C_j\mathcal{R}_i = 0, \qquad i \neq j, \text{ and } i,j \varepsilon \underline{k}\ . \qquad (3.2\text{-}13)$$

Denoting Ker C_i by $\mathcal{N}_i$ we may state the decoupling problem in terms of a set of controllability subspaces [W7]: Given A,B, and $\mathcal{N}_i$, $i \varepsilon \underline{k}$, find conditions for the existence of a feedback map F and a set of c.s. $\{R_i, i \in \underline{k}\}$ such that

$$R_i = \{A+BF|B \cap R_i\} , \qquad i \in \underline{k} ; \tag{3.2-14}$$

$$R_i + N_i = R^n, \qquad i \in \underline{k} ; \tag{3.2-15}$$

$$R_i \subset K_i \triangleq \bigcap_{\substack{j \neq i \\ j \in \underline{k}}} N_j , \qquad i \in \underline{k} . \tag{3.2-16}$$

We note that conditions (3.2-14)-(3.2-16) are equivalent to (3.2-11)-(3.2-13) respectively, and that a set of c.s. satisfying the former constitutes a solution to the decoupling problem. Condition (3.2-14) guarantees that set $\{R_i, i \in \underline{k}\}$ is compatible, while (3.2-15) insures that each R_i is large enough to completely control output y_i, and (3.2-16) assures that R_i is small enough so that it has no effect on other ouputs.

Given A,B and N_i, $i \in \underline{k}$, we may compute $\overline{R}_i$, the maximal c.s. contained in K_i, $i \in \underline{k}$ by Prop. 3.2.5, guaranteeing noninteraction by construction. It then follows that (3.2-15) is true only if

$$\overline{R}_i + N_i = R^n, \qquad i \in \underline{k} . \tag{3.2-17}$$

Furthermore it may be shown (see [M8] that the set of maximal c.s., $\{\overline{R}_i, i \in \underline{k}\}$, is compatible if and only if

$$A \overline{R}* \subset B + \overline{R}* \tag{3.2-18}$$

Thus (3.2-17) and (3.2-18) if true, are sufficient to yield a solution to the decoupling problem, $\{\overline{R}_i, i \in \underline{k}\}$. However, (3.2-18) is not necessary. That is, while the set of maximal c.s. may not be compatible, there may exist a set of smaller c.s., sufficiently large to satisfy (3.2-15), but compatible.

Unfortunately there is no systematic procedure to generate non-maximal solutions to the decoupling problem. Thus the geometric method of Wonham and Morse yields a complete answer to the decoupling problem only for those cases in which the set of maximal set of c.s., $\{\bar{R}_i,\ i \in \underline{k}\}$, provides a solution. At this point a simple example from [M8] may help to clarify this point. Consider the system defined by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, C_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T, C_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}^T.$$

Letting e_i, $i \in \underline{5}$ represents the i^{th} standard unit vector, direct calculation yields $\bar{R}_1 = \text{Span}\{e_1, e_2, e_5\}$ and $\bar{R}_2 = \text{Span}\{e_3, e_4, e_5\}$. Thus $\bar{R}^* = \text{Span}\{e_5\}$ and (3.2-18) fails. However it is easily verified that the set of c.s. $R_1 = \bar{R}_1$, $R_2 = \text{Span}\{e_3,\ e_4 + \alpha e_5\}$ for some scalar $\alpha \neq -1$ constitutes a non-maximal solution to this problem.

Let us now turn our attention to several special cases for which the question of existence of a decoupling feedback law is resolved.

(3.2.7) <u>Proposition</u>: (Wonham and Morse [W7, Theorem 6.1]) If rank $C = n$, a solution to the decoupling problem exists if and only if (3.2-17) holds.

The proof of Prop. 3.2.7 is established by noting that rank C = n is equivalent to $\bigcap_{i \varepsilon \underline{k}} N_i = 0$, whence $\overline{R}^* = 0$ and the set of maximal c.s. is compatible.

Another special case which lends itself to complete solution is when rank B = k, that is when the number of inputs equals the number of output subvectors. In this case it is clear that maximal c.s. must be singley generated, i.e. $\dim(B \cap R_i) = 1$ for all $i \varepsilon \underline{k}$.

(3.2.8) Proposition: (Morse and Wonham [M8, Theorem 8]) If rank B = k, a solution to the decoupling problem exists if and only if

$$B = \sum_{i \varepsilon \underline{k}} B \cap \overline{R}_i \ . \tag{3.2-19}$$

Furthermore, if (3.2-19) holds then $\{\overline{R}_i, i \varepsilon \underline{k}\}$ is the only solution.

Proof: (Sufficiency): Although a proof is exhibited in [M8] it is somewhat brief. For this reason we offer here an expanded version of that construction. To prove compatibility of the set $\{\overline{R}_i, i \varepsilon \underline{k}\}$ we need only establish (3.2-18). Dropping the superbar briefly to simplify notation, we have

$$A\,R^* = A\left(\bigcap_{i \varepsilon \underline{k}} R_i^*\right) \subset \bigcap_{i \varepsilon \underline{k}} A\,R_i^* \subset \bigcap_{i \varepsilon \underline{k}} (B + R_i^*) \ .$$

Now consider $(B \cap R_1^*) \cap (B \cap R_2^*)$. Since

$$B \cap (R_1^* + R_2^* + B) = B \cap R_1^* + B \cap (R_2^* + B)$$

it follows by Lemmas A.1 and A.2 (see Appendix) that

$$(B + R_1^*) \cap (B + R_2^*) = B + R_1^* \cap (B + R_2^*) \qquad (3.2\text{-}20)$$

(let $B = X$, $R_1^* = Y$ and $B + R_2^* = Z$). Again from Lemma A.2, since

$$B \cap (R_1^* + R_2^*) = B \cap R_1^* + B \cap R_2^* = B$$

by assumption, we have

$$R_1^* \cap (B + R_2^*) = R_1^* \cap B + R_1^* \cap R_2^* .$$

Combining with (3.2-20) yields

$$(B + R_1^*) \cap (B + R_2^*) = B + R_1^* \cap R_2^* . \qquad (3.2\text{-}21)$$

Having demonstrated (3.2-21) let us proceed by induction. Assume for some $j \in \underline{k}$, $j \geq 2$

$$\bigcap_{i<j} (B + R_i^*) = B + \bigcap_{i<j} R_i^*$$

and consider

$$\bigcap_{i\leq j} (B + R_i^*) = (B + \bigcap_{i<j} R_i^*) \cap (B + R_j^*) .$$

Since

$$B \cap (\bigcap_{i<j} R_i^* + B + R_j^*) = B$$

it again follows from Lemmas A.1 and A.2 that

$$(B + \bigcap_{i<j} R_i^*) \cap (B + R_j^*) = B + (B + \bigcap_{i<j} R_i^*) \cap R_j^* \qquad (3.2\text{-}22)$$

(let $B = X$, $R_j^* = Y$, and $B + \bigcap_{i<j} R_i^* = Z$). As $R_j \subset \bigcap_{i<j} R_i^*$ and

$$B \subset \sum_{i\in\underline{k}} B \cap R_i \subset B \cap (\sum_{i\in\underline{k}} R_i) \subset B \cap (R_j^* + \bigcap_{i<j} R_i^*)$$

by assumption, another application of Lemma A.2 yields

$$(\mathcal{B} + \bigcap_{i<j} \mathcal{R}_i^*) \cap \mathcal{R}_j^* = \mathcal{B} \cap \mathcal{R}_j^* + \bigcap_{i\leq j} \mathcal{R}_i^* .$$

Combining this with (3.2-22) gives

$$\bigcap_{i\leq j} (\mathcal{B} + \mathcal{R}_i^*) = \mathcal{B} + \bigcap_{i\leq j} \mathcal{R}_i^*$$

and establishes the induction. Thus we have shown that (3.2-19) implies

$$\bigcap_{i\epsilon \underline{k}} (\mathcal{B} + \mathcal{R}_i^*) = \mathcal{B} + \bigcap_{i\epsilon \underline{k}} \mathcal{R}_i^* = \mathcal{B} + \mathcal{R}^*$$

and thus $A\,\mathcal{R}^* \subset \mathcal{B} + \mathcal{R}^*$, establishing the compatibility of the set $\{\mathcal{R}_i,\ i \ \epsilon\ \underline{k}\}$.

To finish the proof of sufficiency we must establish (3.2-17). From the assumed controllability of the pair (A,B)

$$\sum_{i\epsilon \underline{k}} \mathcal{R}_i = \sum_{i\epsilon \underline{k}} \{A + BF|\mathcal{B} \cap \mathcal{R}_i\} = \{A + BF| \sum_{i\epsilon \underline{k}} \mathcal{B} \cap \mathcal{R}_i\} = \{A + BF|\mathcal{B}\} = R^n$$

for any F such that $(A + BF)\mathcal{R}_i \subset \mathcal{R}_i,\ i \ \epsilon\ \underline{k}$. Thus

$$R^n = \sum_{i\epsilon \underline{k}} \mathcal{R}_i = \mathcal{R}_i + \mathcal{R}_i^* \subset \mathcal{R}_i + \mathcal{N}_i \subset R^n$$

proving the desired result.

(Necessity): It may be shown that given a solution set $\{\overline{\mathcal{R}}_i,\ i \ \epsilon\ \underline{k}\}$ $\dim \mathcal{B} \cap \overline{\mathcal{R}}_i = 1$ for all $i \ \epsilon\ \underline{k}$. From this fact necessity and uniqueness follows. See [W7] for more detailed proofs. ☐

Morse and Wonham present existence results for one additional case. If we constrain rank G = m, then

$$\mathcal{B} = B\mathcal{G} = B \sum_{i\varepsilon \underline{k}} \mathcal{G}_i \subset \sum_{i\varepsilon \underline{k}} \mathcal{B} \cap \mathcal{R}_i \subset \mathcal{B}. \qquad (3.2\text{-}23)$$

(3.2.9) Proposition: (Morse and Wonham [M8, Theorem 7]) If rank G = m, then a solution to the decoupling problem exists if and only if (3.2-19) holds.

Sufficiency is proved exactly as for Prop. 3.2.8 and necessity follows from (3.2-23) and the maximality of the set $\{\overline{\mathcal{R}}_i, i \varepsilon \underline{k}\}$.

In general we will be able to arbitrarily assign some poles without destroying the decoupled structure of a system. Roughly speaking, the spectrum of A + BF may vary everywhere except where c.s. intersect. More formally, given a set of c.s. $\{\mathcal{R}_i, i \varepsilon \underline{k}\}$, construct the radical $\mathcal{R}^*$. Then for each $i \varepsilon \underline{k}$, let $\hat{\mathcal{R}}_i$ be any subspace such that

$$\mathcal{R}_i = \hat{\mathcal{R}}_i \oplus (\mathcal{R}_i \cap \mathcal{R}^*).$$

Define P_i to be projection on $\hat{\mathcal{R}}_i$ along $\mathcal{R}^* + \sum_{\substack{j\varepsilon \underline{k} \\ j\neq i}} \hat{\mathcal{R}}_j$, $i \varepsilon \underline{k}$ and P_0 to be the projection on $\mathcal{R}^*$ along $\sum_{i\varepsilon \underline{k}} \hat{\mathcal{R}}_i$.

For some fixed F such that $(A + BF)\mathcal{R}_i \subset \mathcal{R}_i$, $i \varepsilon \underline{k}$, let Λ_i denote the spectrum of $P_i(A + BF)|\hat{\mathcal{R}}_i$, $i \varepsilon \underline{k}$ and Λ_0 the spectrum of $P_0(A + BF)|\mathcal{R}^*$. Define $n_0 = \dim \mathcal{R}^*$, $n_i = \dim \hat{\mathcal{R}}_i$, $i \varepsilon \underline{k}$. Then we have

(3.2.10) Proposition: (Wonham and Morse [W7, Theorem 7.2]) The set Λ_0 and the integers n_0, and n_i, $i \varepsilon \underline{k}$ are fixed for all F such that $(A + BF)\mathcal{R}_i \subset \mathcal{R}_i$, $i \varepsilon \underline{k}$. The sets Λ_i, $i \varepsilon \underline{k}$ may be freely assigned subject only to the requirement of conjugate symmetry.

3.3 Maximal (A,B) Invariant Subspaces

In the previous section a closed form expression for maximal (A,B) i.s. contained in the kernel of a linear form was shown (Corollary 3.2.3). In addition this result provides a first link between the geometric approach and the work of Falb and Wolovich. We shall now go one step further and demonstrate that arbitrary (A,B) i.s. may, under certain circumstances, be decomposable into i.s. of the form (3.2-3).

In general we could not expect the intersection of (A,B) i.s. to be (A,B) invariant. For example, given

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

it is easily demonstrated that $V_1 = \text{Span}\{(1\ 0\ -1)^T, (0\ 1\ 0)^T\}$ and $V_2 = \text{Span}\{(0\ 1\ 1)^T, (1\ 1\ 0)^T\}$ are both (A,B) i.s. However $V_1 \cap V_2 = \text{Span}\{(-1\ 0\ 1)^T\}$ which is clearly not (A,B) invariant. The difficulty, of course, is one of compatibility. If a set $\{V_i, i \in \underline{t}\}$ of (A,B) i.s. is compatible, then $\bigcap_{i \in \underline{t}} V_i$ will be (A,B) invariant. We shall show in this section that given an (A,B) i.s. V, it is often possible to find a set of (A,B) i.s., $\{V_i, i \in \underline{t}\}$ each of the form of (2.2-3) such that $V = \bigcap_{i \in \underline{t}} V_i$.

First we put Corollary 3.2.3 in an alternative form for our purposes. If N is a subspace of dimension n-1, i.e. $N = \text{Ker } H$ for some linear form H, then from (3.2-3) the maximal (A,B) i.s. contained in $\mathcal{N}$ is given by

$$\overline{\mathcal{V}} = (\mathcal{H}^T + A^T\mathcal{H}^T + \dots + (A^T)^d\mathcal{H}^T)^\perp \qquad (3.3\text{-}1)$$

where $\mathcal{H}^T$ is the image of H^T, and d is the least nonnegative integer such that $H A^d B \neq 0$. Since $(\mathcal{H}^T)^\perp = \mathcal{N}$, $((A^T)^j\mathcal{H}^T)^\perp = A^{-j}\mathcal{N}$, and $(\mathcal{X} + \mathcal{Y})^\perp = \mathcal{X}^\perp \cap \mathcal{Y}^\perp$, it follows immediately that the max(A,B) i.s. contained in $\mathcal{N}$ is alternatively written

$$\overline{\mathcal{V}} = \bigcap_{j=0}^{d} A^{-j}\mathcal{N} \,. \qquad (3.3\text{-}2)$$

Now we move on to the general case. Let $\mathcal{N}$ be an arbitrary subspace of R^n, and let $H_1^T, \dots, H_t^T$ be any basis for the orthogonal complement of $\mathcal{N}$. If H is the t x n matrix whose i^{th} row is given by H_i, $i \in \underline{t}$, then $\mathcal{N} = \text{Ker } H$. Interpreting H_i as a linear form on R^n, we may define $\mathcal{N}_i = \text{Ker } H_i$, $i \in \underline{t}$. We note immediately that $\dim \mathcal{N}_i = n - 1$, $i \in \underline{t}$, $\dim \mathcal{N} = n - t$, and that

$$\mathcal{N} = \bigcap_{i \in \underline{t}} \mathcal{N}_i \,.$$

Resorting to basic definitions, x is an element of the maximal (A,B) i.s. contained in $\mathcal{N}$ if and only if for some feedback map F

$$(A + BF)^j x \in \mathcal{N}_i \qquad i \in \underline{t}, \quad j = 0,1,\dots \qquad (3.3\text{-}3)$$

or equivalently

$$H_i(A + BF)^j x = 0 \qquad i \in \underline{t}, \quad j = 0,1,\dots \qquad (3.3\text{-}4)$$

Defining the feedback invariant d_i for each row vector H_i, $i \varepsilon \underline{t}$, as per (2.2-4), it follows that (3.3-4) is equivalent to

$$H_i A^j x = 0, \quad \text{for } j \leq d_i, \quad i \varepsilon \underline{t} \tag{3.3-5a}$$

$$H_i (A + BF)^j x = 0, \quad \text{for } j > d_i, \quad i \varepsilon \underline{t} . \tag{3.3-5b}$$

We find it useful to write (3.3-5a) in the complemented form

$$x \; \varepsilon \; \bigcap_{i \varepsilon \underline{t}} \left(\bigcap_{j=0}^{d_i} (A^{-j} N_i) \right) \triangleq \overline{V} . \tag{3.3-6}$$

By the definition of N_i and d_i, $i \varepsilon \underline{t}$ it is clear that

$$B \subset \bigcap_{i \varepsilon \underline{t}} \left(\bigcap_{j=0}^{d_i - 1} (A^{-j} N_i) \right) . \tag{3.3-7}$$

Furthermore from (3.3-6)

$$Ax \; \varepsilon \; \bigcap_{i \varepsilon \underline{t}} \left(\bigcap_{j=0}^{d_i - 1} (A^{-j} N_i) \right) \tag{3.3-8}$$

where $\bigcap_{j=0}^{d_i - 1} (A^{-j} N_i) \triangleq R^n$ if $d_i = 0$.

Now for each $i \varepsilon \underline{t}$, since H_i is of rank 1, $H_i A^{d_i} B \neq 0$ if and only if

$$A^{-d_i} N_i + B = R^n, \quad i \varepsilon \underline{t}.$$

Assume that the row vectors $H_i A^{d_i} B$, $i \varepsilon \underline{t}$ are mutually independent and consider the subspace

$$M = \text{Span}\{ (A^T)^{d_i} H_1^T, \ldots, (A^T)^{d_k} H_t^T \} .$$

If $x \in M$, then $x = \sum_{i \in \underline{t}} \alpha_i (A^T)^{d_i} H_i^T$. If x is additionally an element of Ker B^T, then $\sum_{i \in \underline{t}} \alpha_i B^T (A^T)^{d_i} H_i^T = 0$ contradicting the independence of the row vectors $H_i A^{d_i} B$, $i \in \underline{t}$ unless the scalars α_i, $i \in \underline{t}$ are identically zero. Thus it follows that

$$M \cap \text{Ker } B^T = 0 ,$$

or equivalently by complementation

$$\bigcap_{i \in \underline{t}} (A^{-d_i} N_i) + B = R^n . \qquad (3.3\text{-}9)$$

By (3.3-9) we may write $A x = b + w$ with $b \in B$ and $w \in \bigcap_{i \in \underline{t}} A^{-d_i} N_i$. From (3.3-7) and (3.3-8) it follows that

$$w \in \bigcap_{i \in \underline{t}} \left(\bigcap_{j=0}^{d_i} (A^{-j} N_i) \right) = \overline{V} ,$$

and hence $\overline{V}$ is the maximal (A,B) i.s. contained in N. For choosing a basis $x = x_1, \ldots x_s$ for $\overline{V}$, with $A x_i = b_i + w_i$, $i \in \underline{s}$ by (3.3-9), we may construct a map F such that $F x_i = - u_i$ where $b_i = Bu_i$ for some unique u_i, $i \in \underline{s}$. Hence (3.3-5b) follows for all $x \in \overline{V}$, and we have proven the following result.

(3.3.1) <u>Proposition</u>: Given a subspace N of dimension, $n - t$, let H_i^T, $i \in \underline{t}$ be a basis for $N^{\perp}$, and define $N_i = \text{Ker } H_i$ with V_i the maximal (A,B) i.s. contained in N_i, $i \in \underline{t}$. For each H_i, $i \in \underline{t}$, let

d_i be the feedback invariant defined by (2.2-4). If the row vectors $H_i A^{d_i} B$ are mutually independent the maximal (A,B) i.s. contained in $\mathcal{N}$ is given by $\overline{\mathcal{V}} = \bigcap_{i \in \underline{t}} \mathcal{V}_i$.

Proposition 3.3.2 allows the reduction of the problem of finding maximal (A,B) i.s. contained in general subspaces to the special case of Corollary 3.2.3 when certain conditions are satisfied. We note however that these conditions are not necessary for the desired result to hold. Indeed let t = 2, and assume $H_1 \neq H_2$, but $H_1 A^{d_1} = H_2 A^{d_2} \neq 0$. Then $\mathrm{Span}\{(A^T)^{d_i} H_i^T\} \cap \mathrm{Ker}\, B^T = 0$, $i \in \underline{2}$ implies $\mathrm{Span}\{(A^T)^{d_1} H_1^T, (A^T)^{d_2} H_2^T\} \cap \mathrm{Ker}\, B^T = 0$, whence (3.3-9) holds and the remainder of the arguments leading to the proposition follow.

It is clear that the sum of (A,B) i.s. is again (A,B) invariant, hence $\mathcal{I}_{A,B}$ forms a join semi-lattice. Prop. 3.3.1 indicates that under certain conditions the intersection of (A,B) i.s. will be (A,B) invariant. Although the proposition is stated for maximal (A,B) i.s., since any subspace is maximal relative to itself, its applicability is general. For example let $\mathcal{V}$ and $\mathcal{W}$ be (A,B) i.s. with $\{H_1,\ldots H_s\}$ and $\{H_{s+1},\ldots H_t\}$ bases for $\mathcal{V}^\perp$ and $\mathcal{W}^\perp$ respectively. For $d_i = d_i(H_i)$ $i \in \underline{t}$, if the row vectors $H_i A^{d_i} B$, $i \in \underline{t}$ are independent, then $\mathcal{V} \cap \mathcal{W}$ is also (A,B) invariant.

We must however be careful about hastily applying Prop. 3.3.1 as the result is dependent upon fixed maps A and B, and the relatively arbitrary linear forms H_i, $i \in \underline{t}$. That is given A,B, and $\mathcal{N}$, we may choose a basis $\{H_i^T, i \in \underline{t}\}$ of $N^{\perp}$ and construct the set of row vectors $\{H_iA^{d_i}B, i \in \underline{t}\}$. Yet the independence of this latter set is not an invariant property of A,B and N. Given two different bases of $N^{\perp}$, $\{H_i^T, i \in \underline{t}\}$ and $\{J_i^T, i \in \underline{t}\}$, it is entirely possible that the set $\{H_iA^{d(H_i)}B, i \in \underline{t}\}$ is independent, while the set $\{J_iA^{d(J_i)}B, i \in \underline{t}\}$ is not.

For example consider (A,B) given by

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

with $N = \text{Span}\{(0\ 0\ 1)^T\}$. Choosing the basis

$$H_1^T = (1\ 0\ 0)^T, \qquad H_2^T = (0\ 1\ 0)^T$$

for $N^{\perp}$, we have $d(H_1) = 0$, $d(H_2) = 1$, with H_1B and H_2AB independent. Now the maximal (A,B) i.s. contained in Ker H_1, Ker H_2 is given by $V(H_1) = \text{Span}\{(0\ 1\ 0)^T, (0\ 0\ 1)^T\}$, $V(H_2) = \text{Span}\{(1\ 0\ 0)^T\}$ respectively. Therefore by the proposition, the maximal (A,B) i.s. in N is $V(H_1) \cap V(H_2) = 0$.

If we were to choose the basis

$$J_1^T = (1\ 0\ 0) \qquad J_2^T = (1\ 1\ 0\ )$$

for $N^{\perp}$, then we would find that $d(J_1) = d(J_2) = 0$ and $J_1B = J_2B$. In this case the maximal (A,B) i.s. contained in N is not given by

$V(J_1) \cap V(J_2) = \text{Span}\{(0\ 1\ 0)^T,\ (0\ 0\ 1)^T\} \cap \text{Span}\{(1\ \ -1\ \ 0)^T,\ (0\ 0\ 1)^T\}$ $= \text{Span}\ \{(0\ 0\ 1)^T\}$. In fact $V(J_1) \cap V(J_2)$ is not even (A,B) invariant.

3.4 Decoupling with Dynamic Compensation

The use of dynamic compensation to decouple where memoryless feedback alone would not work was proposed by Gilbert [G2] in an example. However Gilbert developed no substantive theory for dealing with this more general problem. By considering dynamic compensation as an extension of the state space, Morse and Wonham [M7] were able to formulate this problem in a manner similar to that of decoupling by state feedback alone. With the freedom of this state space extension, complete necessary and sufficient conditions for the existence of decoupling control laws were derived.

Although our primary interest is decoupling by state feedback alone, we will give a brief overview of the results of Morse and Wonham for the sake of completeness, following [M8, Section 5]. In addition, this problem has motivated the work by Fabian and Wonham [F1] on the generic solvability of decoupling problems, which we present and extend in the following section.

Consider the discrete time system represented by (1.4-2) and adjoin to it the set of $\tilde{n}$ simple delays

$$\tilde{x}(j+1) = \tilde{u}(j)$$

where $\tilde{x}$ and $\tilde{u}$ are real $\tilde{n}$ vectors, $\tilde{n}$ to be determined. Letting $\bar{x}$ and $\bar{u}$ denote the extended state and input respectively,

$\bar{x} = \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$, $\bar{u} = \begin{bmatrix} u \\ \tilde{u} \end{bmatrix}$, we have the new system

$$\bar{x}(j+1) = \bar{A}\,\bar{x}(j) + \bar{B}\,\bar{u}(j), \quad y(j) = \bar{C}\,\bar{x}(j) \qquad (3.4\text{-}1)$$

where $\bar{A}$ is $(n+\tilde{n})\times(n+\tilde{n})$, $\bar{B}$ is $(n+\tilde{n})\times(m+\tilde{n})$, $\bar{C}$ is $(q+\tilde{n})\times(n+\tilde{n})$

$$\bar{A} = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right), \qquad \bar{B} = \left(\begin{array}{c|c} B & 0 \\ \hline 0 & I \end{array}\right), \qquad \bar{C} = (C \quad 0).$$

We define the projection P to be on R^n along $R^{\tilde{n}}$, i.e. $P\,\bar{x} = \begin{bmatrix} x \\ 0 \end{bmatrix}$.

As in the formulation of the memoryless feedback decoupling problem (Section 3.2) we allow control laws of the form

$$\bar{u}(j) = F\,\bar{x}(j) + \sum_{i\in\underline{k}} G_i v_i(j)$$

where now F is a mapping from $R^{n+\tilde{n}}$ to $R^{m+\tilde{n}}$, and the v_i's are a set of external input subvectors with associated gains G_i, $i \in \underline{k}$. Using this control law, (3.4-1) becomes

$$\bar{x}(j+1) = (\bar{A} + \bar{B}\,F)\bar{x}(j) + \sum_{i\in\underline{k}} \bar{B}\,G_i v_i(j), \quad y(j) = \bar{C}\,\bar{x}(j).$$

For a given decomposition of the outputs $y_i(j) = C_i x(j)$, $i \in \underline{k}$ of the original system, the suboutputs of the extended system are given by $y_i(j) = \bar{C}_i \bar{x}(j)$, $i \in \underline{k}$, where clearly $\bar{C}_i = (C_i \,\vdots\, 0)$ and $\text{Ker}\,\bar{C}_i = \text{Ker}\,C_i \oplus R^{\tilde{n}}$. Letting N_i denote $\text{Ker}\,C_i$, $i \in \underline{k}$, the extended decoupling problem may be posed as follows: Given A,B, and N_i, $i \in \underline{k}$, find conditions for the existence of an integer $\tilde{n}$, a feedback map F and a set of c.s., $\{S_i, i \in \underline{k}\}$, of $(\bar{A},\bar{B})$ such that

$$\mathcal{S}_i = \{\overline{A} + \overline{B}\,F|\overline{\mathcal{B}} \cap \mathcal{S}_i\}, \qquad i \in \underline{k};$$

$$\mathcal{S}_i + \mathcal{N}_i + R^{\tilde{n}} = R^{n+\tilde{n}}, \qquad i \in \underline{k};$$

$$\mathcal{S}_i \subset R^{\tilde{n}} + \bigcap_{\substack{j \neq i \\ i \varepsilon \underline{k}}} \mathcal{N}_j, \qquad i \in \underline{k}.$$

We note immediately that this formulation differs from that of the memoryless feedback case (3.2-14 through 3.2-16) only in the inclusion of the free variable $\tilde{n}$.

(3.4.1) Theorem: (Morse and Wonham [M7, Theorem 1.1]) The linear system (A,B,C) may be decoupled by dynamic compensation if and only if (3.2-17) holds, i.e.

$$\overline{\mathcal{R}}_i + \mathcal{N}_i = R^n \qquad \text{for } i \in \underline{k}.$$

The proof of Theorem 3.4.1 hinges on the key fact that if $\mathcal{S}_i$ is a c.s. of $(\overline{A},\overline{B})$ then $P\mathcal{S}_i$ is a c.s. of (A,B), and conversely if $P\mathcal{S}_i$ is a c.s. of (A,B), then $\mathcal{S}_i$ is a c.s. of $(\overline{A},\overline{B})$. Using this fact, Morse and Wonham show that given any set of c.s. of (A,B),$\{\mathcal{R}_i,\ i \in \underline{k}\}$, it is always possible to find an extension $R^{\tilde{n}}$ and an independent set of extended c.s., $\{\mathcal{S}_i,\ i \in k\}$, with $P\mathcal{S}_i = \mathcal{R}_i$ for $i \in \underline{k}$. With the problem of compatibility easily dispensed, it remains only to assure output controllability. Roughly speaking, the solution method is based on finding an $\tilde{n}$ sufficiently large ($\tilde{n}$ need never exceed $\sum_{i \varepsilon \underline{k}} \dim \overline{\mathcal{R}}_i$) such that each c.s. $\overline{\mathcal{R}}_i$, $i \in \underline{k}$ may be separately injected into the extended state space $R^{n+\tilde{n}}$ eliminating in this extended space any nonzero intersections which occur in R^n.

However as Morse and Wonham [M7] have shown, it is not generally necessary to have an extension large enough to insure independence of the c.s. of $(\overline{A},\overline{B})$. Solution of the decoupling problem requires only compatibility of the c.s., hence it may be possible to reduce the order of the feedback compensator and still decouple, a most desirable design objective. With $\tilde{n}$ large enough, independence of the c.s. of $(\overline{A},\overline{B})$ and hence complete pole placement is assured. However for a minimal state extension S^* may be nonzero, hence all poles may not be arbitrarily assigned, and a situation described in Prop. 3.2.10 holds.

3.5 Generic Solvability

The problems of decoupling by memoryless feedback or dynamic compensation as formulated in Sections 3.2 and 3.4 respectively, have been reduced to a set of mathematical conditions which must be satisfied to assure the existence of a solution. However these conditions themselves imply nothing about the abundance or dearth of decoupleable systems. The utility of the theory of decoupling is intimately tied to its applicability. Just as the ideas of controllability and observability are powerful results at least partly due to the fact that controllable and observable systems are dense in the Euclidean topology, we might hope that decoupleable systems are typical rather than exceptions or singularities.

In Section 3.2 an example of a decoupleable system, albeit one where the constructions of maximal c.s. were of little avail, was shown. There is no apriori reason however to believe that system is any more typical than one represented by

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}^T, \quad C_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T.$$

It may be readily checked that for this latter example $\text{Ker } C_1 = \text{Span } \{(0\ 0\ 1)^T\} \not\subset I_{A,B}$, thus $\overline{R}_2 = 0$ and the system may not be decoupled.

Fabian and Wonham [F1] have shown that decoupling by the use of dynamic compensation is possible for almost all systems of the form

$$x(j+1) = Ax(j) + Bu(j), \quad y_i(j) = C_i x(j), \quad i \in \underline{k}$$

(or their continuous time analogs) with A $n \times n$, B $n \times m$ and C_i $q_i \times n$, $i \in \underline{k}$, real matrices, if and only if the integers n, m, q_i, $i \in \underline{k}$ satisfy some simple inequality constraints. Building upon this work we then show that the problem of Falb and Wolovich, decoupling an m-input, m-output system into scalar input, scalar output sub-systems by state feedback, is almost always solvable.

In order to present the results of Fabian and Wonham we must formalize the notion of generic solvability, borrowing some elementary facts and terminology from algebraic geometry via [F1]. Consider the ring $R[\lambda_1,\ldots,\lambda_N]$ of polynomials in N indeterminates with real coefficients. An algebraic variety $V \subset R^N$ is the set of common zeros of a finite number of such polynomials. A variety is called proper if it is not equal to R^N, and nontrivial if it is not empty.

A property Π is a function on R^N to a two element set, {true, false} for example. If V is a proper variety of R^N, we say Π is generic relative to V if Π is true everywhere on R^N except for a subset of V. Π is deemed generic if such a V exists. Since a proper variety is closed in the Euclidean topology, it follows that if Π is generic relative to V, for every $x \in V^c$ (the complement of V) Π is true on some neighborhood of x. As a proper variety V cannot contain any open set in R^N, (if this were so, the defining polynomials would all be identically zero) it follows that if Π is false for some $x \in V$, then there exist points arbitrarily close to x such that Π is true at these points.

If we let $N = n(n + m + q_1 + \ldots + q_k)$, then R^N is the real Euclidean space consisting of system parameter sets $(A,B,C_i, i \in \underline{k})$. Each point of R^N represents a linear system with partitioned output vector. In this formulation solvability of the decoupling problem by dynamic compensation is a property Π on R^N, and if Π is generic, this problem is solvable for almost all parameter sets $(A,B,C_i, i \in \underline{k})$.

(3.5.1) <u>Theorem</u>: (Fabian and Wonham [F1]) Π is generic if and only if

$$\sum_{i \in \underline{k}} q_i \leq n \qquad (3.5.1)$$

$$m \geq 1 + \sum_{i \in \underline{k}} q_i - \min\{q_i \mid i \in \underline{k}\} . \qquad (3.5\text{-}2)$$

Basically the theorem states that decoupling by dynamic compensation is almost always possible if there are fewer outputs than states (3.5-1) and enough inputs (3.5-2). Clearly we may never have fewer inputs than output subvectors and still decouple. From (3.5-2) it is easily seen that if $m = k$, decoupling by dynamic compensation is generic only if $q_i = 1$, $i \in \underline{k}$.

A key element in the proof of Theorem 3.5.1 consists of showing that given a parameter set $(A,B,C_i,\ i \in \underline{k})$, the subspace's $K_i = \bigcap_{\substack{j \neq i \\ j \in \underline{k}}} N_j$, $i \in \underline{k}$ are generically controllability subspaces. Since the condition for solvability of decoupling by dynamic compensation (Theorem 3.4.1) coincides with that for the special case rank $C = n$ of the state feedback decoupling problem (Prop. 3.2.7), it follows that Theorem 3.5.1 presents conditions for the generic solvability of the latter case.

(3.5.2) <u>Corollary</u>: For rank $C = n$, the decoupling problem is solvable if and only if $K_i = \bigcap_{\substack{j \neq i \\ j \in \underline{k}}} N_j$ is a c.s. for $i \in \underline{k}$. Furthermore, this is almost always true.

Proof: (Sufficiency): Since the rows of C are independent

$$\Big(\sum_{\substack{j \neq i \\ j \in \underline{k}}} C_j^T\Big) \cap C_i^T = 0$$

which is equivalent by complementation to

$$K_i + N_i = R^n .$$

(Necessity): Rank $C = n$ implies $\bigcap_{j \in \underline{k}} N_j = 0$, hence $\overline{R}_i \cap N_i = 0$, $i \in \underline{k}$, and thus R_i and N_i are direct summands. Since K_i and N_i are direct summands, and $\overline{R}_i \subset K_i$, we must have $\overline{R}_i = K_i$, $i \in \underline{k}$. ∎

Theorem 3.5.1 and the discussion immediately thereafter indicate that m-input, m-output systems considered by Falb and Wolovich, and Gilbert are almost always decoupleable by the use of dynamic compensation. Indeed the theorem implies that systems for which $m = k$, that is the number of inputs equals the number of output subvectors, will be generically decoupleable if and only if the k output subvectors are all scalars. We will now show that for such systems the decoupling problem is generically solvable by the use of state feedback control laws alone.

(3.5.3) <u>Theorem:</u> The m-input, m-output state feedback decoupling problem is generically solvable if $k = m$.

For the proof of Theorem 3.5.3 we first note that by Prop. 3.2.8, an m-input m-output linear system with $k = m$ may be decoupled by a feedback control law (2.2-2) if and only if (3.2-19) holds, that is

$$\mathcal{B} = \sum_{i \varepsilon \underline{m}} \mathcal{B} \cap \overline{\mathcal{R}}_i \ .$$

Then we proceed with a key lemma.

(3.5.4) <u>Lemma:</u> Given a linear system of the form (1.4-2), if

$$n \geq m \geq \sum_{i \varepsilon \underline{k}} q_i \qquad (3.5\text{-}3)$$

then (3.2-19) is generically true.

Proof: We shall make use of the results on the generic dimension of subspaces from [F1] without specific reference. The interested reader should consult [F1] for more complete background details. For convenience of notation we will drop the bar superscript to denote maximal c.s., thus for this proof $\mathcal{R}_i = \overline{\mathcal{R}}_i$. Further, identities which hold everywhere except possibly on a subset of a proper algebraic variety will be indicated by a postscripted (g).

We note that any r x s matrix Q generically has rank $t = \min(r,s)$. For otherwise all t x t minors of Q must vanish identically, in which case the elements of Q constitute a zero for a set of polynomials defined on $R^{r\times s}$. Thus $\dim C_i^T = q_i$ (g) and $\dim (\mathrm{Ker}\ C_i) = n - q_i$ (g), $i \in \underline{k}$. Then from (3.5-3)

$$\dim \Big(\sum_{\substack{j\neq i \\ i\varepsilon \underline{k}}} C_j^T \Big) = \min(n, q_i^*) = q_i^* \ (g), \quad i \in \underline{k}$$

where $q_i^* = \sum_{\substack{j\neq i \\ i\varepsilon \underline{k}}} q_j$, and by complementation

$$\dim \mathcal{K}_i = n - \min(n, q_i^*) = n - q_i^* \ (g), \quad i \in \underline{k} \ .$$

Fabian and Wonham have shown [F1] that the $\mathcal{K}_i$ are generically c.s. whence we have

$$\dim \mathcal{R}_i = n - q_i^* \ (g), \quad i \in \underline{k} \ . \qquad (3.5\text{-}4)$$

Since $\sum_{i\varepsilon \underline{k}} \mathcal{B} \cap \mathcal{R}_i \subset \mathcal{B}$, we need only prove that (3.5-3) implies

$$\dim \Big(\sum_{i\varepsilon \underline{k}} \mathcal{B} \cap \mathcal{R}_i \Big) = \dim \mathcal{B} = m \quad (g) \ . \qquad (3.5\text{-}5)$$

Using the geometric identity

$$\dim (\mathcal{S} \cap \mathcal{J}) = \dim(\mathcal{S}) + \dim(\mathcal{J}) - \dim(\mathcal{S} + \mathcal{J})$$

to expand the left side of (3.5-5) results in

$$\dim(\sum_{i\in\underline{k}} \mathcal{B}\cap\mathcal{R}_i) = \dim(\mathcal{B}\cap\mathcal{R}_k) + \dim(\sum_{i\in\underline{k-1}} \mathcal{B}\cap\mathcal{R}_i) - \dim((\mathcal{B}\cap\mathcal{R}_k)\cap(\sum_{i\in\underline{k-1}} \mathcal{B}\cap\mathcal{R}_i)).$$

But

$$\dim((\mathcal{B}\cap\mathcal{R}_k)\cap(\sum_{i\in\underline{k-1}} \mathcal{B}\cap\mathcal{R}_i)) = \dim(\mathcal{R}_k\cap(\sum_{i\in\underline{k-1}} \mathcal{B}\cap\mathcal{R}_i))$$

$$= \dim \mathcal{R}_k + \dim(\sum_{i\in\underline{k-1}} \mathcal{B}\cap\mathcal{R}_i) - \dim(\mathcal{R}_k + \sum_{i\in\underline{k-1}} \mathcal{B}\cap\mathcal{R}_i),$$

which yields

$$\dim(\sum_{i\in\underline{k}} \mathcal{B}\cap\mathcal{R}_i) = \dim(\mathcal{B}\cap\mathcal{R}_k) - \dim \mathcal{R}_k + \dim(\mathcal{R}_k + \sum_{i\in\underline{k-1}} \mathcal{B}\cap\mathcal{R}_i) \ . \tag{3.5-6}$$

Now

$$\dim(\mathcal{B}\cap\mathcal{R}_k) = \dim \mathcal{B} + \dim \mathcal{R}_k - \dim(\mathcal{B}+\mathcal{R}_k)$$

$$= m + (n-q_k^*) - \min(n, m+n-q_k^*) \quad (g)$$

$$= m - q_k^* \quad (g) \tag{3.5-7}$$

as $m > q_k^*$ by (3.5-3). Since

$$\dim(\mathcal{B}+\mathcal{R}_k) = \min(n, \dim \mathcal{B} + \dim \mathcal{R}_k) \quad (g)$$

$$= \min(n, m+n-q_k^*) = n \quad (g),$$

it follows from (3.5-7) that

$$\dim(\mathcal{R}_k + \sum_{i\in\underline{k-1}} \mathcal{B}\cap\mathcal{R}_i) = \min(n, n-q_k^* + \sum_{i\in\underline{k-1}} (m-q_i^*)) \quad (g). \tag{3.5-8}$$

But

$$q_k^* + \sum_{i\in\underline{k-1}} q_i^* = (k-1)\sum_{i\in\underline{k}} q_i$$

whence (3.5-8) becomes

$$\dim(\mathcal{R}_k + \sum_{i\in\underline{k-1}} \mathcal{B} \cap \mathcal{R}_i) = \min(n, n + (k-1)(m - \sum_{i\in\underline{k}} q_i)) \quad (g)$$

$$= n. \quad (g) \qquad (3.5\text{-}9)$$

Combining (3.5-4),(3.5-7) and (3.5-9) together with (3.5-6) gives

$$\dim(\sum_{i\in\underline{k}} \mathcal{B} \cap \mathcal{R}_i) = m - q_k{}^* - (n-q_k{}^*) + n = m \quad (g)$$

proving the lemma. ∎

Proof of Theorem 3.5.3: For the m-input, m-output state feedback decoupling problem we have $n \geq m = \sum_{i\in\underline{k}} q_i$ by definition. Hence the result follows immediately from Lemma 3.5.4. ∎

CHAPTER 4

CONNECTIONS BETWEEN THE INPUT-OUTPUT AND GEOMETRIC APPROACHES TO DECOUPLING

4.1 Introduction

In the preceding chapters, two very different approaches to the problem of decoupling a linear time invariant system by a state feedback control law have been highlighted. The input-output method of Falb and Wolovich [F2] and Gilbert [G2] made use of feedback invariants and matrix manipulations, while Wonham and Morse [M8],[W7] chose a geometric setting for their theory of decoupling. As each approach led to a disimilar, yet complete solution of the m-input, m-output decoupling problem, we know that the two methods must be connected. It is the primary aim of this chapter to develop that connection and then use it to extend our knowledge about decoupling problems.

Morse and Wonham [M8] have directly shown the equivalence of their necessary and sufficient condition for decoupling an m-input, m-output linear system into scalar input, scalar output subsystems (Prop. 3.2.8) with the result of Falb and Wolovich (Theorem 2.2.3) for a simple case $k = m = 2$. Of course these two results are implicitly equivalent as they represent necessary and sufficient conditions for solving the same problem. However by explicitly demonstrating this equivalence we begin to explore the connections between the matrix D and the c.s. $\{\overline{R}_i, i \in \underline{k}\}$. For this reason, using the characterization of maximal (A,B) i.s. given by Prop. 3.3.1, we will give an expanded version of the Morse and Wonham proof, generalizing to the case of arbitrary k.

Following these connections we return to the canonically decoupled form of Gilbert (Chapter 2, Section 2) and reinterpret it in light of our familiarity with c.s. It will be shown that the constructions used by Gilbert to develop this form are naturally related to the methods of Wonham and Morse.

We have already seen that Prop. 3.2.8 gives conditions for the solvability of a larger class of decoupling problems than originally considered by Falb and Wolovich. Building upon the equivalence between Prop. 3.2.8 and Theorem 2.2.3 for the case $k = m = q$, we will derive a strong necessary condition, in the form of the original Falb and Wolovich result, for decoupling of linear systems into single input, multiple output subsystems, and then show that it applies to more general decoupling problems. The result developed, in terms of an augmented D matrix, is deemed strong as we shall show in an example that there is no further information pertaining to decoupling to be extracted from such a formulation. Indeed, in this example, we shall demonstrate systems with identical augmented D matrices, some of which are decoupleable, others not.

Our attempt in Section 4.3 is not to duplicate, albeit in a different form previously established results. Wonham and Morse [W7], Sato and LoPresti [S4], Silverman and Payne [S6] have solved this problem. However the geometric condition $\mathcal{B} = \sum_{i \in \underline{k}} \mathcal{B} \cap \overline{\mathcal{R}}_i$ is not easily verified given a system (A,B,C). The conditions of Sato and LoPresti, Silverman and Payne involve a complicated algorithm and are also not easily applied. On the other hand, the original result of Falb and Wolovich is easily implemented. Our aim in this section is to exploit the geometric result (3.2.19) to obtain a strong, yet readily applied, necessary condition for this problem.

4.2 The m-Input, m-Output Decoupling Problem

Falb and Wolovich have shown that the decoupling problem for $k = m = q$ is solvable if and only if the matrix

$$D = [(A^T)^{d_1} C_1^T; \ldots, (A^T)^{d_m} C_m^T]^T B$$

is nonsingular (Theorem 2.2.3). Wonham and Morse have shown this problem solvable if and only if $\mathcal{B} = \sum_{i \in \underline{k}} \mathcal{B} \cap \overline{\mathcal{R}}_i$ (Prop. 3.2.8). In this section, following a result from [M8], we will establish the direct equivalence of these two conditions, illustrating the connections between these two highly varying methods.

In Section 3 of Chapter 3 it was shown (Prop. 3.3.1) that under certain circumstances the intersection of maximal (A,B) i.s. would again be a maximal (A,B) i.s. Preparatory to the proof of equivalence to follow, we demonstrate a rather obvious, but nevertheless useful fact.

(4.2.1) Lemma: Let $\mathcal{V}_i$ be the maximal (A,B) i.s. contained in $\mathcal{N}_i$, $i \in \underline{t}$, and $\mathcal{V}$ the maximal (A,B) i.s. contained in $\mathcal{N} = \bigcap_{i \in \underline{t}} \mathcal{N}_i$. Then $\mathcal{V} \subset \bigcap_{i \in \underline{t}} \mathcal{V}_i$.

Proof: Let $x \in \mathcal{V}$. Then $(A+BF)^j x \in \mathcal{N}$ for $j \geq 0$ and some F. But clearly then $(A+BF)^j x \in \mathcal{N}_i$, $j \geq 0, i \in \underline{t}$, so by the definition of $\mathcal{V}_i$, $x \in \mathcal{V}_i$, $i \in \underline{t}$. □

Now let us proceed to the main result of this section.

(4.2.2) Proposition: The necessary and sufficient conditions for solvability of the decoupling problem for $k = m = q$,

$$\text{rank } D = m$$

$$\mathcal{B} = \sum_{i \varepsilon \underline{m}} \mathcal{B} \cap \overline{\mathcal{R}}_i$$

are equivalent.

Proof: To demonstrate this equivalence we will rely on the characterization of maximal (A,B) i.s. contained in a subspace of dimension n-1 (Corollary 3.2.3). The geometric result, $\mathcal{B} = \sum_{i \varepsilon \underline{m}} \mathcal{B} \cap \overline{\mathcal{R}}_i$, may be written in terms of maximal (A,B) i.s., which by Prop. 3.3.1 may be decomposed into intersections of (A,B) i.s. of the form given by the corollary. Taking orthogonal complements, we arrive at a condition based on one dimensional subspaces which are related to the rows of D.

Let $\mathcal{V}_i$ be the maximal (A,B) i.s. contained in $\mathcal{N}_i = \text{Ker } C_i$, $i \varepsilon \underline{m}$. From Corollary 3.2.3 we may write $\mathcal{V}_i^{\perp} = \mathcal{S}_i + z_i$, $i \varepsilon \underline{m}$

$$\mathcal{S}_i = \text{span}\{C_i^T + \ldots + (A^T)^{d_i - 1} C_i^T\}, \qquad i \varepsilon \underline{m}$$

$$z_i = \text{span}\{(A^T)^{d_i} C_i^T\}, \qquad i \varepsilon \underline{m},$$

where $\mathcal{S}_i \stackrel{\Delta}{=} 0$ if $d_i = 0$.

We denote Ker B^T by $\mathcal{N}$, and it follows from the definition of d_i, $i \varepsilon \underline{m}$ that

$$\mathcal{S}_i \subset \mathcal{N}, \qquad i \varepsilon \underline{m} \tag{4.2-1}$$

$$z_i \cap \mathcal{N} = 0, \qquad i \varepsilon \underline{m}. \tag{4.2-2}$$

Define $\overline{V}_i$ to be the maximal (A,B) i.s. contained in $K_i = \bigcap_{\substack{j \neq i \\ j \in \underline{m}}} N_j$, $i \in \underline{m}$ and assume rank D = m. Then since the i^{th} row of D is given by $C_i A^{d_i} B$, it follows from Prop. 3.3.1 that

$$\overline{V}_i = \bigcap_{\substack{j \neq i \\ j \in \underline{m}}} V_j, \qquad i \in \underline{m} . \tag{4.2-3}$$

Since $\overline{R}_i$ is the maximal c.s. contained in K_i, $i \in \underline{m}$, by Prop. 3.2.5 we have $B \cap \overline{R}_i = B \cap \overline{V}_i$, $i \in \underline{m}$, and hence the condition $B = \sum_{i \in \underline{m}} B \cap \overline{R}_i$ is equivalent to

$$B = \sum_{i \in \underline{m}} B \cap \overline{V}_i . \tag{4.2-4}$$

Taking compliments of (4.2-4) yields

$$N = \bigcap_{i \in \underline{m}} (N + \overline{V}_i^{\perp}) = \bigcap_{i \in \underline{m}} (N + \sum_{\substack{j \neq i \\ j \in \underline{m}}} V_j^{\perp}) = \bigcap_{i \in \underline{m}} (N + Z_i^*) \tag{4.2-5}$$

with $Z_i^* = \sum_{\substack{j \neq i \\ j \in \underline{m}}} z_j$, $i \in \underline{m}$ and where we have implicitly used the relations (4.2-1) through (4.2-3). It then follows (4.2-4) and (4.2-5) are equivalent conditions when rank D = m, and we need only demonstrate (4.2-5).

Now rank D = rank D^T = rank$(B^T[(A^T)^{d_1} C_1^T : \ldots : (A^T)^{d_m} C_m^T])$,

hence the subspaces N, z_i, $i \in \underline{m}$ are mutually independent. Then if

$x \in \bigcap_{i \in \underline{m}} (N + Z_i^*)$ we may write

$$x = w_i + \sum_{\substack{j \neq i \\ j \in \underline{m}}} z_{ij}, \qquad \text{for } i \in \underline{m},$$

where $w_i \in N$, and $z_{ij} \in Z_j$, $i \in \underline{m}$. Letting $i = 1,2$ we have

$$w_1 + z_{12} + \dots + z_{1m} = w_2 + z_{21} + z_{23} + \dots + z_{2m}$$

or

$$z_{21} = w_1 - w_2 + z_{12} + (z_{13} - z_{23}) + \dots + (z_{1m} - z_{2m}).$$

However the independence of N, Z_i, $i \in \underline{m}$ implies that $z_{21} = z_{12} = 0$, $w_1 = w_2$, and $z_{1j} = z_{2j}$, for $j = 3,\dots,m$. Continuing in a similar manner we may show that $z_{ij} = 0$ for all $j \neq i$, $i,j \in \underline{m}$, whence it follows that $x \in N$. Thus we have shown

$$\bigcap_{i \in \underline{m}} (N + Z_i^*) \subset N,$$

and since the other inclusion is obvious, (4.2-5) follows.

Now let us assume that $B = \sum_{i \in \underline{m}} B \cap \overline{R}_i$, or equivalently, that (4.2-4) holds. By Lemma 4.2.1 we have

$$\overline{V}_i \subset \bigcap_{\substack{j \neq i \\ j \in \underline{m}}} V_j, \quad i \in \underline{m}$$

whence we have

$$B \subset \sum_{i \in \underline{m}} (B \cap (\bigcap_{\substack{j \neq i \\ j \in \underline{m}}} V_j)) \ . \tag{4.2-6}$$

But since the reverse inclusion is obvious we may assume equality in (4.2-6). Taking complements and noting that $V_i^{\perp} = S_i + z_i$, $i \in \underline{m}$, we again obtain (4.2-5), that is

$$N = \bigcap_{i \in \underline{m}} (N + Z_i^*) .$$

From (4.2-2) we have

$$z_i \cap N = z_i \cap \left(\bigcap_{j \in \underline{m}} (N + Z_j^*) \right) = 0, \quad i \in \underline{m} .$$

This latter expression may be expanded

$$z_i \cap \left(\bigcap_{\substack{j \neq i \\ j \in \underline{m}}} (N + Z_j^*) \right) \cap (N + Z_i^*) = 0. \qquad (4.2\text{-}7)$$

But clearly $z_i \subset Z_j^*$ for $j \neq i$ and $i,j \in \underline{m}$ by definition, and so (4.2-7) implies.

$$z_i \cap (N + Z_i^*) = 0, \qquad i \in \underline{m}. \qquad (4.2\text{-}8)$$

Therefore

$$\dim(z_1+\ldots+z_m + N) = \dim(z_1) + \dim(z_2+\ldots+z_m + N) - \dim(z_1 \cap (z_2+\ldots+z_m + N))$$

where the last term is zero by (4.2-8). Continuing in a like manner, it is readily established that

$$\dim(z_1+\ldots+z_m + N) = \dim(z_1)+\ldots+ \dim(z_m) + \dim N .$$

As $\dim N = n-m$, and the subspaces z_i, $i \in \underline{m}$ are nonzero by construction, $\dim(z_1 + \ldots + z_m + N) = n$, hence

$$z_1+\ldots+z_m + N = R^n. \qquad (4.2\text{-}9)$$

Since $\mathcal{N} = \text{Ker } B^T$ we have

$$\mathcal{B}^T = B^T(z_1+\ldots+z_m).$$

Thus $\dim \mathcal{B}^T = \text{rank}(B^T[(A^T)^{d_1} C_1^T:\ldots:(A^T)^{d_m} C_m^T]) = \text{rank } D^T = m,$ proving the desired equivalence. ∎

In Section 3 of Chapter 2 we explored the problem of decoupling a system with more inputs than outputs into multiple input, scalar output subsystems. Having defined an augmented D matrix, it was shown this problem is solvable if and only if D contains a q x q nonsingualr submatrix. In terms of the geometric formulation this result, Theorem 2.3.5, implies that the decoupling problem with $m \geq q = k$ is solvable only if there exists a set of singly generated c.s., i.e. $\dim(\mathcal{B} \cap \mathcal{R}_i) = 1$ for all $i \in \underline{k}$, satisfying (3.2-14) thru (3.2-16). This we now briefly demonstrate.

Assume there exists a compatible set of c.s., $\{\mathcal{R}_i,\ i \in \underline{q}\}$ which constitutes a solution to this decoupling problem. Consider any c.s. $\mathcal{R}_j$ for which $\dim(\mathcal{B} \cap \mathcal{R}_j) > 1$ and choose a basis $b_{j1},\ldots,b_{jr}$ for $\mathcal{B} \cap \mathcal{R}_j$. Then

$$\mathcal{R}_j = \{A+BF|\mathcal{B} \cap \mathcal{R}_j\} = \sum_{i\in\underline{r}} \{A+BF|b_{ji}\};$$

and since $\mathcal{R}_j + \mathcal{N}_j = R^n$, it follows that for some $s \in \underline{r}$

$$c_j(A+BF)^{d_j} b_{js} \neq 0 ,$$

and furthermore

$$\hat{\mathcal{R}}_j = \{A+BF|b_{js}\} \subset \mathcal{R}_j.$$

We may then choose G_j such that $BG_j = b_{js}$.

Continuing in a similar manner for all other $\mathcal{R}_i$ with $\dim(\mathcal{B} \cap \mathcal{R}_i) > 1$, we may find a new set of c.s., $\{\hat{\mathcal{R}}_i, i \in \underline{q}\}$, where $\hat{\mathcal{R}}_i = \mathcal{R}_i$ if $\dim(\mathcal{B} \cap \mathcal{R}_i) = 1$, which are singly generated and satisfy (3.2-14) thru (3.2-16). Defining $G = [G_1,\ldots,G_q:0]$ it follows that $D(F,G) = DG$ is of the form $[R : 0]$ where R is q x q and nonsingular. Since rank G = q by construction, it follows that D must have rank of at least q, and thus contain a q x q nonsingular submatrix.

4.3 The Canonical Form of Gilbert

In the second section of Chapter 2 we briefly presented the definition of canonically decoupled m-input, m-output linear systems by Gilbert [G2], and stated the result that every integrator decoupled (i.d.) system is similar to a canonically decoupled (c.d.) system. (Remember, a system (A,B,C) is i.d. if D is diagonal and nonsingular, and $C_i(A+BF)^{d_i+1} = 0$, $i \in \underline{m}$.) Having gained familiarity with the geometric method of decoupling in Chapter 3, we return to Gilbert's canonical form for decoupled systems and re-examine it in terms of invariant subspaces.

In proving the result that every i.d. system is similar to a c.d. system, Gilbert resorted to certain subspace constructions. We will show that these subspaces tie in quite naturally to the geometric problem formulation. As it is more convenient to deal with subspaces of the state space rather than its dual, we shall redefine Gilbert's decomposition to effect that desire.

For a linear system (A,B,C) let B_i, $i \in \underline{m}$ denote the i^{th} column of B, C_i, $i \in \underline{m}$ the i^{th} row of C. Then define the subspaces

$$\mathcal{Q}_i = \{x \mid x^T A^j B_s = 0,\ s \neq i,\ s \in \underline{m},\ j = 0,\ldots,n-1\},\ i \in \underline{m}.$$

If $\mathcal{R}_i$ is the subspace reachable from the i^{th} input (or the i^{th} controllability subspace),

$$\mathcal{R}_i = \{A \mid \mathcal{B}_i\}, \quad i \in \underline{m}$$

then it follows that $\mathcal{Q}_i \subset \mathcal{R}_j^{\perp}$ $j \neq i$, $i,j \in \underline{m}$, and hence

$$\mathcal{Q}_i = \bigcap_{\substack{j \neq i \\ j \in \underline{m}}} \mathcal{R}_j^{\perp}, \qquad i \in \underline{m}. \tag{4.3-1}$$

Having defined the subspaces $\mathcal{Q}_i$, $i \in \underline{m}$, we show some of their properties via a lemma.

(4.3.1) Lemma: (Gilbert [G2, Lemma 1]) Assume the system (A,B,C) is i.d. and controllable. Then for $i \in \underline{m}$:

(i) $\mathcal{Q}_i$ is an A^T invariant subspace;

(ii) $\mathcal{Q}_i \cap \mathcal{Q}_j = 0$ for $j \neq i$ $i,j \in \underline{m}$;

(iii) $C_i^T, A^T C_i^T, \ldots, (A^T)^{d_i} C_i^T$ are linearly independent elements of $\mathcal{Q}_i$, $i \in \underline{m}$.

A formal proof of Lemma 4.3.1 may be found in [G2]. We do note however that the proof of condition (i) requires neither the assumptions that (A,B,C) is i.d. nor controllable; the proof of (ii) requires solely the controllability of (A,B); to prove (iii) we need only that (A,B,C) is i.d.

It should be pointed out that the subspaces $\mathcal{Q}_i$, $i \in \underline{m}$ are not in general invariant under feedback. For example, $x^T AB_j = 0$ does not imply $x^T(A+BF)B_j = 0$. However we do have the following invariant property of the $\mathcal{Q}_i$'s.

(4.3.2) Lemma: $\mathcal{Q}_i$ is the maximal A^T invariant subspace contained in $\bigcap_{\substack{j\neq i\\ j\in\underline{m}}} B_j^{\perp}$, $i \in \underline{m}$.

Proof: By Lemma 4.3.1 $\mathcal{Q}_i$, $i \in \underline{m}$ is A^T invariant. For $x \in \mathcal{Q}_i$, $x^T B_j = 0$ for $j \neq i$, $i,j \in \underline{m}$ by definition whence $x \in \bigcap_{\substack{j\neq i\\ j\in\underline{m}}} B_j^{\perp}$ which proves $\mathcal{Q}_i \subset \bigcap_{\substack{j\neq i\\ j\in\underline{m}}} B_j^{\perp}$, $i \in \underline{m}$.

Now assume to the contrary that there exists a subspace $\overline{\mathcal{Q}}_i \neq \mathcal{Q}_i$, such that $\mathcal{Q}_i \subset \overline{\mathcal{Q}}_i$, $A^T \overline{\mathcal{Q}}_i \subset \overline{\mathcal{Q}}_i$ and $\overline{\mathcal{Q}}_i \subset \bigcap_{\substack{j\neq i\\ j\in\underline{m}}} B_j^{\perp}$, for some $i \in \underline{m}$. Choose $x \in \overline{\mathcal{Q}}_i$, $x \notin \mathcal{Q}_i$. Then $x^T B_j = 0$ and $x^T A^s B_j$ for $s = 0,\ldots,n-1$, $j \neq i$, $j \in \underline{m}$. But then $x \in \mathcal{Q}_i$ by definition, a contradiction.

Gilbert uses the subspaces $\mathcal{Q}_i$, $i \in \underline{m}$ to produce a similarity transformation which takes a controllable i.d. system (A,B,C) into a c.d. form [G2, Prop. 5]. Indeed, define $p_i = \dim \mathcal{Q}_i$, $i \in \underline{m}$, and let

$\mathcal{Q}_{m+1}$ be any subspace such that $\mathcal{Q}_1 \oplus \cdots \oplus \mathcal{Q}_m \oplus \mathcal{Q}_{m+1} = R^n$. Let p_{m+1} denote $\mathcal{Q}_{m+1}$. Using Lemma 4.3.1 (iii), Gilbert demonstrates a basis in which the controllable i.d. system is c.d.

We have seen from Lemma 4.3.1, that if (A,B,C) is i.d. then $c_i^T, \ldots, (A^T)^{d_i} c_i^T$ are independent elements of $\mathcal{Q}_i$ for $i \in \underline{m}$. Let s_i be the largest nonnegative integer such that

$$\mathcal{W}_i = \text{Span}\{c_i^T, \ldots, (A^T)^{s_i} c_i^T\}$$

has dimension s_i+1, $i \in \underline{m}$. We shall call $\mathcal{W}_i$ the observability subspace of c_i, $i \in \underline{m}$, or simply the i^{th} observability subspace. For (A,B,C) i.d. it is clear that $s_i = d_i$, $i \in m$, and any initial condition $x_0 \in \mathcal{W}_i$ may be uniquely determined by observing solely output i under the free action of the system.

For a system (A,B,C) i.d. and controllable, we have

$$\mathcal{Q}_1 \oplus \cdots \oplus \mathcal{Q}_m \oplus \mathcal{Q}_{m+1} = R^n,$$

whence we may form a basis $\{q_{11}, \ldots, q_{1p_i}, q_{21}, \ldots\}$ of R^n with $q_{ij} \in \mathcal{Q}_i$ for $j \in \underline{p_i}$, $i \in \underline{m+1}$. Let $x \in \mathcal{R}_1$. Then by (4.3-1) $x^T q_{ij} = 0$ for $j \in \underline{p_i}$, $i \neq 1$, $i \in \underline{m}$, and hence $x \in \mathcal{Q}_1 \oplus \mathcal{Q}_{m+1}$. It follows immediately that $\mathcal{R}_1 \subset \mathcal{Q}_1 \oplus \mathcal{Q}_{m+1}$, and hence generally

$$\mathcal{R}_i \subset \mathcal{Q}_i \oplus \mathcal{Q}_{m+1}, \qquad i \in \underline{m} . \tag{4.3-2}$$

We note that we may always choose

$$\mathcal{Q}_{m+1} = (\sum_{i\varepsilon\underline{m}} \mathcal{Q}_i)^{\perp} = \bigcap_{i\varepsilon\underline{m}} \mathcal{Q}_i^{\perp} = \bigcap_{i\varepsilon\underline{m}} \mathcal{R}_i{}^* = \mathcal{R}^*$$

where the third equality follows from (4.3-1). For (A,B,C) c.d. and controllable, the particular form of A,B and C from (2.2-11) thru (2.2-14) immediately yields

$$\mathcal{R}_i = \mathcal{Q}_i \oplus \mathcal{R}_i \cap \mathcal{R}^* \qquad i \varepsilon \underline{m} \tag{4.3-3}$$

Since every i.d. system is similar to a c.d. system [G2, Theorem 2], it follows that (4.3-3) holds for controllable i.d. systems.

As (A,B,C) is assumed controllable, $\sum_{i\varepsilon\underline{m}} \mathcal{R}_i = R^n$. For any $x \varepsilon R^n$, let $\bar{x}$ denote its coset in $R^n/\mathcal{R}^*$, and define the induced map $\bar{A} : R^n/\mathcal{R}^* \to R^n/\mathcal{R}^*$ in the usual manner, $\bar{A}\,\bar{x} = \overline{Ax}$. Then by [M7, Theorem 1.2] we have

$$\mathcal{Q}_i \approx \bar{\mathcal{R}}_i \triangleq (\mathcal{R}_i + \mathcal{R}^*)/\mathcal{R}^*, \qquad i \varepsilon \underline{m}; \tag{4.3-4}$$

$$\mathcal{Q}_i + \mathcal{N}_i = R^n, \qquad i \varepsilon \underline{m}; \tag{4.3-5}$$

$$A\mathcal{Q}_i \subset \mathcal{Q}_i \oplus \mathcal{R}^*, \qquad i \varepsilon m. \tag{4.3-6}$$

Furthermore, as was shown in Prop. 3.2.10, the spectrum of $\bar{A}|\bar{\mathcal{R}}_i$, $i \varepsilon \underline{m}$ may be freely assigned subject only to the requirement of conjugate symmetry.

Equation (4.3-4) shows that the subspace $\mathcal{Q}_i$, $i \varepsilon \underline{m}$ are canonically isomorphic to the independent position of the c.s. $\mathcal{R}_i$, $i \varepsilon \underline{m}$.

As $\mathcal{R}_i + \mathcal{N}_i = R^n$ for $i \in \underline{m}$ and $\mathcal{R}^* \subset \bigcap_{i\in\underline{m}} \mathcal{N}_i$, (4.3-5) follows directly from (4.3-3). Finally (4.3-6) follows from (4.3-3) and the invariant properties of the $\mathcal{R}_i$, $i \in \underline{m}$. Further, the subspaces $\mathcal{W}_i \subset \mathcal{Q}_i$, $i \in \underline{m}$ are canonically isomorphic to the controllable and observable portion of the state space of the i^{th} subsystem, (A_i, b_i, c_i). This fact is easily seen for a c.d. system by the special form of A,B and C in (2.2-11) thru (2.2-14).

We have already mentioned that a system (A,B,C) in c.d. form is generally neither controllable nor observable. Since every c.d. system is i.d., the subsystems (A_i, b_i, c_i) are easily seen to have transfer functions $h_i(\lambda) = \gamma_i \lambda^{-d_i-1}$, $i \in \underline{m}$. As Gilbert shows [G2, Theorem 4], for (A,B,C) c.d., a control law of the form (2.2-15) and (2.2-16) preserves decoupling, and results in subsystem transfer functions

$$h_i(\lambda; F, G) = \frac{\alpha_i(\lambda)\gamma_i\delta_i}{\psi_i(\lambda;\theta_i)}, \qquad i \in \underline{m} \tag{4.3-7}$$

where $\psi_i(\lambda;\theta_i) = \det(\lambda I_{p_i} - A_i - b_i\theta_i)$ and $\alpha_i(\lambda) = \det(\lambda I_{r_i} - \Phi_i)$ with $\alpha_i(\lambda) \triangleq 1$ if $r_i = p_i - (d_i+1) = 0$, for $i \in \underline{m}$. Since it is clear that (A_i, b_i) is a controllable pair (see(2.2-14)), the coefficients of $\psi_i(\lambda;\theta_i)$ may be arbitrarily altered by the feedback θ_i. For $\theta_i = 0$, $\delta_i = 1$, $i \in \underline{m}$, it follows that

$$h_i(\lambda;0) = \frac{\alpha_i(\lambda)\gamma_i}{\lambda^{d_i+1}\alpha_i(\lambda)} = h_i(\lambda), \qquad i \in \underline{m}$$

i.e., there is a total cancellation of numerator dynamics.

If the c.d. system (A,B,C) is both controllable and observable, then it follows from Chandrasekharan [C1] that $\alpha_i(\lambda) \equiv 1$, $i \in \underline{m}$, and the c.d. form is completely determined by the feedback invariants d_i, $i \in \underline{m}$. It is clearly true in this case that every feedback law which decouples (A,B,C) preserves observability. If

$$m + \sum_{i\in\underline{m}} d_i \leq \sum_{i\in\underline{m}} p_i = n$$

then it follows that pole-zero cancellations may occur, and hence observability of the decoupled system depends upon the particular feedback law used. For $\sum_{i\in\underline{m}} p_i < n$, no feedback law will leave the decoupled system observable, for it is easily seen in this case that $\mathcal{R}^*$ will be nonzero.

At this point an example would be of value. Consider the controllable c.d. system (A,B,C) where

$$A = \left(\begin{array}{cc|c|c} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 \end{array}\right), \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

It follows of course that $d_1 = d_2 = 0$, $p_1 = 2$, $p_2 = 1$ and $p_3 = 1$. Further we have $\mathcal{R}_1 = \text{Span}\{e_1, e_2, e_4\}$ and $\mathcal{R}_2 = \text{Span}\{e_3, e_4\}$ where e_i is the i^{th} standard unit vector in R^4. From (4.3-1) we see that $\mathcal{Q}_1 = \text{Span}\{e_1, e_2\}$ and $\mathcal{Q}_2 = \text{Span}\{e_3\}$, and we may choose $\mathcal{Q}_3 = \mathcal{R}^* = \mathcal{R}_1 \cap \mathcal{R}_2 = \text{Span}\{e_4\}$. It is observed that the $\mathcal{Q}_i$'s are canonically isomorphic to the independent portion of the $\mathcal{R}_i$'s, $i \in \underline{2}$.

The transfer function matrix $H(\lambda)$ for (A,B,C) may be determined

$$H(\lambda) = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

and it is immediate that the subsystem (A_1, b_1, c_1) is unobservable, while (A_2, b_2, c_2) is controllable and observable. The observability subspaces $\mathcal{W}_1$ and $\mathcal{W}_2$ are given by $\text{Span}\{e_1\}$ and $\text{Span}\{e_3\}$ respectively. From (2.2-15) and (2.2-16), it follows that the control law (F,G) decouples (A,B,C) if and only if

$$F = \begin{pmatrix} \theta_1 & \theta_2 & 0 & 0 \\ 0 & 0 & \theta_3 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}.$$

With (F,G) so defined, it is easily verified that we have

$$h_1(\lambda; F, G) = \frac{(\lambda-1)\delta_1}{(\lambda-\theta_1)(\lambda-1-\theta_2)+\theta_2(1+\theta_1)}$$

$$h_2(\lambda; F, G) = \frac{1}{\lambda - \theta_3} .$$

For almost all values of θ_1 and θ_2, it follows that the subsystem $(A_1 + b_1(\theta_1\theta_2),\ b_1\delta_1,\ c_1)$ will be observable.

4.4 Decoupling into Single-Input, Multiple Output Subsystems

In the first section of this chapter we proved the equivalence of the geometric condition ($\mathcal{B} = \sum_{i \in \underline{m}} \mathcal{B} \cap \overline{\mathcal{R}}_i$), and the input-output condition (D nonsingular) for the problem of decoupling an m-input, m-output linear system into single input, single output subsystems. However, we are well aware that this geometric result is a necessary and sufficient condition for the existence of a solution to the problem of decoupling a linear system into single input, multiple output subsystems, i.e. $m = k$, $q_i \geq 1$ for all $i \in \underline{k}$, (see Prop. 3.2.8). Hence, it seems perfectly reasonable that we might extend the original Falb and Wolovich condition to cover this more general decoupling problem formulation. In this section we will develop a strong necessary condition of this form, in terms of submatrices of an augmented D matrix, for such a problem, and then show it applies to an even more general formulation.

We shall derive the desired condition in four steps. First, as in the proof of Prop. 4.2.2, we will transform the geometric condition (3.2-19) into one involving (A,B) i.s. and then by Prop. 3.3.1 show that this implies a condition involving i.s. of the form of Corollary 3.2.3. By a simple manipulation we arrive at an equality explicitly involving the rows of the matrix C.

Before demonstrating this first step, let us establish some notation. Assume we are given a linear system (1.4-2) with desired decomposition of the outputs into m subvectors $y_i(j) = C_i x(j)$, where C_i is $q_i \times n$,

$$C_i = \begin{bmatrix} C_{i_1} \\ \vdots \\ C_{iq_i} \end{bmatrix} \qquad i \in \underline{m} .$$

Define $N_{is} = \text{Ker } C_{is}$ for $s \in \underline{q_i}$, and $i \in \underline{m}$, with $N_i = \text{Ker } C_i$, for $i \in \underline{m}$. Then $N_i = \bigcap_{s \in \underline{q_i}} N_{is}$. As before we let $K_i = \bigcap_{\substack{j \neq i \\ j \in \underline{m}}} N_j$, and denote the maximal (A,B) i.s. contained in K_i by $\overline{V}_i$, $i \in \underline{m}$. Finally, let V_{is} be the maximal (A,B) i.s. contained in N_{is} for $s \in \underline{q_i}$, and $i \in \underline{m}$. Then by Corollary 3.2.2 we have

$$V_{is}^{\perp} = S_{is} + z_{is} \qquad \text{for } s \in \underline{q_i}, \text{ and } i \in \underline{m},$$

$$S_{is} = \text{Span}\{C_{is}^T + \ldots + (A^T)^{d_{is}-1} C_{is}^T\},$$

$$z_{is} = \text{Span}\{(A^T)^{d_{is}} C_{is}^T\}$$

where d_{is} is the least non negative integer such that $C_{is}A^{d_{is}} B \neq 0$ for $s \in \underline{q_i}$, and $i \in \underline{m}$, and $S_{is} \triangleq \{0\}$ if $d_{is} = 0$. Letting $N = \text{Ker } B^T$, we note that $S_{is} \subset N$ and $z_{is} \cap N = 0$ for $s \in \underline{q_i}$, and $i \in \underline{m}$.

Let us now prove the first step.

(4.4.1) <u>Lemma</u>: The geometric condition $B = \sum_{i \in \underline{m}} B \cap \overline{R}_i$ implies

$$N = \bigcap_{i \in \underline{m}} (N + \sum_{\substack{j \neq i \\ j \in \underline{m}}} (\sum_{s \in \underline{q_j}} z_{js})) . \qquad (4.4\text{-}1)$$

Proof: By Prop. 3.2.5 the condition $B = \sum_{i \in \underline{m}} B \cap \overline{R}_i$ is equivalent to

$$B = \sum_{i \in \underline{m}} B \cap \overline{V}_i \ . \tag{4.4-2}$$

From Lemma 4.2.1, together with the definitions of K_i and N_i, $i \in \underline{m}$

$$\overline{V}_i \subset \bigcap_{\substack{j \neq i \\ j \in \underline{m}}} \left(\bigcap_{s \in \underline{q_i}} V_{js} \right), \qquad i \in \underline{m} \ ,$$

whence (4.4-2) becomes

$$B = \sum_{i \in \underline{m}} \left(B \cap \left(\bigcap_{\substack{j \neq i \\ j \in \underline{m}}} \left(\bigcap_{s \in \underline{q_i}} V_{js} \right) \right) \right), \tag{4.4-3}$$

where equality holds as one inclusion is obvious. Taking complements of (4.4-3) yields

$$N = \sum_{i \in \underline{m}} \left(N + \sum_{\substack{j \neq i \\ j \in \underline{m}}} \left(\sum_{s \in \underline{q_i}} V_{js}^{\perp} \right) \right). \tag{4.4-4}$$

But since $V_{js}^{\perp} = S_{js} + Z_{js}$ for $s \in \underline{q_i}$ and $j \in \underline{m}$, (4.4-1) follows immediately from (4.4-4). ∎

At this point we wish to emphasize that the converse of Lemma 4.4.1 is not generally true. That is, (4.4-3) need not imply (4.4-2). Of course, if $\overline{V}_i = \bigcap_{\substack{j \neq i \\ j \in \underline{m}}} \left(\bigcap_{s \in \underline{q_i}} V_{js} \right)$ for all $i \in \underline{m}$, then the converse would hold, as was the case in the proof of Prop. 4.2.2.

To simplify notation in (4.4-4) let us define

$$Z_j = \sum_{s \varepsilon \underline{q_j}} z_{js}, \quad j \varepsilon \underline{m}$$

$$Z_i^* = \sum_{\substack{j \neq i \\ j \varepsilon \underline{m}}} Z_j, \qquad i \varepsilon \underline{m}$$

whence (4.4-1) becomes

$$N = \bigcap_{i \varepsilon \underline{m}} (N + Z_i^*). \tag{4.4-5}$$

Then the next step in our development is a technical, but relatively straightforward lemma which reduces (4.4-5) to a series of simpler subspace inclusions.

(4.4.2) <u>Lemma</u>: Condition (4.4-5) is true if and only if

$$Z_i \cap (N + Z_i^*) \subset N, \quad i \varepsilon \underline{m}\ . \tag{4.4-6}$$

Proof: (Necessity): Assume to the contrary that there exists an $x \varepsilon Z_i$ such that $x \varepsilon N + Z_i^*$, and $x \notin N$. But then since $Z_i \subset Z_j^*$ for $j \neq i$, with $i,j \varepsilon \underline{m}$

$$x \varepsilon (N + Z_i^*) \cap (\bigcap_{j \neq i} (N + Z_j^*) = N$$

by (4.4-5), a contradiction.

(Sufficiency): Assuming (4.4-6) is true, choose x such that

$$x \varepsilon \bigcap_{i \varepsilon \underline{m}} (N + Z_i^*).$$

Therefore

$$x = w_i + \sum_{\substack{j \neq i \\ j \in \underline{m}}} z_{ij}, \qquad i \in \underline{m}$$

where $w_i \in N$ and $z_{ij} \in Z_j$, $i,j \in \underline{m}$. In particular

$$x = w_1 + z_{12} + \ldots + z_{1m} = w_2 + z_{21} + z_{23} + \ldots z_{2m}.$$

Thus it follows that $z_{12} \in (N + Z_2^*) \cap Z_2$, implying that $z_{12} \in N$ by (4.4-6). Similarly, we may show $z_{1s} \in N$ for $s \neq 1$, $s \in \underline{m}$, which implies $x \in N$, proving the result. ∎

The next step in our development consists of showing that (4.4-6) may be reduced to a series of statements about the one dimensional subspaces z_{js} for $s \in \underline{q_j}$, and $j \in \underline{m}$. Indeed we will show that (4.4-6) is equivalent to

$$z_{js} \cap (N + Z_j^*) = 0 \quad \text{for} \quad s \in \underline{q_j}, \text{ and } j \in \underline{m}. \qquad (4.4\text{-}7)$$

We note that (4.4-7) does not follow immediately from (4.4-6) for arbitrary subspaces; it will be necessary to exploit the particular structure of the z_{js} to arrive at the desired conclusion.

(4.4.3) Lemma: Condition (4.4-6) is true if and only if (4.4-7) holds,

$$z_{js} \cap (N + Z_j^*) = 0, \qquad \text{for } s \in \underline{q_j}, \quad j \in \underline{m}\ .$$

Proof: (Necessity): By the definition of z_{js},

$$z_{js} \cap N = 0 \ \text{ for } s \in \underline{q_j}, \ j \in \underline{m}.$$

Assuming (4.4-6) holds we have by the previous lemma

$$N = \bigcap_{i \in \underline{m}} (N + Z_i^*)$$

whence

$$z_{js} \cap N = z_{js} \cap (\bigcap_{\substack{i \neq j \\ i \in \underline{m}}} (N + Z_i^*)) \cap (N + Z_j^*) = 0, \text{ for } s \in \underline{q_j},\ i \in \underline{m}.$$

But since $z_{js} \subset Z_i^*$ for $j \neq i$, $i,j \in \underline{m}$, it follows that

$$z_{js} \cap (N + Z_j^*) = 0, \quad \text{for } s \in \underline{q_j}, \quad j \in \underline{m}.$$

(Sufficiency): Consider the subspaces $N + Z_i^*$, $i \in \underline{m}$. Since rank $B = m$ we have $\dim N = n-m$. Furthermore, it follows from (4.4-7) that Z_i^*, for each $i \in \underline{m}$, contains at least $m-1$ independent vectors which are also independent of N, and hence $\dim(N + Z_i^*) \geq n-1$, $i \in \underline{m}$.

To demonstrate this fact let us fix $i = 1$, and choose the set of $m-1$ vectors in Z_1^*,

$$z_{j1} = (A^T)^{d_{j1}} c_{j1}^T \in Z_j, \quad \text{for } j \neq 1, \quad j \in \underline{m}.$$

We note that $z_{j1} = \text{Span}\{z_{j1}\}$, $j \in \underline{m}$. Choose a basis $w_1, \ldots, w_{n-m}$ of N. Since $z_{21} \cap N = 0$, it follows that the vectors $\{z_{21}, w_1, \ldots, w_{n-m}\}$ are independent. Then by (4.4-7), $z_{31} \cap (N + Z_3^*) = 0$, implying $z_{31} \cap (N + Z_2) = 0$ whence the vectors $\{z_{21}, z_{31}, w_1, \ldots, w_{n-m}\}$ are independent. Continuing (4.4-7) implies $z_{41} \cap (N + Z_2 + Z_3) = 0$ and hence the vectors $\{z_{21}, z_{31}, z_{41}, w_1, \ldots, w_{n-m}\}$. By repeated application of (4.4-7) we achieve an independent set of $n-1$ vectors in $N + Z_1^*$,

$\{z_{21},\ldots,z_{2m},w_1,\ldots,w_{n-m}\}$. Using a similar construction we may show an identical result for any $i \in \underline{m}$, i.e. $\dim(N + Z_i^*) \geq n-1$. Appealing once more to (4.4-7), we see that we must have strict equality in the result above, hence

$$\dim(N + Z_i^*) = n-1, \qquad i \in \underline{m}. \tag{4.4-8}$$

Now

$$\dim(Z_i \cap (N + Z_i^*)) = \dim Z_i + \dim(N + Z_i^*) - \dim(N + \sum_{j\in\underline{m}} Z_j), \qquad i \in \underline{m}.$$

From (4.4-7) and (4.4-8) it follows that

$$\dim(N + \sum_{j\in\underline{m}} Z_j) = n$$

whence

$$\dim(Z_i \cap (N + Z_i^*)) = \dim Z_i - 1, \qquad i \in \underline{m}. \tag{4.4-9}$$

Consider now the subspaces $N + Z_i$, $i \in \underline{m}$. Since $z_{i1} \in Z_i$, and $z_{i1} \cap N = 0$, it follows that

$$\dim(N + Z_i) \geq n - m + 1, \qquad i \in \underline{m}. \tag{4.4-10}$$

For concreteness, choose $i = 1$. Then by (4.4-7) there exists $z_{21} \in Z_2$ such that $z_{21} \cap (N + Z_1) = 0$ and thus

$$\dim(N + Z_1 + Z_2) - \dim(N + Z_1) \geq 1.$$

Continuing it follows that

$$\dim(N + Z_1 + Z_2 + Z_3) - \dim(N + Z_1 + Z_2) \geq 1$$

whence

$$\dim(N + Z_1 + Z_2 + Z_3) - \dim(N + Z_1) \geq 2.$$

Applying (4.4-7) repeatedly, we may readily establish

$$\dim(N + \sum_{j \in \underline{m}} Z_j) - \dim(N + Z_1) \geq m - 1. \qquad (4.4\text{-}11)$$

Comparing (4.4-10) for $i = 1$, and (4.4-11) it is immediately clear that strict equality must hold in each. Since the construction yielding (4.4-11) is valid for $i \neq 1$, $i \in \underline{m}$, it follows that

$$\dim(N + Z_i) = n - m + 1, \quad i \in \underline{m}. \qquad (4.4\text{-}12)$$

Therefore, from (4.4-12) we have

$$\dim(Z_i \cap N) = \dim Z_i + \dim N - \dim(Z_i + N) = \dim Z_i - 1, \quad i \in \underline{m}. \qquad (4.4\text{-}13)$$

Comparing (4.4-9) with (4.4-13) and noting

$$Z_i \cap N \subset Z_i \cap (N + Z_i^*), \qquad i \in \underline{m}$$

it is immediate that

$$Z_i \cap (N + Z_i^*) = Z_i \cap N \subset N$$

which was to be proved. ◻

Now we are ready to demonstrate the final step in our development. First we will define an augmented D matrix for this system and then show that (4.4-7) is equivalent to statements about the ranks of submatrices of this matrix D. Define the $q \times m$ matrix D as

$$D = \begin{bmatrix} C_{11}A^{d_{11}}B \\ \vdots \\ C_{1q_1}A^{d_{1q_1}}B \\ C_{21}A^{d_{21}}B \\ \vdots \\ C_{mq_m}A^{d_{mq_m}}B \end{bmatrix} .$$

We shall designate the row $C_{js_j}A^{d_{js_j}}B$ of D by D_{js_j}, for $s_j \in \underline{q_j}$, $j \in \underline{m}$.

For each $i \in \underline{m}$ the $q_i \times m$ submatrix D_i of D is given by

$$D_i = \begin{bmatrix} C_{i1}A^{d_{i1}}B \\ \vdots \\ C_{iq_i}A^{d_{iq_i}}B \end{bmatrix} ,$$

and the $(n-q_i) \times m$ submatrix $D_i{}^*$ of D

$$D_i{}^* = \begin{bmatrix} C_{11}A^{d_{11}}B \\ \vdots \\ C_{i-1,q_{i-1}}A^{d_{i-1,q_{i-1}}}B \\ C_{i+1,1}A^{d_{i+1,1}}B \\ \vdots \\ C_{mq_m}A^{d_{mq_m}}B \end{bmatrix} = \begin{bmatrix} D_1 \\ \vdots \\ D_{i-1} \\ D_{i+1} \\ \vdots \\ D_m \end{bmatrix} .$$

Finally for each set of m integers $(s_1, \ldots, s_m)$ with $s_j \in \underline{q_j}$, $j \in \underline{m}$, we define the m x m submatrix $D_{(s_1,\ldots,s_m)}$ of D

$$D_{(s_1,\ldots,s_m)} = \begin{pmatrix} C_{1s_1} A^{d_{1s_1}} B \\ \vdots \\ C_{ms_m} A^{d_{ms_m}} B \end{pmatrix} = \begin{pmatrix} D_{1s_1} \\ \vdots \\ D_{ms_m} \end{pmatrix} .$$

That is, the submatrix $D_{(s_1,\ldots,s_m)}$ consists of one row from each submatrix D_i, $i \in \underline{m}$.

(4.4.4) Proposition: The condition (4.4-7),

$$Z_{js} \cap (N + Z_j^*) = 0, \quad \text{for } s \in \underline{q_j}, \text{ and } j \in \underline{m}$$

is true if and only if

i) for every set $(s_1,\ldots,s_m)$ with $s_j \in \underline{q_j}$, $j \in \underline{m}$, the submatrix $D_{(s_1,\ldots,s_m)}$ has rank m;

ii) for every $i \in \underline{m}$, the submatrix D_i^* has rank $m - 1$.

Proof: (Necessity): For any set $(s_1,\ldots,s_m)$ with $s_j \in \underline{q_j}$, $j \in \underline{m}$,

choose the set of m vectors $\{z_{1s_1},\ldots,z_{ms_m}\}$, $z_{js_j} = (A^T)^{d_{js_j}} C^T_{js_j}$.

From (4.4-7) we have $Z_{js_j} \cap (N + Z_j^*) = 0$ for $s_j \in \underline{q_j}$, $j \in \underline{m}$, where of course $Z_{js_j} = \text{Span}\{z_{js_j}\}$. Hence by a construction similar to that used to show $\dim(N + Z_i^*) = n-1$, (4.4-8), we may show

$$\dim(z_{1s_1} + \ldots + z_{ms_m} + N) = n \ . \tag{4.4-14}$$

Indeed

$$\dim(z_{1s_1} + \ldots + z_{ms_m} + N) = \dim z_{1s_1} + \dim(z_{2s_2} + \ldots + z_{ms_m} + N)$$

$$- \dim(z_{1s_1} \cap (z_{2s_2} + \ldots + z_{ms_m} + N)),$$

but the last term is zero by (4.4-7). Continuing in a like manner (4.4-14) is established.

Now from (4.4-14) and the fact $N = \text{Ker } B^T$,

$$\dim B^T = \dim(B^T(z_{1s_1} + \ldots + z_{ms_m})) = m,$$

which of course is equivalent to

$$\text{rank } (B^T[z_{1s_1}; \ldots ; z_{ms_m}]) = m. \tag{4.4-15}$$

But $B^T[z_{1s_1}; \ldots ; z_{ms_m}] = D^T_{(s_1, \ldots, s_m)}$, and thus (4.4-15) implies i).

From (4.4-8) we have $\dim(N + Z_i^*) = n-1$, $i \in \underline{m}$, whence

$$\dim B^T(N + Z_i^*) = \dim B^T Z_i^* = \text{rank}(D_i^*)^T \leq m-1, \ i \in \underline{m}.$$

But from (4.4-15)

$$\text{rank}(B^T[z_{1s_1}; \ldots ; z_{i-1,s_{i-1}}; z_{i+1,s_{i+1}}; \ldots ; z_{ms_m}]) = m-1, \quad i \in \underline{m}. \tag{4.4-16}$$

Since the matrix in (4.4-16) is a submatrix of $(D_i^*)^T$, the desired result, $\text{rank } (D_i^*)^T = \text{rank } D_i^* = m-1$, $i \in \underline{m}$ follows.

(sufficiency): If $D_{(s_1,\ldots,s_m)}$ is of full rank for all sets $(s_1,\ldots,s_m)$ with $s_j \in \underline{q_j}$, $j \in \underline{m}$, then (4.4-14) holds, i.e.

$$\dim(z_{1s_1} + \ldots + z_{ms_m} + N) = n.$$

For $D^T_{(s_1,\ldots,s_m)} = B^T[z_{1s_1};\ldots;z_{ms_m}]$, hence the vectors $B^Tz_{1s_1},\ldots,B^Tz_{ms_m}$ must be independent. But this implies $z_{1s_1},\ldots,z_{ms_m}$ are independent of $N = \text{Ker } B^T$, and thus (4.4-14) holds. Also

$$\text{rank } D_i^* = \text{rank}(D_i^*)^T = \dim(B^TZ_i^*) = m-1, \quad i \in \underline{m}.$$

Now from (4.4-14) we have

$$\dim(N + Z_i^*) \geq n-1, \quad i \in \underline{m}.$$

But since

$$\dim(B^T(N + Z_i^*)) = \dim(B^TZ_i^*) < \dim B^T, \quad i \in \underline{m},$$

it follows that $N + Z_i^* \neq R^n$, and thus

$$\dim(N + Z_i^*) = n-1 \qquad i \in \underline{m}.$$

As

$$z_{is_i} + \sum_{\substack{j\neq i\\ j\in\underline{m}}} z_{js_j} + N \subset z_{is_i} + Z_i^* + N, \quad i \in \underline{m}$$

(4.4-14) implies

$$\dim(z_{is_i} + Z_i^* + N) = n, \quad i \in \underline{m}.$$

Then we have

$$\dim(Z_{is_i} \cap (N + Z_i^*)) = \dim Z_{is_i} + \dim(N + Z_i^*) - \dim(Z_{is_i} + N + Z_i^*)$$

$$= 1 + n - 1 - n = 0, \quad \text{for } s_i \in \underline{q_i}, \quad i \in \underline{m}, \quad \blacksquare$$

which establishes (4.4-7) and the proposition.

Combining the preceeding three results we have a strong necessary condition for decoupling.

(4.4.5) <u>Theorem</u>: A linear system may be decoupled into m single input, (possibly) multiple output subsystems only if conditions i) and ii) of Prop. 4.4.4 hold.

The proof of Theorem 4.4.5 follows directly from Prop. 3.2.8 and the preceeding four results of this section.

Proposition 4.4.4 provides us with a readily implementable yet strong test which must be satisfied before a linear system may be decoupled into single input, multiple output subsystems. Condition i) says that every m-input, m-output subsystem of the original system, consistent with the desired partition of the outputs must be decoupleable into m single input, single output subsystems itself. Indeed the transfer function for a system decoupled into single input, multiple output subsystems would necessarily be of the form

$$H(\lambda;F,G) = \text{block diagonal } [h_1(\lambda;F,G),\ldots,h_m(\lambda;F,G)]$$

with $h_i(\lambda;F,G)$ $q_i \times 1$, $i \in \underline{m}$. Control of the outputs requires that for each $i \in \underline{m}$, every component of $h_i(\lambda,F,G)$ is non-zero. Thus for every m element subset of the q outputs, $(y_{1s_1},\ldots,y_{ms_m})$, where $s_i \in \underline{q_i}$, $i \in \underline{m}$, the corresponding rows of $H(\lambda;F,G)$ form an m x m diagonal nonsingular matrix.

From Chapter 2, Section 2, we recall that the feedback invariant $d_{is_i} + 1$, $s_i \in \underline{q_i}$, $i \in \underline{m}$ denotes the minimum time delay for the effect of any input to be visible at output y_{is_i}, while the row vector D_{is_i} represents the first nontrivial pointwise mapping from inputs to output y_{is_i}. Clearly these quantities are unaltered by feedback. Now if D_i^* has rank m for any $i \in \underline{m}$, it follows that the initial non-zero responses of all m inputs affect the m-1 output subvectors $(y_1,\ldots,y_{i-1}, y_{i+1},\ldots,y_m)$. Since this situation cannot be remedied by a feedback control law (F,G), it follows that the system may not be decoupled.

From the results on generic solvability of decoupling problems, Theorem 3.5.1, we recognize that linear systems are not generically decoupleable into single input, multiple output subsystems. (We showed decoupling is almost always possible only if $q_i = 1$, $i \in \underline{k}$ when $k = m$.) This we would expect as the requirements of Prop. 4.4.4 are less likely to hold as the number of outputs, q, and hence the number of components per suboutput vector, q_i, $i \in \underline{m}$, increases.

Unlike Prop. 4.2.2 where the equivalence of the geometric and input-output conditions was shown, Theorem 4.4.5 states only a necessary condition. Indeed there is not sufficient information inherent in an augmented D matrix to develop a complete solution based only on the matrix D, for this more general decoupling problem. In Chapter 2, Section 2, we showed that for the problem considered by Falb and Wolovich, D nonsingular was sufficient to guarantee the existence of a feedback law (F,G) such that

$$\{A+BF|B_i*G\} \subset \text{Ker } C_i, \quad i \in \underline{m} \tag{4.4-16}$$

That is the effect of all inputs other than i could be localized to Ker C_i, $i \in \underline{m}$. This could be accomplished, even though the effect of feedback on A was limited by the image of B, as the subspaces Ker C_i, were all of dimension n-1.

However for the problem considered in this section, conditions i) and ii) of Prop. 4.4.4 are not sufficient to yield a feedback law such that (4.4-16) holds. The augmented D matrix contains information about the initial non-zero pointwise input-output maps, and if i) and ii) hold, then the initial non-zero responses at the outputs may be "decoupled". But since dimension Ker C_i may be less than n-1, the number of inputs may not be sufficient to afford enough feedback freedom to guarantee that subsequent outputs will be "decoupled". That is, the initial non-zero response of output y_{is_i}, $y_{is_i}(d_{is_i}+1)$, will be due solely to the effect of input i, but subsequent outputs, $y_{is_i}(d_{is_i}+j)$, $j > 1$, may be affected by inputs other than i.

In the second example to follow, we demonstrate systems with indentical augmented D matrices satisfying i) and ii) of Prop. 4.4.4. However we shall see that by changing parameters which do not affect D, we may have some of these systems decoupleable, others not. This example shows the matrix D itself contains insufficient information to yield a complete solution to this problem.

At this point, let us examine several examples to help clarify the results presented here. Consider the system given in Chapter 3, Section 5:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}^T, \quad C_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T .$$

We may readily construct the augmented D matrix as $d_{11} = d_{21} = 0$ and $d_{12} = 1$,

$$D = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ \hline 0 & 1 \end{pmatrix} .$$

However as the submatrix $D_2{}^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has rank 2, this system cannot be decoupled. Indeed, as was shown in Chapter 3, Section 5, for this example $\overline{R}_2 = 0$.

Let us now consider the system represented by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}^T, \quad C_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}^T$$

where the a_{ij}'s are temporarily unspecified. We may readily construct the augmented D matrix as it is independent of A, (all the d_{ij}'s are zero)

$$D = \begin{pmatrix} 1 & 0 \\ \hdashline 0 & 1 \\ 0 & 1 \end{pmatrix} \quad ,$$

and note that both requirements of Prop. 4.4.4 are satisfied. As we might expect, the existence of a decoupling feedback law will hinge on the values given to the elements of A.

From the D matrix we may note that the initial non-zero response of output y_{11}, is due solely to input 1, while those of outputs y_{21} and y_{22} are due solely to output 2. Hence to decouple this system we need only find a matrix F such that $\{A+BF|\mathcal{B}_1\} \subset \text{Ker } C_2$, and $\{A+BF|\mathcal{B}_2\} \subset \text{Ker } C_1$, as any decoupling input gain G will necessarily be diagonal. Assume a most general feedback map

$$F = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \\ f_5 & f_6 & f_7 & f_8 \end{bmatrix} \quad .$$

Then it is immediately established that no F exists such that $\{A+BF|\mathcal{B}_1\} \subset \text{Ker } C_2$ for all possible A as

$$C_2(A+BF)B_1 = \begin{pmatrix} a_{32} + a_{42} + f_6 \\ a_{42} + f_6 \end{pmatrix} \quad ,$$

and hence in particular this system cannot be decoupled if $a_{32} \neq 0$.

Now fix $a_{12} = a_{14} = a_{21} = a_{24} = a_{34} = a_{44} = 1$, and set all the other elements of A to zero. It follows that (A,B) is controllable. Now set $f_1 = f_2$, $f_4 = -2$, $f_3 = f_5 = f_6 = 0$. Then

$$A + BF = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1+f_1 & f_1 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & f_7 & 1+f_8 \end{pmatrix}$$

and we have

$$\{A+BF|\mathcal{B}_1\} = \text{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ f_1 \\ 0 \\ 0 \end{pmatrix}\right\} \subset \quad \text{Ker } C_2$$

$$\{A+BF|\mathcal{B}_2\} = \text{Span}\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1+f_8 \end{pmatrix}, \begin{pmatrix} f_8 \\ -f_8 \\ 1+f_8 \\ f_7+(1+f_8)^2 \end{pmatrix}\right\} \subset \text{Ker } C_i.$$

Clearly then the system (A+BF,B,C) is decoupled.

As a final example consider the fifth order, three input, four output linear system represented by the matrix triple (A,B,C),

$$A = \begin{pmatrix} 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 1 & 2 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 & 1 \\ -3 & 0 & 1 & 0 & 2 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 3 \end{pmatrix}, \; C_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}^T,$$

$$C_2 = \begin{pmatrix} 0 \\ 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}^T, \quad C_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 2 \\ 0 & 1 \end{pmatrix}^T.$$

It may be verified that this system is controllable, and that the augmented D matrix is

$$D = \begin{pmatrix} 1 & -1 & 0 \\ \hline 0 & 1 & 2 \\ \hline 1 & 0 & 2 \\ 1 & 1 & 4 \end{pmatrix}$$

with $d_{11} = d_{21} = d_{32} = 0$, $d_{31} = 1$. We note that the submatrices $D_i{}^*$, $i \in \underline{3}$ are all of rank 2, hence condition ii) of Prop. 4.4.4 is satisfied. However, $D_{(1,1,1)}$ is singular as is $D_{(1,1,2)}$ and thus this system may not be decoupled.

Theorem 4.4.5 extends in a natural way to the problem of decoupling a linear system into k multiple input, multiple output subsystems. In this extension of the problem originally considered in this section, we again assume k output subvectors, $y_i = C_i x$, $i \in \underline{k}$, but now we allow the number of inputs, m, to exceed k, $m \geq k$. For these more general problem formulations to be solvable, conditions similar to i) and ii) of Prop. 4.4.4 must hold.

Indeed for such a system we may readily construct an augmented $q \times m$ D matrix, with submatrices $D_i{}^*$, $i \in \underline{m}$ and $D_{(s_1,\ldots,s_k)}$, $s_i \in \underline{q_i}$, $i \in \underline{k}$. Since the rows of D are feedback invariants, it follows that if the system in question is decoupleable, then the initial responses of any set of k outputs $(y_{s_1},\ldots,y_{s_k})$ with $s_i \in \underline{q_i}$, $i \in \underline{k}$ must be determined by k independent inputs. In other words the submatrix $D_{(s_1,\ldots,s_k)}$ of D must contain a $k \times k$ nonsingular submatrix.

In light of our results on decoupling systems with an excess of inputs in Chapter 2, Section 2, this is quite reasonable. For condition i) of Prop. 4.4.4 says that every problem of the type considered by Falb and Wolovich consistently imbedded in the original problem must be solvable. For the case $m \geq k$, this becomes every m-input, k-output subproblem consistently imbedded must be solvable. By Theorem 2.3.5 it follows that every submatrix $D_{(s_1,\ldots,s_k)}$ must have rank k.

By reasoning identical to that of the discussion following Theorem 4.4.5, if for any $i \in \underline{k}$ the submatrix D_i^* has rank m, then the initial non-zero responses of all m inputs affect the k-1 output subvectors $(y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_k)$. Of course, if this happens, the system may not be decoupled.

(4.4.6) Corollary: A linear system with $m \geq k$, $q_i \geq 1$, $i \in \underline{k}$ may be decoupled into k multiple input, multiple output subsystems only if

i) for every set $(s_1,\ldots,s_k)$ with $s_i \in \underline{q_i}$, $i \in \underline{k}$, the submatrix $D_{(s_1,\ldots,s_k)}$ has rank k.

ii) for every $i \in \underline{k}$, the submatrix D_i^* has rank not exceeding m-1.

The proof of Corollary 4.4.6 follows directly from the preceeding discussion.

CHAPTER 5

TOWARDS STRUCTURES OF (A,B) INVARIANT AND CONTROLLABILITY SUBSPACES

5.1 Introduction

Although the geometric theory of Wonham and Morse has proved to yield significant insight into the fundamental aspects of decoupling, the basic elements of this theory, (A,B) invariant and controllability subspaces, are not well characterized. Indeed there has been little developed in the way of structural properties of the sets of these subspaces, except for the calculation of specified elements of those sets. If we wish to address questions about the sensitivity of decoupling solutions to variations in the parameters of the linear maps A,B,C, then we must pursue these questions of structure. If we wish to understand why systems may not be developed, then we must ascertain the constraints, if any, on the construction of controllability subspaces.

In this chapter we shall begin to build a structure for (A,B) invariant and controllability subspaces. We will examine the space of all i.s. of a given pair of matrices (A,B,), and show how this is associated with that of pairs $(\bar{A},\bar{B})$ related to (A,B) by similarity, feedback or input transformations. Further, we will show that controllability subspaces have a natural representation in terms of elements of the kernel of the polynomial matrix $[\lambda I-A \;\vdots\; -B]$. Using this analogy we are able to determine the possible dimensions of the c.s. of a given system in terms of the Kronecker invariants of the matrix pair (A,B). These constraints tie in naturally to the problem of generic solvability of the decoupling problem as explored by Fabian and Wonham [F1]. Finally, we indicate a method for generating minimal dimension c.s. which cover or contain subspaces of $\mathcal{B}$.

5.2 The Structure of $\mathcal{I}_{A,B}$

The space of (A,B) invariant subspaces (i.s.), which we designate $\mathcal{I}_{A,B}$ affords us little of the wealth of structure associated with the invariant subspaces of a single linear transformation ([B5], [D4]). Clearly, the sum of (A,B) i.s. is again an element of $\mathcal{I}_{A,B}$, but as we have already seen (Prop. 3.3.1) only under certain conditions will the intersection of (A,B) i.s. again be (A,B) invariant.

So far our interest in the structure of invariant subspaces has been primarily algebraic in nature. However at this point we shall introduce a metric on $\mathcal{I}_{A,B}$. It is not that we wish to exploit topological structure of $\mathcal{I}_{A,B}$; indeed as we show, very little exists. Rather we find this metric useful in relating the invariant subspaces of different matrix pairs (A,B) themselves related by certain transformations. It then becomes straightforward to show that we need only consider systems with A and B in a particular canonical form to determine the decoupleability of all linear systems.

A standard metric on the subspaces of a Hilbert space is called the gap ([B3], [D4], [K4]) and defined as follows. Let $\mathcal{M}$, $\mathcal{N}$ be subspaces of a finite dimensional Euclidean space $\mathcal{H}$ with $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$

the orthogonal projections on M and N respectively. Then $\Theta(M,N)$, the gap between M and N, is given by $||P_M - P_N||$, where $||\cdot||$ is the usual linear operator norm,

$$||A|| = \sup_{||x|| < 1} ||Ax||$$

We note immediately that $0 \leq \Theta(M,N) \leq 1$ and that $\Theta(M,N) < 1$ implies $\dim M = \dim N$.

With the gap as the metric, the space of i.s. of a given linear transformation forms a complete metric space. (See [D4] for further details.) In contrast, it is easily established that $(I_{A,B},\Theta)$ need not be closed, and hence not complete. Consider the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and the sequence of subspaces

$$M_j = \left\{ \begin{pmatrix} 2^{-j} \\ 1 \end{pmatrix} \right\} \qquad j = 1,2,\ldots$$

For every j, $M_j \in I_{A,B}$, but $\lim_{j\to\infty} M_j \notin I_{A,B}$. Since the sequence $\{M_j\}$ is easily shown to be Cauchy, the space is not complete.

Further $I_{A,B}$ need not be open in the space of subspaces of R^n. Indeed if we let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

then $V_1 = \{(0\ 0\ 1)^T\} \in I_{A,B}$, but $V_2 = \{(\alpha\ 0\ 1)^T\} \notin I_{A,B}$ for any $\alpha \neq 0$. Note that this implies that the linear system defined by A and B above, with

$$C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad C_2 = (0 \quad 0 \quad 1)$$

may be decoupled, whereas if

$$C_1 = \begin{pmatrix} 1 & 0 & -\alpha \\ 0 & 1 & 0 \end{pmatrix}, \qquad C_2 = (0 \quad 0 \quad 1), \qquad \alpha \neq 0$$

then the resulting system cannot be decoupled.

We can however derive a relationship between I_{A_1,B_1} and I_{A_2,B_2} when the matrix pairs (A_1,B_1) and (A_2,B_2) are related by a similarity transformation, that is, there exists a nonsingular S such that $A_2 = S\,A_1S^{-1}$, $B_2 = S\,B_1$.

(5.2.1) Proposition: Let the matrix pairs (A_1,B_1) and (A_2,B_2) be related by a similarity transformation S. Then I_{A_1,B_1} is homeomorphic to I_{A_2,B_2}.

To prove the proposition we make use of a technical lemma from [D4].

(5.2.2) Lemma: (Douglas and Pearcy): For i = 1,2, let M_i be a subspace of a Hilbert space H, and let T_i be linear operators on H satisfying $||T_ix|| \geq \varepsilon_i\ ||x||$ $(\varepsilon_i > 0)$ for all $x \in M_i$. If $N_i = T_iM_i$, then N_i is closed, and

$$||P_{N_1}-P_{N_2}|| \leq \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)||T_1 - T_2|| + \left(\frac{1}{\varepsilon_1}||T_2|| + \frac{1}{\varepsilon_2}||T_1||\right)||P_{M_1}-P_{M_2}||.$$

Proof of Prop. 5.2.1: Consider the map $S: I_{A_1,B_1} \to I_{A_2,B_2}$ defined $S(M) = S\,M$. This map is well defined and into and from Lemma 5.2.2 with $T_1 = T_2 = S$, we have

$$||P_{SM_1}-P_{SM_2}|| \leq 2||S||\ ||S^{-1}||\ ||P_{M_1}-P_{M_2}||,$$

$$||P_{M_1}-P_{M_2}|| \leq 2||S||\ ||S^{-1}||\ ||P_{SM_1}-P_{SM_2}||,$$

which implies that S is an homeomorphism. ∎

Now let us broaden our investigation slightly, and consider triples of matrices (A,B,C) and $(\overline{A},\overline{B},\overline{C})$ related by a similarity transformation S. If C_i is a $q_i \times n$ submatrix of C, then the corresponding submatrix $\overline{C}_i$ of $\overline{C}$ is given by C_iS^{-1}. Further if N_i is the kernel of C_i, then SN_i is the kernel of $\overline{C}_i$. Nothing that $S(N_i \cap N_j) = SN_i \cap SN_j$ since S is 1:1, it is clear that $M \subset \bigcap_{j\varepsilon \underline{k}} N_j$ implies $SM \subset \bigcap_{j\varepsilon \underline{k}} SN_j$. Combining this fact with Prop. 5.2.1 we have the following result.

(5.2.3) Corollary: Let the triples (A,B,C) and $(\overline{A},\overline{B},\overline{C})$ be related by a similarity transformation S. If $\{R_1,\ldots,R_k;F\}$ is a set of c.s. and a compatible feedback map which solves the decoupling problem for A,B, and a given partition of C, C_i, $i\ \varepsilon\ \underline{k}$, then $\{SR_1,\ldots,SR_{\underline{k}};FS^{-1}\}$ solves the decoupling problem for $\overline{A},\overline{B}$, and a partition $\overline{C}_i = C_iS^{-1}$, $i\ \varepsilon\ \underline{k}$ of $\overline{C}$.

Proof: Follows directly from preceding arguments and the fact that $R_i = \{A + BF|B \cap R_i\}$ if and only if $SR_i = \{\overline{A} + \overline{B}FS^{-1}|\overline{B} \cap SR_i\}$, $i\ \varepsilon\ \underline{k}$. ∎

Turning our attention to input change of basis transformations, we see immediately that for $\overline{B} = BG$ with det $G \neq 0$, $I_{A,B} \equiv I_{A,\overline{B}}$. Thus it follows that if $\{R_1,\ldots,R_k;F\}$ is a solution to the decoupling problem for (A,B,C), then $\{R_1,\ldots,R_k;G^{-1}F\}$ is a solution for $(A,\overline{B},C)$.

Finally, let us consider the effect of state feedback on a system representation. We say two matrix triples (A,B,C) and $(\overline{A},\overline{B},\overline{C})$ are feedback equivalent if $\overline{A} = A + BT$, $\overline{B} = B$, $\overline{C} = C$ for any map $T:R^n \to R^m$. Then again it follows that $I_{A,B}$ and $I_{\overline{A},\overline{B}}$ are identical and if $\{R_1,\ldots,R_k;F\}$ is a solution to the decoupling problem for (A,B,C), then $\{R_1,\ldots,R_k;F-T\}$ is a solution for $\{\overline{A},\overline{B},\overline{C}\}$.

If we expand slightly the approach of Kalman [K2], Brunovsky [B7], and Popov [P1], we may consider similarity, input, and feedback transformations as acting on matrix triples (A,B,C):

$$(A,B,C) \to (SAS^{-1},SB,CS^{-1}) \quad \det S \neq 0 \quad \text{(similarity)}$$

$$(A,B,C) \to (A,BG,C) \quad \det G \neq 0 \quad \text{(input)}$$

$$(A,B,C) \to (A+BF,B,C). \quad \text{(feedback)}$$

The combined action of these transformations constitute a group Γ acting on matrix triples (A,B,C) and defines equivalence classes of elements called orbits of Γ. In this context, the preceding arguments imply

(5.2.4) Corollary: If (A,B,C) and $(\overline{A},\overline{B},\overline{C})$ are elements of the same orbit of Γ then the (A,B,C) is decoupeable if and only if $(\overline{A},\overline{B},\overline{C})$ is.

Brunovsky, Kalman and Popov have shown that if the group Γ is restricted to controllable pairs (A,B) with B of full column rank, then the resulting orbits are completely characterized by a set of integers $\{\nu_1,\ldots,\nu_m\}$ known as Kronecker invariants. This leads directly to a canonical form for such pairs (A,B) in which

$$A = \text{block diagonal } [A_1,\ldots,A_m]$$

$$B = \text{block diagonal } [B_1,\ldots,B_m].$$

The blocks A_i are of dimension $\nu_i \times \nu_i$ $i \in \underline{m}$, and consist of ones on the super diagonal and zeros elsewhere. The blocks B_i are of dimension $\nu_i \times 1$ $i \in \underline{m}$, and are given by $B_i = [0 \;.\;.\;.\; 0\; 1]^T$.

The structure of the Brunovsky form immediately yields a decomposition of the state space R^n into singly generated, independent controllability subspaces. This fact has previously been observed by Wonham and Morse [W8]. In the succeeding sections of this chapter we will further exploit these connections.

At this point we note again the canonical form for integratator decoupled systems suggested by Gilbert [G2] (see Chapter 2, Section 2). It is clear that if a system is integrator decoupled, controllable and observable, then the A and B matrices of Gilbert's canonical form coincide with Brunovsky's canonical form, and the set of feedback invariants $\{d_i + 1,\; i \in \underline{m}\}$ is identical to the set of Kronecker invariants $\{\nu_i,\; i \in m\}$, save perhaps order. As we have already seen that any decoupleable m-input, m-output linear system may be integrator decoupled, it follows that for such controllable and observable systems, if decoupling does not destroy observability, the Kronecker invariants are a complete set of invariants of the triple (A,B,C) modulo action of the group Γ.

5.3 A Characterization of Controllability Subspaces

(The majority of the results in the remaining sections of this chapter have been derived jointly with A.E. Eckberg. See [W1].)

In this section we will develop an algebraic characterization of the controllability subspaces of a matrix pair (A,B), where for concreteness we assume that B has full column rank and (A,B) is controllable. We shall show that the concept of a c.s. has a natural analog in terms of the kernel of a particular polynomial matrix. Furthermore, certain invariants of this polynomial matrix are shown to lead quite naturally to the canonical form for controllable pairs developed by Brunovsky [B7], Kalman [K2], Rosenbrock [R1], and Wolovich and Falb [W4].

(5.3.1) Lemma: If $\mathcal{R}$ is a c.s., then for every nonzero $b \in \mathcal{B} \cap \mathcal{R}$ there exists a matrix F such that

$$\{A + BF | b\} = \mathcal{R} \tag{5.3-1}$$

and

$$(A + BF)^{r} b = 0 \tag{5.3-2}$$

where $r = \dim \mathcal{R}$.

Proof: This is a special case of Theorem 4.2 in [W7]. We are choosing F so that $\mathcal{R}$ is cyclic with respect to A + BF, with generator b, and so that $(A + BF)|\mathcal{R}$ is nilpotent. The latter condition is possible by the pole assignment properties of c.s. ∎

Lemma 5.3.1 leads to an interesting characterization of controllability subspaces.

(5.3.2) Proposition: A subspace $\mathcal{R} \subset R^n$ of dimension r is a c.s. if and only if there exist $x(\lambda) \in R^n[\lambda]$ and $u(\lambda) \in R^m[\lambda]$ such that

(i) $\deg u(\lambda) = k$ and $\deg x(\lambda) = k - 1$, for some $k \geq r$

(ii) $(\lambda I - A)x(\lambda) = Bu(\lambda)$

(iii) If $x(\lambda) = \sum_{i \in \underline{k}} \lambda^{i-1} x_{i-1}$, then $\mathcal{R} = \text{Span}\,\{x_{i-1}, i \in \underline{k}\}$.

Proof: (Necessity): Suppose $\mathcal{R}$ is a c.s. of dimension r. Let $b \in \mathcal{B} \cap \mathcal{R}$ and F be chosen to satisfy (5.3-1) and (5.3-2). Define

$u(\lambda) = \sum_{i=0}^{r} \lambda^i u_i \in R^m[\lambda]$ and $x(\lambda) = \sum_{i=0}^{r-1} \lambda^i x_i$ so that

$$Bu_r = b$$

$$u_i = F(A + BF)^{r-i-1} b, \quad 0 \leq i \leq r-1$$

$$x_i = (A + BF)^{r-i-1} b, \quad 0 \leq i \leq r-1$$

Then (i) is trivially satisfied; (ii) follows by comparing coefficients of powers of λ, and from (5.3-2); (iii) follows from (5.3-1).

(Sufficiency): Let $u(\lambda) \in R^m[\lambda]$ and $x(\lambda) \in R^n[\lambda]$ satisfy (i) - (iii). We shall demonstrate that

$$A\mathcal{R} \subset \mathcal{R} + \mathcal{B} \qquad (5.3\text{-}3)$$

and that

$$R = W_k \tag{5.3-4}$$

where $W_0 = 0$, and $W_i = (AW_{i-1} + B) \cap R$ for $i \in \underline{k}$. The result will then follow from Theorem 4.1 in [W7], which shows that this iterative procedure will converge to a c.s. (See also (3.2.6).)

From (ii) it is easily seen that $Ax_i = x_{i-1} - Bu_i$ for $1 \leq i \leq k-1$, and that $Ax_0 = - Bu_0$; thus (5.3-3) follows from (iii).

To demonstrate (5.3-4), define subspaces S_i as $S_i = \text{Span}\ \{x_{k-1}, x_{k-2}, \ldots, x_{k-i}\}$, for $i \in \underline{k}$. Since $x_{k-1} = Bu_k$ it follows that $S_1 = \text{Span}\ \{x_{k-1}\} \subset B \cap R = W_1$. Moreover, from (ii) it is easily seen that $S_i \subset (AS_{i-1} + B) \cap R$ for $2 \leq i \leq k$, whence it follows inductively that $S_i \subset W_i$ for all $i \in \underline{k}$. But clearly, $S_k = R$, and (5.3-4) follows. ∎

5.3.3 Remark: If a pair $(x(\lambda), u(\lambda))$ can be found which satisfies conditions (i) - (iii) of Prop. 5.3.2, and additionally the coefficients of $x(\lambda)$ are independent, then one can find a feedback matrix F such that $Fx_{i-1} = u_{i-1}$ for all $i \in \underline{k}$. It then follows that $x_{k-1} \in B \cap R$ is a cyclic generator for R with respect to the matrix $A + BF$.

If we define the set $S \subset R^n[\lambda] \times R^m[\lambda]$ as $S = \{x(\lambda), u(\lambda) \mid (\lambda I - A)x(\lambda) = Bu(\lambda)$ and the coefficients of $x(\lambda)$ are independent$\}$, then each element of S defines a unique c.s. Conversely, every c.s. determines at least one member of S, the nonuniqueness arising from the choice of the generating element $b \in B \cap R$.

We have thus established a characterization of controllability subspaces in terms of elements of $\mathrm{Ker}(\lambda I - A; -B)$, where the matrix $(\lambda I - A; - B)$ is to be interpreted as representing an $R[\lambda]$-module morphism: $R^{m+n}[\lambda] \to R^{n}[\lambda]$. Elements in $\mathrm{Ker}(\lambda I - A; -B)$ generate a submodule of $R^{m+n}[\lambda]$ which may in turn be characterized by the minimal column indices $\{\nu_i, i \in \underline{m}\}$ and a fundamental series $\{z_i(\lambda), i \in \underline{m}\}$ associated with the singular pencil of matrices $(\lambda I - A; -B)$. These two sets are determined as follows (see [Gl, Vol. II, Ch. 12]):

(i) Let ν_1 be the least degree of all nonzero elements of $\mathrm{Ker}(\lambda I - A; - B)$, and choose $z_1(\lambda) \in \mathrm{Ker}(\lambda I - A; -B)$ so that $\deg z_1(\lambda) = \nu_1$

(ii) For each i, $1 \leq i \leq m-1$, after having chosen $\{z_j(\lambda), j \in \underline{i}\}$ we define ν_{i+1} to be the least degree of all elements $z(\lambda) \in \mathrm{Ker}(\lambda I - A; -B)$ such that $z(\lambda)$ is not an element of the submodule generated by the set $\{z_j(\lambda), j \in \underline{i}\}$. Then choose $z_{i+1}(\lambda) \in \mathrm{Ker}(\lambda I - A; -B)$ so that $\deg z_{i+1}(\lambda) = \nu_{i+1}$ and so that $z_{i+1}(\lambda)$ is not an element of the submodule generated by $\{z_j(\lambda), j \in \underline{i}\}$.

We shall call the set $\{\nu_i, i \in \underline{m}\}$, so obtained, the Kronecker invariants of the pair (A,B). Note that by the construction of this set, the ν_i's are ordered as $0 \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_m$. The sets $\{\nu_i, i \in \underline{m}\}$ and $\{z_i(\lambda), i \in \underline{m}\}$ enjoy other properties, which we now state below.

(5.3.4) Proposition: Let $(A,B) \in R^{nxn} \times R^{nxm}$ be a controllable pair such that rank $B = m$. Then the Kronecker invariants and the fundamental series, as determined above, satisfy:

(i) The set $\{\nu_i, i \in \underline{m}\}$ is well-defined and unique;

(ii) $\nu_i > 0$, all $i \in \underline{m}$;

(iii) $\sum_{i \in \underline{m}} \nu_i = n$;

(iv) $\{z_i(\lambda), i \in \underline{m}\}$ is a set of free generators for $\mathrm{Ker}(\lambda I - A; -B)$, and any $z(\lambda) \in \mathrm{Ker}(\lambda I - A; -B)$ can be uniquely written as

$$z(\lambda) = \sum_{i:\nu_i \leq \deg z(\lambda)} z_i(\lambda)\alpha_i(\lambda)$$

for appropriate $\alpha_i(\lambda) \in R[\lambda]$ such that $\deg \alpha_i(\lambda) \leq \deg z(\lambda) - \nu_i$;

(v) The fundamental series $\{z_i(\lambda), i \in \underline{m}\}$ is not uniquely determined; however, for each i such that $\nu_i < \nu_{i+1}$, the submodule $M_i \triangleq$ (submodule generated by $\{z_j(\lambda), j \in \underline{i}\}$) is invariant with respect to the choice of fundamental series;

(vi) If each $z_i(\lambda)$ is partitioned as $z_i(\lambda) = (s_i^T(\lambda); t_i^T(\lambda))^T$, where $t_i(\lambda) \in R^m[\lambda]$ and $s_i(\lambda) \in R^n[\lambda]$, then $\deg s_i(\lambda) = \nu_i - 1$ and the collections of coefficients $\{t_{i,\nu_i}, i \in \underline{m}\}$ and $\{s_{ij}; 0 \leq j \leq \nu_i - 1, i \in \underline{m}\}$ are bases for R^m and R^n.

Proof: (i) This is clear from the statement of the algorithm.

(ii) If $\nu_1 = 0$, then there exists a $z_1(\lambda)$ of degree 0, hence $z_1(\lambda) = (s_1^T; t_1^T)^T$ with $s_1 \varepsilon R^n$, $t_1 \varepsilon R^m$, such that $[\lambda I-A; -B]z_1 = 0$. But this requires that $s_1 = 0$ and $Bt_1 = 0$, a contradiction since B is of full column rank. By the inherent ordering of the ν_i's (ii) is true.

(iii) This follows from the assumption that (A,B) is a controllable pair, and is most easily seen via an alternative derivation of the Kronecker invariants (See [B7], [K2], or [P1].)

(iv) See Eckberg [E1] or Forney [F4] for a proof.

(v) We note that if $j > 1$ then $\overline{z}_j(\lambda) = \sum_{i \varepsilon \underline{j}} z_i(\lambda)$ is an element of Ker $[\lambda I - A; -B]$, independent of M_{j-1}, and of degree ν_j. Thus we may replace $z_j(\lambda)$ by $\overline{z}_j(\lambda)$ and retain an acceptable fundamental series. If $\nu_i = \nu_{i+1}$, then we may interchange $z_i(\lambda)$ and $z_{i+1}(\lambda)$ without affecting the validity of the fundamental series, but obviously changing the submodule M_i. (See [G1] for further details.)

(vi) The first part follows trivially from $(\lambda I - A)s_i(\lambda) = Bt_i(\lambda)$. The formation of bases for R^n and R^m has been shown indirectly by Wolovich and Falb [W4] and directly by Eckberg [E1]. ∎

Now consider the pair $(s_i(\lambda), t_i(\lambda))$, as determined by $z_i(\lambda)$. This pair of polynomial vectors satisfies $(\lambda I - A)s_i(\lambda) = Bt_i(\lambda)$. Thus, from statement (vi) of Prop. 5.3.4 and from Remark 5.3.3, it follows that Span $\{s_{ij}, 0 \leq j \leq \nu_i-1\}$ is a c.s. generated by s_{i,ν_i-1}.

We call this c.s. $\mathcal{R}_i$:

$$\mathcal{R}_i \triangleq \text{Span}\{s_{ij},\ 0 \le j \le \nu_i - 1\},\ \text{for } i \in \underline{m}. \qquad (5.3\text{-}5)$$

Since $\{s_{ij};\ 0 \le j \le \nu_i - 1,\ i \in \underline{m}\}$ is a basis for R^n by Prop. 5.3.4 (vi)

$$R^n = \mathcal{R}_1 \oplus \ldots \oplus \mathcal{R}_m. \qquad (5.3\text{-}6)$$

However, because the fundamental series $\{z_i(\lambda),\ i \in \underline{m}\}$ is not unique, the decomposition of R^n via (5.3-6) is not unique. In spite of this fact, there are certain properties of the decomposition (5.3-6) which are invariant with respect to the choice of fundamental series $\{z_i(\lambda),\ i \in \underline{m}\}$.

(5.3.5) <u>Proposition</u>: The subspaces $\mathcal{V}_i = \mathcal{R}_1 \oplus \ldots \oplus \mathcal{R}_i$ for which $\nu_i < \nu_{i+1}$ are invariant with respect to the choice of fundamental series $\{z_i(\lambda),\ i \in \underline{m}\}$.

Proof: From Prop. 5.3.4 (iv)-(vi), it is easily seen that when $\nu_i < \nu_{i+1}$, Span $\{s_{kj};\ 0 \le j \le \nu_k - 1,\ k \in \underline{i}\}$ is invariant with respect to the choice of fundamental series $\{z_i(\lambda),\ i \in \underline{m}\}$. This proves the proposition. ∎

We note that the subspace $\mathcal{V}_i = \mathcal{R}_1 \oplus \ldots . \oplus \mathcal{R}_i$ for which $\nu_i < \nu_{i+1}$ has another interpretation as the maximal c.s. contained in the subspace $A^{-\nu_i}(\mathcal{B} + \ldots + A^{\nu_i - 1}\mathcal{B})$. This latter subspace consists of all $x \in R^n$ which may be driven to zero in at most ν_i steps. Clearly $\mathcal{V}_i$ is a c.s. and every element of $\mathcal{V}_i$ may be driven to zero in ν_i or less steps. In contrast, by (5.3-6) any c.s. not contained in V_i contains an element which may not be driven to zero in ν_i steps.

Finally, from the fundamental series $\{z_i(\lambda) = (s_i^T(\lambda);t_i^T(\lambda))^T, i \in \underline{m}\}$ we may determine a feedback matrix F such that the fundamental series associated with the controllable pair (A + BF, B) is of a particularly simple form; this will then lead to the Brunovsky canonical form for (A,B). We define F as follows. Since $\{s_{ij}\}$ is a basis for R^n, there exists a matrix F such that

$$Fs_{i,j} = t_{i,j};\ 0 \leq j \leq \nu_i-1,\ i \in \underline{m}.$$

It now follows easily that

$$(\lambda I - A - BF)s_i(\lambda) = B\,\lambda^{\nu_i}\, t_{i,\nu_i},\ \text{for each } i \in \underline{m}.$$

This last relation completely specifies the maps $A + BF : R^n \to R^n$ and $B : R^m \to R^n$ with respect to the bases $\{s_{i,j}\}$ (in R^n) and $\{t_{i,\nu_i}\}$ (in R^m). That is,

$$Bt_{i,\nu_i} = s_{i,\nu_i-1},\ i \in \underline{m}$$

while

$$(A + BF)s_{i,j} = \begin{cases} s_{i,j-1};\ \text{if } 1 \leq j \leq \nu_i-1,\ i \in \underline{m} \\ 0;\ \ \text{if } j = 0,\ i \in \underline{m}. \end{cases}$$

Thus, with nonsingular matrices S and G defined as

$$S = (s_{1,0};s_{1,1};\ \ldots;\ s_{1,\nu_1-1};s_{2,0};\ldots;s_{m,\nu_m-1})$$

and

$$G = (t_{1,\nu_1};\ldots;t_{m,\nu_m})$$

it follows that

$$S^{-1}BG = (e_{\nu_1+};\ e_{\nu_2+};\ \ldots;\ e_{\nu_m+})$$

$$S^{-1}(A + BF)S = \text{Block diagonal } (H_{\nu};\ldots;H_{\nu_m})$$

where H_k is k x k with ones on the superdiagonal and zeros elsewhere. (Recall that for an ordered set $\{\nu_i,\ i \in m\}$, $\nu_j^+ = \sum_{i \leq j} \nu_i$.)

We note immediately that the pair $(S^{-1}(A + BF)S,\ S^{-1}BG)$ is the Brunovsky canonical form for the pair (A,B). In the remaining sections of this chapter it will be convenient to work with this canonical form. We note that the fundamental series associated with $(S^{-1}(A + BF)S,\ S^{-1}BG)$ is $\{z_i(\lambda) = (s_i^T(\lambda);t_i^T(\lambda))^T,\ i \in \underline{m}\}$ with

$$t_i(\lambda) = \lambda^{\nu_i}\hat{e}_i \quad i \in \underline{m}$$

$$s_i(\lambda) = \lambda^{\nu_i - 1} e_{\nu_i+} + \ldots + e_{\nu_i+ - \nu_i + 1} \quad i \in \underline{m} \tag{5.3-7}$$

where $\hat{e}_i$ is the i^{th} standard unit vector in R^m. For this choice of fundamental series the subspaces $\mathcal{R}_i \subset R^n$ are given as

$$\mathcal{R}_i = \text{Span}\,\{e_{\nu_i+},\ldots,\ e_{\nu_i+ - \nu_i + 1}\}, \quad i \in \underline{m}. \tag{5.3-8}$$

5.4 Dimensions of Controllability Subspace

We now consider a controllable pair (A,B) in the Brunovsky canonical form. Define the projection on $\mathcal{R}_i$ along $\bigoplus_{j\neq i} \mathcal{R}_j$ by P_i, $i \in \underline{m}$, and the set $\{j \in \underline{m} | P_j\mathcal{R} \neq 0\}$ by $M(\mathcal{R})$ for any subspace $\mathcal{R}$. Then we have the following bound on the dimension of a c.s.

(5.4.1) Lemma: Let $\mathcal{R}$ be a c.s. of the pair (A,B). Then $\dim \mathcal{R} \geq \max\{\nu_j \mid j \in M(\mathcal{R})\}$.

Proof: If $j \in M(\mathcal{R})$, then $P_j\mathcal{R} \neq 0$. However by (5.3-6), (5.3-8) and the assumed canonical form for (A,B), this implies $P_j\mathcal{R} = \mathcal{R}_j$. Since the projection P_j cannot raise dimension, $\nu_j = \dim \mathcal{R}_j \leq \dim \mathcal{R}$, proving the lemma. ■

We note that this lemma is the state space analog of Prop. 5.3.4 (iv). As the $z_i(\lambda)$ $i \in \underline{m}$ are a set of free generators of $\mathrm{Ker}[\lambda I - A; -B]$, the c.s. $\mathcal{R}_i$, $i \in \underline{m}$ constitute a set of independent "building blocks" for constructing c.s. of (A,B). To carry the analogy one step further, we may compare addition of elements of the fundamental series with addition of the c.s. they spawn. Also, we may liken the product of λ and $z_i(\lambda)$ to the c.s. $\mathcal{R}_i$ advanced one unit of time.

The dimension of a c.s. may be similarly bounded from above.

(5.4.2) Lemma: Let $\mathcal{R}$ be a c.s. of the pair (A,B). Then $\dim \mathcal{R} \leq \sum_{j\in M(\mathcal{R})} \nu_j$.

Proof: Clearly $P_i\mathcal{R} = 0$ for $i \notin M(\mathcal{R})$. Then $\sum_{i\notin M(\mathcal{R})} P_i \mathcal{R} = 0$ or equivalently $\mathcal{R} \subset \mathrm{Ker}\left(\sum_{i \notin M(\mathcal{R})} P_i\right) = \sum_{j \in M(\mathcal{R})} \mathcal{R}_j$. Since the $\mathcal{R}_j$'s are independent, this latter subspace has dimension $\sum_{j \in M(\mathcal{R})} \nu_j$ and contains $\mathcal{R}$, proving the lemma. ∎

The preceding two lemmas allow us now to state and prove the main result of this section.

5.4.3 Theorem: Let $V = \{\nu_1, \ldots, \nu_m\}$ be the set of Kronecker invariants of the controllable pair (A,B). Then there exists a c.s. $\mathcal{R}$ of dimension p if and only if

$$\max\{\nu_i \mid \nu_i \in U\} \leq p \leq \sum_{\nu_i \in U} \nu_i \tag{5.4-1}$$

for some subset U of V.

Proof: (Necessity): Let $U = \{\nu_i \mid i \in M(\mathcal{R})\}$. Then the result is immediate from the preceeding two lemmas.

(Sufficiency): Given a subset $U \subset V$ and a p satisfying (5.4-1) we shall construct a c.s. $\mathcal{R}$ of dimension p by summing the c.s. $\mathcal{R}_i$, $i \in \underline{m}$ possibly with some "overlap." First order the elements of U in decreasing size $(\nu_{i_1}, \ldots, \nu_{i_k})$ and define s as the smallest integer such that $\eta_s = \sum_{j \leq s} \nu_{i_j}$ is greater than or equal to p. If $p = \eta_s$ then we may construct a c.s. of dimension p by forming the direct sum of c.s. $\mathcal{R}_{i_1} \oplus \ldots \oplus \mathcal{R}_{i_s}$. If η_s exceeds p, then for $s > 2$, we shall construct a c.s. of the form $\mathcal{R}_{i_1} \oplus \ldots \oplus \mathcal{R}_{i_{s-2}} \oplus \mathcal{Q}$, where $\mathcal{Q}$ is a c.s. of dimension

$p - \eta_{s-2}$ obtained by "overlapping" the c.s. $\mathcal{R}_{i_{s-1}}$ and $\mathcal{R}_{i_s}$. Finally if $\eta_s > p$ and $s = 2$, then we shall construct a c.s. of the form $\mathcal{Q}$ above. Hence it suffices to consider only the cases $p = \eta_s$ for some s, or $\eta_1 < p < \eta_2$.

To prove the first case we need only show how to form the direct sum of two c.s., say $\mathcal{R}_i \oplus \mathcal{R}_j$. Consider the feedback which changes the $(\nu_j+, \nu_i+ -\nu_i + 1)$ element of A from zero to one. With this feedback, $(A + BF)^{\nu_i} e_{\nu_i+} = e_{\nu_j+}$; that is the c.s. $\mathcal{R}_i \oplus \mathcal{R}_j$ is generated by e_{ν_i+}, the generator of $\mathcal{R}_i$.

To prove the second case let $p < \nu_i + \nu_j$. By choosing feedback such that the $(\nu_j+, \nu_i+ - (p - \nu_j - 1))$ element of A is changed from zero to one, it is easily seen that the resulting c.s. generated by e_{ν_i+} is spanned by set of p independent vectors

$$\Big\{e_{\nu_i+},\ e_{\nu_i+ -1},\dots,e_{\nu_i+ -p+\nu_j} + e_{\nu_j+},\dots,e_{\nu_i+ -\nu_i+1}$$

$$+ e_{\nu_j+ -\nu_i-\nu_j+p+1},\ e_{\nu_j+ -\nu_i-\nu_j+p},\dots,e_{\nu_j+ -\nu_j+1}\Big\}$$

and hence is of dimension p.

As an example consider the matrix pair (A,B) with Kronecker invariants 2 and 3,

$$A = \left(\begin{array}{cc|ccc} 0 & 1 & & \underline{0} & \\ 0 & 0 & & & \\ \hline & & 0 & 1 & 0 \\ & \underline{0} & 0 & 0 & 1 \\ & & 0 & 0 & 0 \end{array}\right) \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Let the feedback map F be given by

$$F = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

Then for $b = (0\ 0\ 0\ 0\ 1)^T$, $\{A + BF|b\}$ is a c.s. of dimension four.

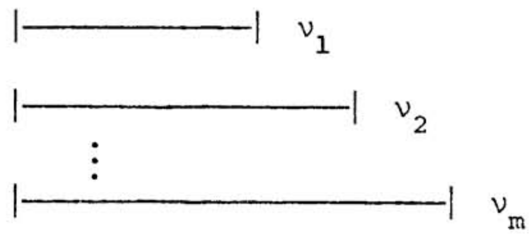
The constructions used to prove sufficiency of Theorem 5.4.3 also have an analog in the context of Prop. 5.3.2. Let $[s_i^T(\lambda);\ t_i^T(\lambda)]^T$ be the i^{th} free generator of Ker$[\lambda I - A; -B]$, that is, R_i is the span of the coefficients of $s_i(\lambda)$. Then for any $k \geq 0$ it is easily seen that

$$(\lambda I - A)(\lambda^k s_i(\lambda) + s_j(\lambda)) = B(\lambda^k t_i(\lambda) + t_j(\lambda)).$$

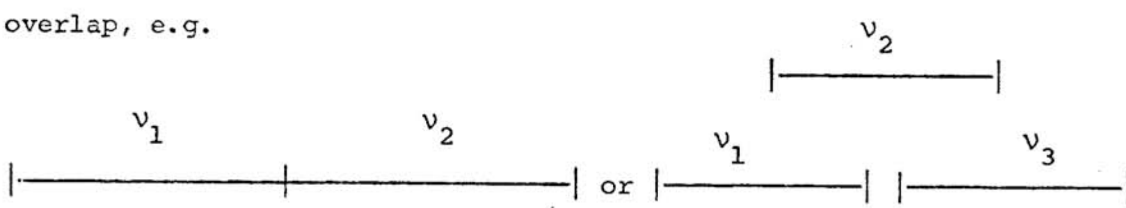
When $k = \nu_j$, the span of the coefficients of $\lambda^{\nu_j} s_i(\lambda) + s_j(\lambda)$ yield the c.s. $R_i \oplus R_j$. For $0 < k < \nu_j$ the span of the coefficients of $\lambda^k s_i(\lambda) + s_j(\lambda)$ is a c.s. of dimension $\nu_i + k$ of the form Q in the proof of the theorem. It should be clear that these constructions are not unique. We may replace λ^k by $\alpha(\lambda)$, any polynomial of degree k, and achieve controllability subspaces, albeit possibly different ones, of the appropriate dimensions.

Theorem 5.4.3 indicates that the possible dimensions of controllability subspaces of a pair (A,B) in Brunovsky form are directly determined by the Kronecker invariants of (A,B). However, the construction of a canonical representation for any controllable pair (A,B) with B of full rank is accomplished solely by similarity, input and feedback transformations. From the results of Section 5.2 we know that the dimensions of (A,B) i.s. are unchanged by such transformations, hence the dimensions of c.s. remain invariant, and the theorem is equally applicable to the more general pairs (A,B).

A more basic visual analogy of Theorem 5.4.3 might be useful at this point. If one considers the Kronecker invariants to be represented by line segments of length ν_i, $i \in \underline{m}$



then the theorem states that the corresponding dimensions of possible c.s. are given by the lengths of line segments obtained by joining together some of the above line segments, with the possibility of integral overlap, e.g.



etc.

(5.4.4) <u>Corollary</u>: If for some $j \in \underline{m-1}$, $\nu_{j+1} > \nu_j^+ + 1$, then there exists no c.s. of dimension p, where p is any integer satisfying $\nu_j^+ < p < \nu_{j+1}$.

Proof: This follows directly from Theorem 5.4.3 as any subset of the set $U = \{\nu_i | i \leq j\}$ clearly fails the upper bound in (5.4-1), while if U includes any elements ν_k for $k > j$, it likewise fails the lower bound in (5.4-1). For example, if the Kronecker invariants of (A,B) are (1,2,5) then c.s. of dimensions 1,2,3,5,6,7,8 exist, whereas no c.s. of dimension 4 exists. ∎

The results of the theorem together with Prop. 5.3.5 point to the uniqueness of c.s. of particular dimensions.

(5.4.5) <u>Corollary</u>: Let $\mathcal{R}$ be a c.s. of dimension p. Then $\mathcal{R}$ is the unique c.s. of dimension p iff $p = \nu_j^+ < \nu_{j+1}$ for some $j \in \underline{m-1}$. In particular, if $\nu_2 \neq \nu_1$, then $\mathcal{R}_1$ is the unique c.s. of dimension ν_1.

Proof: (Sufficiency): Assume $\dim \mathcal{R} = p = \nu_j^+ < \nu_{j+1}$. By Lemma 5.4.1 $P_i\mathcal{R} = 0$ for $i > j$, so $\mathcal{R} \subset \bigcap_{i>j} \operatorname{Ker} P_i = \bigoplus_{k\leq j} \mathcal{R}_k$. Since $\dim \bigoplus_{k\leq j} \mathcal{R}_k = \nu_j^+$ we must have $\mathcal{R} = \bigoplus_{k\leq j} \mathcal{R}_k$. However, regardless of the non-uniqueness of the set of c.s. $\mathcal{R}_k$, the subspace $\bigoplus_{k\leq j} \mathcal{R}_k$ is unique by Prop. 5.3.5 since $\nu_{j+1} > \nu_j$.

(Necessity): We will show that if there exists any c.s. of dimension p such that either $p \neq \nu_j^+$ or $p = \nu_j^+ \nless \nu_{j+1}$ for any $j \in \underline{m}$, then there are, in general, many different c.s. of identical dimension.

Consider first the case $p \neq \nu_j^+$ for any $j \in \underline{m}$. Assume there exists a c.s. of dimension p. Define k to be the largest integer such that $\nu_k^+ < p$, and let $q = p - \nu_k^+$. Then by the remarks following Theorem 5.4.3 it is clear that given the element $u_\alpha(\lambda)$ of $R^m[\lambda]$

$$u_\alpha(\lambda) = \lambda^{\nu^+_{k-1}+q} t_k(\lambda) + \lambda^{\nu^+_{k-2}+q} t_{k-1}(\lambda) + \ldots + \lambda^q t_1(\lambda) + \alpha t_{k+1}(\lambda)$$

for any non-zero $\alpha \in R$, the span of the coefficients of the corresponding $x_\alpha(\lambda)$ is a c.s. of dimension p. Further, since $q < \nu_{k+1}$, it is clear that for $\alpha \neq \beta$, both non-zero, the spans of the coefficients of $x_\alpha(\lambda)$ and $x_\beta(\lambda)$ differ.

Now assume $p = \nu_j^+$ for some $j \in \underline{m}$, but $\nu_j^+ \geq \nu_{j+1}$. Clearly the polynomial vector

$$u_0(\lambda) = \lambda^{\nu^+_{j-1}} t_j(\lambda) + \lambda^{\nu^+_{j-2}} t_{j-1}(\lambda) + \ldots + t_1(\lambda)$$

has an associated $x_0(\lambda)$, the span of whose coefficients is given by $\mathcal{R}_1 \oplus \ldots\ldots \oplus \mathcal{R}_j$, a c.s. of dimension p. However the polynomial vector

$$u_\alpha(\lambda) = \lambda^{\nu^+_{j-1}} t_j(\lambda) + \lambda^{\nu^+_{j-2}} t_{j-1}(\lambda) + \ldots + t_1(\lambda) + \alpha t_{j+1}(\lambda)$$

where $\alpha \in R$, has an associated $x_\alpha(\lambda)$ whose coefficients span a different c.s. of dimension p. Further, if $\alpha \neq \beta$, then the c.s. associated with $u_\alpha(\lambda)$ differs from that associated with $u_\beta(\lambda)$. ∎

Note that we have shown that for systems defined over the reals (or any other uncountably infinite field), if there exists more than one c.s. of a given dimension, then there exists an uncountable number. In the example with Kronecker invariants 1,2, and 5, the c.s. of dimension 1,3, and 8 are unique , while there are nondenumerably many c.s. of dimension 2,5,6, and 7.

5.5 Dimensional Constraints and Generic Decoupling

The results of the previous section indicate the constraints placed upon the construction of controllability subspaces of arbitrary dimension. The conditions necessary for decoupling such as non-interaction and output control put additional constraints of dimension on sets of c.s. which might form a solution. It is clear that whenever the former constraints are in conflict with the latter set, no decoupling solution exists.

The work of Fabian and Wonham [F1] on the generic solvability of decoupling by dynamic compensation (see Chapter 3, Section 4) has shown that given the conditions

$$n \geq \sum_{i \in \underline{k}} q_i \qquad (5.5\text{-}1)$$

$$m \geq 1 + \sum_{i \in \underline{k}} q_i - \min_{i \in \underline{k}} q_i \qquad (5.5\text{-}2)$$

where q_i is the rank of C_i, $i \in \underline{k}$, it is possible to find a set of c.s. $\{\hat{\mathcal{R}}_i \ i \in \underline{k}\}$ such that

$$\hat{\mathcal{R}}_i \subset \bigcap_{\substack{j \in \underline{k} \\ j \neq i}} \mathcal{N}_j \triangleq \mathcal{K}_i \,, \qquad i \in \underline{k} \qquad (5.5\text{-}3)$$

$$\hat{R}_i + N_i = R^n \qquad i \in \underline{k} \qquad (5.5\text{-}4)$$

where $N_i = \text{Ker } C_i$, for almost all parameter sets (A,B,C) of suitable dimension. Now for a given set $\{q_i,\ i \in \underline{k}\}$ there may exist a set of Kronecker invariants $\{\nu_i,\ i \in \underline{m}\}$ such that (5.5-3) or (5.5-4) cannot hold for some $i \in \underline{k}$. Thus it follows that whenever the Kronecker invariants determine dimensions of c.s. in conflict with the requirement imposed by the set $\{q_i,\ i \in \underline{k}\}$, the systems in question are not generically decoupleable in the sense of Fabian and Wonham, i.e. either (5.5-1) or (5.5-2) fails.

From Corollaries 5.4.4 and 5.4.5 we know there is no c.s. of the pair (A,B) of dimension p if for some $j \in \underline{m-1}$

$$\nu_j^+ < p < \nu_{j+1} \qquad (5.5\text{-}5)$$

and there is a unique c.s. of dimension p if for some $j \in \underline{m-1}$

$$\nu_j^+ = p < \nu_{j+1}. \qquad (5.5\text{-}6)$$

Let q_i^* denote $\sum_{\substack{j \in \underline{k} \\ j \neq i}} q_j$. Then since it is assumed that C has full row rank,

$$\dim K_i = n - \dim K_i = n - q_i^*. \qquad (5.5\text{-}7)$$

As $\dim N_i = n - q_i$, the dimensional constraints implied by (5.5-3) and (5.5-4) are given by

$$n - q_i^* \geq \dim \hat{R}_i \geq q_i \qquad i \in \underline{k}. \qquad (5.5\text{-}8)$$

It then follows from (5.5-5) that there exists no c.s. satisfying (5.5-8) for some $i \in \underline{k}$ if for some $j \in \underline{m-1}$

$$\nu_{j+1} > n - q_i^* \geq q_i > \nu_j^+ \tag{5.5-9}$$

and there exists a unique c.s. if for some $j \in \underline{m-1}$

$$\nu_{j+1} > n - q_i^* = q_i = \nu_j^+. \tag{5.5-10}$$

(5.5.1) Proposition: If (5.5-9) or (5.5-10) hold, then (5.5-2) cannot be satisfied. Conversely if (5.5-2) holds, then (5.5-9) and (5.5-10) both fail.

Proof: (Note that we have dispensed with condition (5.5-1) as it is trivially implied by the assumption that C has full row rank.) If (5.5-9) or (5.5-10) hold for some $i \in \underline{k}$ and some $j \in \underline{m-1}$, then we have

$$\nu_{j+1} > n - q_i^*. \tag{5.5-11}$$

Defining ν_j^* as $\sum_{\substack{i \in \underline{m} \\ i \neq j}} \nu_i$, (5.5-11) becomes

$$q_i^* > n - \nu_{j+1} = \nu_{j+1}^*. \tag{5.5-12}$$

Since $\nu_j \geq 1$ for all $j \in \underline{m}$, clearly $\nu_j^* \geq m-1$, $j \in \underline{m}$, whence (5.5-12) becomes

$$q_i^* > m-1 \tag{5.5-13}$$

in direct contradiction to (5.5-2).

Assume (5.5-2) holds now. Then

$$m > q_i^* \qquad i \in \underline{k}, \tag{5.5-14}$$

whence

$$\nu_j^* \geq m-1 \geq q_i^* \qquad i \in \underline{k}, \quad j \in \underline{m}. \qquad (5.5\text{-}15)$$

But since $\nu_j^* = n - \nu_j$, (5.5-15) implies

$$n - q_i^* \geq \nu_j \qquad i \in \underline{k}, \quad j \in \underline{m}, \qquad (5.5\text{-}16)$$

and thus (5.5-9) and (5.5-10) cannot hold. ∎

The condition (5.5-11), when true, means that there is a basic c.s. (one whose dimension may not be diminished by feedback) which is too large for the desired output decomposition, hence noninteraction may not be achieved. Generically, all the Kronecker invariants will be either $\left[\frac{n}{m}\right]$ or $\left[\frac{n}{m}\right] + 1$, where [a] denotes the greatest integer less than or equal to a, and there will be considerable freedom in the construction of controllability subspaces. As long as the number of inputs is sufficiently large vis-a-vis the number of independent outputs, i.e. (5.5-2) is satisfied, this freedom will be sufficient to guarantee c.s. that allow output control (5.5-4) while achieving noninteraction (5.5-3).

It should be pointed out that we have not shown that generic solvability of the decoupling problem in the sense of Fabian and Wonham follows whenever the sets $\{\nu_j, j \in \underline{m}\}$ and $\{q_i, i \in \underline{k}\}$ yield no conflicting dimensional constraints. Indeed this need not be the case as a simple example with $m = k = \nu_1 = \nu_2 = q_1 = q_2 = 2$ will readily verify.

5.6 Minimal Dimension Controllability Subspaces

In Section 5.3 a characterization of controllability subspaces in terms of the free generators of the kernel of the singular pencil of matrices $[\lambda I - A; -B]$ was developed. Using this representation, requirements on the possible dimensions of c.s. were derived in Section 5.4. In this final section we wish to explore c.s. constrained to contain or cover a given subspace. In particular we will construct minimal dimension c.s. covering subspaces of $\mathcal{B}$.

Our motivation for examining this problem is twofold. First Wonham and Morse [W8] have considered the construction of minimum dimension (A,B) i.s. which contain a given subspace and then have shown as an application, how one may determine an observer of minimal dimension for a single linear functional of the state. The main result of this section is actually encompassed by Theorem 2.1 of [W8], but the approach here is sufficiently distinct to merit exposition.

Secondly, one might wish to consider a variation on the decoupling problem where output controllability ($\mathcal{R}_i + \mathcal{N}_i = R^n$, $i \in \underline{k}$) was required, but strict noninteraction was not essential. That is, we might search for a compatible set of c.s. $\{\mathcal{R}_i, i \in \underline{k}\}$ which allow complete control of the outputs and then determine if the interaction associated with this set is acceptable. Although we cannot characterize all c.s. $\mathcal{R}_i$ satisfying $\mathcal{R}_i + \mathcal{N}_i = R^n$, it is clear that if $\mathcal{R}_i$ contains $\mathcal{N}_i$ output control is assured. To minimize interaction we seek to find minimal dimension c.s. covering given subspaces.

Consider an element $b \in \mathcal{B}$. If $\mathcal{R}$ is an c.s. containing b, then by Lemma 5.3.1 there exists a feedback map F such that

$$\mathcal{R} = \text{Span } \{b,(A + BF)b,\ldots (A + BF)^{n-1}b\},$$

and $(A + BF)^n b = 0$. Combining this fact with the characterization of c.s. in terms of the pencil of matrices, we may view any c.s. $\mathcal{R}$ containing b as the span of a trajectory generated by driving b to zero. Clearly then, the minimal dimension c.s. containing b are in 1:1 correspondence with the spans of the trajectories arising from driving b to zero in a minimal number of steps, i.e. the spans of trajectories $\{b,(A + BF)b,\ldots\}$ that contain a minimal number of non-zero vectors.

It should be noted that driving a vector x to zero in r steps implies the construction of an input string $\{u_{r-1},\ldots,u_0\}$ such that if

$$x_{r-1} = x \quad \text{and} \quad x_{r-i-1} = Ax_{r-i} + Bu_{r-i} \qquad 1 \leq i \leq r$$

then $x_{-1} = 0$. If $x \in \mathcal{B}$, this is of course equivalent to finding

$u(\lambda) = \sum_{i=0}^{r} \lambda^i u_i$ and $x(\lambda) = \sum_{i=0}^{r-1} \lambda^i x_i$ such that $Bu_r = x$ and

$(\lambda I - A)x(\lambda) = Bu(\lambda)$. It surely suffices to find a feedback map F such that if

$$x_{r-1} = x \quad \text{and} \quad x_{r-i-1} = (A + BF)x_{r-i} \qquad 1 \leq i \leq r$$

then $x_{-1} = 0$.

If we wish to drive an element $b \in \mathcal{B}$ to zero in a minimal number of steps, it seems natural that the span of the trajectory excluding b, should be independent of $\mathcal{B}$. This is indeed the case.

(5.6.1) <u>Lemma:</u> Let $b \in \mathcal{B}$. If there exists a feedback map F and a trajectory $\{b, (A + BF)b, \dots, (A + BF)^{n-1}b\}$ such that

$\sum_{i=1}^{r} \alpha_i (A + BF)^i b$, $\alpha_r \neq 0$, is an element of β for $1 \leq r \leq n-1$, then b may be driven to zero in r or fewer steps.

Proof: We write $\hat{b} = \sum_{i=1}^{r} \alpha_i (A + BF)^i b$ and assume without loss of generality that $\alpha_r = 1$. Since B has full rank there exist unique elements $\hat{u}$ and u such that $\hat{u} = B^{-1}\hat{b}$, $u = B^{-1}b$. Consider the input string u_i, $r - 1 \geq i \geq 0$ defined by

$$u_{r-1} = Fb + \alpha_{r-1} u$$

$$u_{r-2} = F(A + BF)b + \alpha_{r-1}Fb + \alpha_{r-2} u$$

$$\vdots$$

$$u_0 = F(A + BF)^{r-1}b + \alpha_{r-1}F(A + BF)^{r-2}b + \dots + \alpha_1 Fb - \hat{u}.$$

Now let $x_{r-1} = b$, and consider the sequence generated by the recursion

$$x_{r-i-1} = Ax_{r-i} + Bu_{r-i} \qquad 1 \leq i \leq r.$$

Then it follows that

$$x_{-1} = (A + BF)^r b + \alpha_{r-1}(A + BF)^{r-1}b + \dots + \alpha_1(A + BF)b - \hat{b} = 0,$$

and hence b may be driven to zero in r steps. ∎

Lemma 5.6.1 implies that for $b \in \mathcal{B}$, if $\mathcal{R}$ is a c.s. of minimum dimension containing b, then $\mathcal{R} \cap \mathcal{B} = b$, and thus $\mathcal{R}$ is uniquely generated by the element b. It is now natural to ask: What is the minimum dimension of a c.s. covering an element $x \in R^n$? For $x \in \mathcal{B}$, we can easily answer this query.

Recall that $M(\mathcal{R})$ was defined as the set $\{j \in \underline{m} | P_j\mathcal{R} \neq 0\}$ for any subspace $\mathcal{R}$.

(5.6.2) Lemma: Let $b \in \mathcal{B}$. Then the minimum dimension of a c.s. containing b is given by $\mu = \max \{\nu_j | j \in M(b)\}$.

Proof: If $\mathcal{R}$ is a c.s. containing b, then by Lemma 5.4.1 dim $\mathcal{R} \geq \mu$. Now consider the trajectory $(b, Ab, \ldots A^{n-1}b)$, where A,B are assumed in the Brunovsky canonical form. Since $b = \sum_{j \in M(b)} \gamma_j e_{\nu_j^+}$ and $A^{\nu_j} e_{\nu_j^+} = 0$, it follows that $A^{\mu}b = 0$, yielding a covering c.s. of dimension μ. ∎

We now can turn our attention to the case where we desire to cover an arbitrary subspace of $\mathcal{B}$. As we shall need a minor construction, we first prove a lemma to motivate that construction.

(5.6.3) Lemma: Let b_1 and b_2 be elements of $\mathcal{B}$ such that $M(b_1) \cap M(b_2) = \phi$, and denote $\max\{\nu_j | j \in M(b_i)\}$ by μ_i, $i \in \underline{2}$. Then if $\mathcal{R}$ is a c.s. covering b_1 and b_2, dim $\mathcal{R} \geq \mu_1 + \mu_2$.

Proof: If $\mathcal{R}$ contains b_1 and b_2, then $\mathcal{R}$ contains c.s. which may be generated by b_1 and b_2 respectively (by Lemma 5.3.1). Then it follows that for some F_1, F_2

$$\text{Span}\{b_1,(A+BF_1)b_1,\ldots,(A+BF_1)^{n-1}b_1,b_2,\ldots,(A+BF_2)^{n-1}b_2\} \subset \mathcal{R}.$$

Recalling that $\{s_{i,\nu_i-1},\ i \in \underline{m}\}$ is a basis for $\mathcal{B}$, we may write

$$b_1 = \sum_{j\in M(b_1)} \gamma_{1j} s_{j,\nu_j-1} \text{ and } b_2 = \sum_{j\in M(b_2)} \gamma_{2j} s_{j,\nu_j-1}.$$

Since the lemma obviously follows for $\mu_1 = \mu_2 = 1$, we may assume that for some $i \in \underline{2}$ $\mu_i > 1$. Now for $\mu_i > 1$ we have

$$(A+BF_i)b_i = \sum_{j\in M(b_i)} \gamma_{ij} s_{j,\nu_j-2} + \sum_{k\in\underline{m}} \alpha_{ik1} s_{k,\nu_k-1} \neq 0 \quad (5.6\text{-}1)$$

for some α_{ik1}, $k \in \underline{m}$. Note that the second term on the right is an element of $\mathcal{B}$ and represents the arbitrary nature of the feedback map F_i. Continuing, we have for $\mu_i > r$

$$(A+BF_i)^r\, b_i = \sum_{j\in M(b_i)} \gamma_{ij} s_{j,\nu_j-r-1} + \sum_{p\in\underline{r}}\ \sum_{k\in\underline{m}} \alpha_{ikp} s_{k,\nu_k-(r+1-p)} \neq 0 \quad (5.6\text{-}2)$$

for some α_{ikp}, $k \in \underline{m}$, $p \in \underline{r}$, where $s_{i,j} \triangleq 0$ for $j < 0$. Comparing the forms of elements from (5.6-1) and (5.6-2), it follows from the hypothesis $M(b_1) \cap M(b_2) = \phi$ and the fact that $\{s_{i,j};\ 0 \le j \le \nu_i-1,\ i \in \underline{m}\}$ is a basis for R^n, that the vectors

$$\{b_1,(A+BF_1)b_1,\ldots,(A+BF_1)^{\mu_1-1}b_1,b_2,\ldots,(A+BF_2)^{\mu_2-1}b_2\}$$

are independent, and hence $\dim \mathcal{R} \ge \mu_1 + \mu_2$, proving the lemma. ∎

Let b_1 and b_2 be elements of $\mathcal{B}$ such that $M(b_1) \cap M(b_2) = \phi$. If $\mathcal{V}_1$ and $\mathcal{V}_2$ are minimal dimension covering c.s. for b_1 and b_2 respectively, then by Lemmas (5.6.2) and (5.6.3) it follows that $\mathcal{V}_1 \cap \mathcal{V}_2 = 0$, and $\mathcal{V}_1 \oplus \mathcal{V}_2$ is a minimal dimension c.s. covering $\mathrm{Span}\{b_1, b_2\}$.

(5.6.4) <u>Theorem</u>: Let $\mathcal{D} \subset \mathcal{B}$ and $\{b_1,\ldots,b_k\}$ be a basis for $\mathcal{D}$ such that $M(b_i) \cap M(b_j) = \phi$ for $i \neq j$, $i, j \in \underline{k}$. Then if $\mathcal{R}$ is a c.s. covering $\mathcal{D}$, $\dim \mathcal{R} \geq \sum_{i \in \underline{k}} \mu_i$. Furthermore, if $\mathcal{V}_i$ is a minimal dimension c.s. covering b_i, $i \in \underline{k}$, then $\{\mathcal{V}_i, i \in \underline{k}\}$ is an independent set of subspaces, and $\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_k$ is a minimal dimension c.s. containing $\mathcal{D}$.

Proof: First we note that any $\mathcal{D} \subset \mathcal{B}$ has such a basis. Let $\{d_1,\ldots,d_k\}$ be any basis for $\mathcal{D}$ and let D be a matrix whose columns are given by the d_i, $i \in \underline{k}$, with respect to the basis for $\mathcal{B}$, $\{s_{i,\nu_i-1}, i \in \underline{m}\}$.

Then by applying elementary column operations, it is possible to transform D to a matrix D_0 whose columns are a basis for D and have the desired property (only one non-zero entry per row).

By expanding the argument of Lemma 5.6.3 to the case where for appropriate $F_1,\ldots,F_k$

$$\mathrm{Span}\{b_1,\ldots,(A + BF_1)^{n-1}b,\ldots,b_k,\ldots,(A + BF_k)^{n-1}b_k\} \subset \mathcal{R},$$

it is straightforward to show that $\mathcal{R}$ contains a subspace of dimension

$\sum_{i \varepsilon \underline{k}} \mu_i$. Furthermore, it is clear that $\sum_{i \varepsilon \underline{k}} \mathcal{V}_i$ is a c.s. covering $\mathcal{D}$, hence $\dim (\sum_{i \varepsilon \underline{k}} \mathcal{V}_i) \geq \sum_{i \varepsilon \underline{k}} \mu_i = \sum_{i \varepsilon \underline{k}} \dim \mathcal{V}_i$, which implies that the $\mathcal{V}_i$, $i \varepsilon \underline{k}$ are independent subspaces. ∎

The general case of finding a minimal dimension c.s. containing an arbitrary subspace of R^n seems a more formidable task, even if we are restricted to arbitrary one dimensional subspaces. The key to the solution of the problem when we consider subspaces of $\mathcal{B}$, is Lemma 5.6.1 which allows us to focus attention to c.s. which are generated by a unique element. There seems to be no direct analog of this lemma when arbitrary one dimensional subspaces are permitted.

CHAPTER 6

CONCLUSIONS

In this dissertation we have considered the problem of decoupling linear, time invariant, multivariable systems by state feedback control laws. The basic fruitful approaches of Falb and Wolovich, and Wonham and Morse have been presented, and extended in some cases. The major contributions of this research are:

1) The development of a strong, yet easily implemented necessary condition in the form of the original Falb and Wolovich result, for the decoupling of linear systems into single input, multiple output subsystems;

2) A characterization of controllability subspaces in terms of the elements of the kernel of the singular pencil of matrices $(\lambda I - A; -B)$, and the determination of the possible dimensions of controllability subspaces of a matrix pair (A,B) in terms of the Kronecker invariants of (A,B).

Other lesser results include:

3) An extension of the results of Falb and Wolovich to the case of decoupling a system with more inputs than outputs;

4) A method for decomposing arbitrary (A,B) invariant subspaces into simpler invariant subspaces contained in the kernel of a linear form;

5) A proof that it is possible to decouple almost all m-input, m-output linear systems into scalar input, scalar output subsystems by a state feedback control law;

6) A method for constructing minimal dimension controllability subspaces which contain arbitrary subspaces of the image of the input gain map B.

The necessary conditions for decoupling into single input, multiple output subsystems (Prop. 4.4.4) are not surprising. It would be nice however, to extend that result and achieve a complete, easily implemented analog of the geometric condition (Prop. 3.2.8). Indeed the general problem of decoupling by state feedback remains unsolved and constitutes an important area for future research.

Another area where more work would be of value involves determining the sensitivity of decoupling solutions to parameter perturbations as well as the sensitivity of the algorithms used to compute maximal invariant and controllability subspaces.

For systems which cannot be decoupled it might prove fruitful to quantize the interaction between unassociated subsets of inputs and outputs, and then consider finding feedback control laws which minimize this interaction. Efforts in this direction have been without significant success.

The results of Chapter 5 begin to explore the structure of, and provide a convenient characterization for controllability subspaces. In addition they provide a basic link between the notion of controllability subspaces and basic structural properties of linear systems. Indeed it is not surprising that the Kronecker invariants completely determine the possible dimensions of c.s. For as was shown in Chapter 5, Section 2, the dimension of c.s. was invariant under the action of similarity, feedback, and input change of basis transformations.

The ease by which the canonical form of Brunovsky arose from the characterization of c.s., hints that such an approach might be useful in determining the invariants of, and hence a canonical form for triples (A,B,C). Indeed it is intriguing that for controllable and observable systems, Gilbert's c.d. form presents a partial extension of that of Brunovsky, where the feedback invariants $\{d_i, i \in \underline{m}\}$ completely specify the triple (A,B,C). These connections merit further study.

Finally, the result on the dimensions of c.s. (Theorem 5.4.3) is clearly related to a theorem of Rosenbrock [R1] concerning the limits of linear feedback in modifying the dynamics of time invariant linear systems. This too is worthy of further consideration.

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APPENDIX

SEVERAL MODULAR IDENTITIES

(A.1) <u>Lemma</u>: If $Y \subset X$, then $X \cap (Y + Z) = X \cap Y + X \cap Z$

Proof: Clearly $X \cap Y + X \cap Z \subset X \cap (Y + Z)$. To show the other inclusion let $a \in X \cap (Y + Z)$. Then $a = x = y + z$ for some $x \in X$, $y \in Y$, and $z \in Z$. Since $Y \subset X$, $y \in X$, and $z = x - y \in X$ yielding $a = y + z$, with $y \in X \cap Y$, and $z \in X \cap Z$.

(A.2) <u>Lemma</u>: If $X \cap (Y + Z) = X \cap Y + X \cap Z$, then $Y \cap (X + Z) = Y \cap X + Y \cap Z$

Proof: Again $Y \cap X + Y \cap Z$ is always contained in $Y \cap (X + Z)$ so we need only show the reverse inclusion. Let $a \in Y \cap (X + Z)$. Then $a = y = x + z$ for some $y \in Y$, $x \in X$, and $z \in Z$. Since $x = y - z$, it follows that $x \in X \cap (Y + Z) = X \cap Y + X \cap Z$, and so $x = x_1 + x_2$ with $x_1 \in X \cap Y$, $x_2 \in X \cap Z$. Then $a = y = x + z = x_1 + x_2 + z = x_1 + z_1$, where $x_2 + z = z_1 \in Z$. However $z_1 = y - x_1$, thus $z_1 \in Y + X \cap Y = Y$, yielding $z_1 \in Y \cap Z$. Thus we have shown $a = y = x_1 + z_1 \in Y \cap X + Y \cap Z$.

BIOGRAPHICAL NOTE

Michael Edward Warren was born on 5 February 1947 in New York City, New York. He attended public schools in Valley Stream and Syosset, New York, and was graduated salutatorian from Syosset High School in June 1965.

The following September, Mr. Warren entered M.I.T. and received the degree of Bachelor of Science from the Department of Aeronautics and Astronautics in June 1969. He was active in undergraduate life at M.I.T., serving as president of his fraternity, an editor of the campus newspaper, and chairman of student committees, among other activities. As an upperclassman he was elected to Tau Beta Pi, Sigma Gamma Tau, and Sigma Xi.

Mr. Warren began graduate studies in the Department of Aeronautics and Astronautics as a National Science Foundation Fellow in the fall of 1969, and was awarded the degree of Master of Science in June 1971. He transferred to the Department of Electrical Engineering at M.I.T. during the summer of 1971 and resumed graduate work in the Electronic Systems Laboratory. Since 1971 he has been both a teaching assistant and a research assistant in the laboratory.

In June 1970 he married the former Phyllis Lynn Paltrowitz of Elmira, New York. They presently have one son, Matthew, born 24 December 1971.