AN ALGEBRAIC APPROACH TO LINEAR AND MULTILINEAR SYSTEMS THEORY

by

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Abstract

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This thesis presents an algebraic approach to the study of linear and multilinear systems. The linear system is defined as an R-module of input-output pairs. This definition unifies the study of systems found in Automata Theory, Sequential Circuit Theory, and Control Theory.

The output-algebraic linear system is defined and studied. The concepts of attainability, controllability, and observability are introduced, and it is shown that the Nerode equivalence relation for the system in the zero-state can be used to construct a realization whose state module is both attainable and observable.

Associated with every discrete-time output-algebraic linear system is a minimal system. This system has the minimum number of input and output terminals necessary for the controllability and observability of the state module. The minimal system can also be used as a template for control system design.

The multilinear system is defined as a multilinear relation of input-output pairs over a commutative ring. A multilinear response separation is obtained. The internal structure of a class of these systems is shown to consist of component linear systems which are

interconnected by tensor product maps. The tensor product of two linear systems is shown to be a linear as well as a multilinear system.

Dedication

Most blessed of women be יָעֵאל ..

Song of Deborah; Book of Judges.

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CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1. Objectives of the Research

The objectives of this research are 1) to provide an axiomatic development of both Linear and Multilinear Systems Theory within the framework of Windeknecht's [1,2] theory of General Time Systems, and 2) to use the R-module theory of Modern Algebra to unify the study of linear systems found in Automata Theory, Sequential Circuit Theory, and Control Theory.

Using this axiomatic approach, one obtains results based on as few axioms and as little mathematical structure as possible. In fact, certain results such as the response separation property, zero-input linearity, and zero-state linearity are shown to be derived, rather than postulated properties (see, for example, Zadeh and Desoer [3]). Moreover, the axiomatic approach may lead to further generalizations of the theory.

Related work in this area has been done by Arbib [4,5]. In the first paper he suggests a "rapprochement" between Automata Theory and Control Theory, and discusses related concepts from both areas. In the second paper he attempts to formulate these concepts within a common mathematical framework. In particular, Arbib shows that an "additive"(linear) system can be defined on sets having additive abelian group structure.

Kalman [6,7,8] has studied discrete-time, linear time-invariant systems defined over K-vector spaces, where K is an arbitrary field. Thus, systems found in Sequential Circuit Theory (see Gill [9,10]; and Cohn and Even [11]) and Control Theory can be studied within the same mathematical framework.

Kalman's linear system admits a K[z]-module representation, where K[z] denotes the ring of polynomials in z with pointwise sum and convolution product. By abstracting the system to a K[z]-module, Kalman is able to study the notions of transfer functions, time-domain convolution, and the rational canonical decomposition of the state space within a single algebraic framework. Kalman restricts his study to discrete-time systems.

In this thesis both continuous and discrete-time systems are studied, and many of Kalman's results are obtained using the General Time Systems formalism developed by Windeknecht.

2. Summary of Thesis Content

The remainder of this chapter is devoted to a section on notation and terminology, and a section dealing with the General Time Systems formalism.

Chapter 2 contains an exposition of Linear Systems Theory. Properties such as response separation, zero-state linearity, and zero-input linearity are shown to be derived rather than postulated properties. The notion of an index module is introduced and it is shown that the system admits a morphism representation.

In Chapter 3 the output-algebraic linear system is studied. The concepts of attainability, controllability, and observability are introduced. The Nerode equivalence relation for the system in the zero state is used to construct an attainable and observable state module. The resulting reduced system is shown to be a minimal realization.

Chapter 4 studies discrete-time, linear systems, defined over a field K. In particular, it is shown that the zero-input portion of the algebraic linear system contains all the information required to put the state module into Rational Canonical Form.

The minimal system is introduced and provides a link between free and forced systems. Moreover, the minimal system may be used as a template for the design of completely controllable and observable control systems.

The Theory of Multilinear Systems is presented in Chapter 5. The internal structure of a class of multilinear systems is given, and the tensor product map is used to put the system into a tensor canonical form. Lastly, the Linear Tensor Product System is studied. This system exhibits a response separation property, and is linear as well as multilinear.

Chapter 6 contains a brief summary and suggestions are made for future research. Appendix 1 contains a review of the R-module theory which is relevant to this thesis.

3. Contributions of the Thesis

The major contributions of this thesis are presented under respective chapter headings.

Chapter 2 - The linear system is defined as an input-output relation, and the response separation property, zero-input linearity and zero-state linearity are derived. The notion of the index module for free responses is useful in obtaining the morphism representation of Theorem 2.6.5.

Chapter 3 - The relation between the concepts of attainability and controllability is summarized by Theorems 3.3.7 and 3.3.8. Theorem 3.3.12 shows that the notions of a reduced index module and an observable state module are equivalent.

The construction of a reduced system by means of The Nerode equivalence relation is given in Theorem 3.4.5. Also, Theorem 3.4.7 shows that the constructed state module is minimal.

Chapter 4 - The notion of the minimal system is a conceptual as well as practical contribution. It provides insight into the structure of linear systems and also specifies the minimum number of input and output terminals necessary for complete controllability and observability.

Chapter 5 - The multilinear system is defined, and Theorem 5.2.3 is the multilinear version of the response separation property. Theorem 5.3.2 which characterizes the internal structure of a class of multilinear systems is due to Arbib [12] . The tensor canonical representation in Theorem 5.3.3 was suggested by S. K. Mitter. The Linear Tensor Product System and related theory provide a means to construct and decompose multilinear systems.

Notation and Terminology

The chapters of this thesis are divided into sections, and each section is numbered consecutively. Each section contains numbered statements such as definitions, theorems, and propositions. Within a particular chapter, the j th statement of section i is referred to as Statement i.j. If a statement outside that chapter is referenced, then it is denoted by Statement h.i.j, where h is the chapter number. Thus, Theorem 3.2 of Chapter 6 is written as Theorem 6.3.2.

Equations are numbered consecutively in each chapter. Thus (3.16) denotes equation 16 of Chapter 3. An unnumbered statement called the Remark is used to discuss the results of a theorem or to present pertinent additional material. The completion of a proof, or the end of a unit of material is denoted by the symbol '■'.

The set theoretical and the algebraic notation used here is consistent with MacLane-Birkhoff [13]. Some of the more important module notation is given below.

Let A and B be <u>modules</u>, then

$A + B$	denotes	$\{ a+b \mid a \in A \ \& \ b \in B \}$
$A \oplus B$	denotes	the direct sum of A and B.
$A \otimes B$	denotes	the tensor product of A and B.
A / B	denotes	the quotient module.

The following list of symbols is used to denote special sets:

T	-	the time set.
$\mathbb{R}$	-	the field of real numbers.
$\mathbb{R}^+$	-	the non-negative real numbers.
$\mathbb{Q}$	-	the field of rational numbers.
$\mathbb{N}$	-	the set of natural numbers.

Z - the ring of integers.

$\mathbb{C}$ - the field of complex numbers.

Z_r - the field of integers modulo some prime, r.

R^n - the cartesian product of R taken n times.

K - a commutative ring or a field.

$K[z]$ - the ring of polynomials in z with coefficients from K.

$\subset$ and $\subseteq$ denote proper and improper containment, respectively.

4. General Time Systems

This section presents the important concepts from the theory of General Time Systems. These notions will be used in the study of Linear and Multilinear Systems. A more complete exposition of this theory may be found in [2].

4.1 Definition Let T denote the time set of natural numbers, N, or non-negative real numbers, $\mathbb{R}^+$.

1. A set V is a T-object iff there exists some set A such that $V \subset A^T$, where $A^T = \{ v | v:T \longrightarrow A \}$ denotes the set of T-functions from T into A.

2. Let V be a T-object. For any $v \in V$ and $t \in T$, the t-segment of v and the t-section of V are defined by the respective sets.

$$v^t = \{(t',v(t'))|t' \in T \ \&\ 0 \le t' < t \} \text{ and } v_t = \{(t',v(t+t'))|t' \in T\}.$$

The sets $\tilde{V}^t = \{v^t | v \in V\}$, $\tilde{V} = \{v^t | v \in V \ \&\ t \in T\}$, and $V_t = \{v_t | v \in V\}$ denote the set of initial segments of V at time t, the set of initial segments of V, and the t-section of V, respectively.

3. Let V be a T-object. The sets

$V[t] = \{v(t) \mid v \in V\}$ and $AV = \{v(t) \mid v \in V \ \& \ t \in T\}$

denote the attainable space of V at time t, and the attainable space of V, respectively.

Remark Clearly, $\tilde{V} = \bigcup_{t \in T} \tilde{V}^t$ and $AV = \bigcup_{t \in T} V[t]$.

Also it can be shown that $V_{t+t'} = (V_t)_{t'}$. ∎

4.2 Definition Let A^T and B^T be T-objects. A T-system is any relation

$$S \subset A^T \times B^T$$

The T-system is multivariable iff A and B are the cartesian product of sets, i.e.,

$A = A_1 \times A_2 \times \ldots \times A_m$ and $B = B_1 \times B_2 \times \ldots \times B_p$,

respectively.

Remark The T-system is discrete-time or continuous-time if and only if $T = N$ or $T = \mathbb{R}^+$, respectively.

The multivariable T-system has m input terminals and p output terminals. The input set (of T-functions) is the domain of S, $\mathcal{D}S = \{x \mid (\exists y): xSy\}$, and the output set (of T-functions) is the range of S, $\mathcal{R}S = \{y \mid (\exists x): xSy\}$. The set $\mathcal{I}S = A(\mathcal{D}S)$ is the input space of S and the set $\mathcal{O}S = A(\mathcal{R}S)$ is the output space of S.

A T-system S is free iff the input space, $\mathcal{I}S$, contains only one element. Thus, S is free iff $\mathcal{D}S = \{a^T\}$, a constant T-function.

A T-system S is functional iff S is a function, $S: \mathcal{D}S \longrightarrow \mathcal{R}S$. ∎

Of particular importance to the study of T-systems are the notions of non-anticipation, transition systems, static systems, and system evolution. These concepts are introduced in the next series of definitions and theorems.

4.3 Definition Let S be a T-system. Consider the relations

$$na^t S = \{(x^t, y(t)) \mid xSy\} \quad (1.2)$$

$$na\, S = \{(x^t, y(t)) \mid xSy \;\&\; t \in T\} \quad (1.3)$$

$$tr_t^{t'} S = \{((y(t), x_t^{t'}), y(t+t')) \mid xSy\} \quad (1.4)$$

$$tr^{t'} S = \{((y(t), x_t^{t'}), y(t+t')) \mid xSy \;\&\; t \in T\} \quad (1.5)$$

$$st_t\, S = \{(x(t), y(t)) \mid xSy\} \quad (1.6)$$

$$st\, S = \{(x(t), y(t)) \mid xSy \;\&\; t \in T\} \quad (1.7)$$

1. S is non-anticipatory at time t iff $na^t S$ is a function. S is non-anticipatory iff naS is a function.

2. For each $t' \in T$, S is weakly transitional iff $tr_0^{t'} S$ is a function; S is transitional iff $\forall t \in T$, $tr_t^{t'} S$ is a function; S is uniformly transitional iff $tr^{t'} S$ is a function.

3. S is weakly static iff $st_0 S$ is a function. S is static iff $\forall t \in T$, $st_t S$ is a function. S is uniformly static iff st S is a function.

Remark 1. Clearly, $naS = \bigcup_{t \in T} na^t S$. Also, S is non-anticipatory iff for each $t \in T$, S is non-anticipatory at time t.

2. This definition of transition systems differs from Windeknecht's in that the final time t' is explicitly stated. As

will be seen in the next chapter, this definition preserves certain algebraic structure.

It is easy to show that if S is transitional, then S is weakly transitional. Furthermore, if S is uniformly transitional, then S is transitional.

3. The static system is simply a pointwise map. Note that if S is static, then it is weakly static. Also, if S is uniformly static, then it is static. $\blacksquare$

4.4 <u>Definition</u> Let V be a T-object. V is left-translatable iff $\forall t \varepsilon T$, $V_t \subset V$. V is translation-free iff $\forall t \varepsilon T$, $V_t = V$.

It is easy to prove the

4.5 <u>Lemma</u> Let V be a T-object. If V is left-translatable then

1. $\forall t \varepsilon T$, $V[t] \subset V[0]$.
2. $\forall t, t' \varepsilon T$, $t \leq t' \implies V[t] \subset V[t']$.
3. $AV = V[0]$.

If V is translation-free then $\forall t \varepsilon T$, $V[t] = V[0] = AV$.

4.6 <u>Definition</u> Let S be a T-system. The sets

$$S_t = \{(x_t, y_t) \mid xSy\} \text{ and } \overline{S} = \{(x_t, y_t) \mid xSy \;\&\; t \varepsilon T\}$$

denote the t-section of S and the translation closure of S, respectively.

S is <u>contracting</u> iff $\forall t \varepsilon T$, $S_t \subset S$.

S is <u>stationary</u> iff $\forall t \varepsilon T$, $S_t = S$.

Remark The t-section of S represents the evolution of S. Clearly S_t is a T-system.

Note that a stationary system is a special form of contracting system. The contracting system implies a form of time-invariance for input-output pairs. Moreover, it induces the more familiar notion of time-invariance associated with transition systems and static systems. This is shown in the following theorem which is easy to prove.

4.7 Theorem Let S be a T-system. If S is contracting, then the following statements are equivalent:

1. S is uniformly transitional [uniformly static].
2. S is transitional [static]
3. S is weakly transitional [weakly static].

4.8 Corollary Let S be a T-ststem. If S is contracting, then $\mathcal{I}S=\mathcal{D}S[0]$ and $\mathcal{O}S=\mathcal{R}S[0]$.

Remark This concludes the introduction to General Time Systems. The formalism describes a large variety of T-systems, but only linear and multilinear systems are considered in this work. ∎

CHAPTER 2

LINEAR SYSTEMS THEORY

1. Introduction

This chapter studies the linear T-system in the General Time Systems framework. In order to define the linear T-system, certain T-objects must be endowed with R-module structure. A review of the R-module theory used in this thesis is found in Appendix I.

Basic linear system properties such as response separation, zero-state linearity, and zero-input linearity are derived from the system definition.

The concept of an index module for the free linear system allows the linear T-system to be represented as a morphism. Moreover, the functions associated with transition and static systems are specialized to the linear T-system.

2. The Linear T-System (LT-System)

Let the ring R and the sets A and B be denoted by the triple (R,A,B). The triple has R-module structure if and only if R, A, and B are R-modules. Note that every ring is itself an R-module. The following definition is due to S.K. Mitter.

2.1 Definition Let S be a T-system. Let (R, A, B) have R-module structure. S is linear iff for all $x,x' \in \mathcal{D}S$, $y,y' \in \mathcal{R}S$, $\alpha,\beta \in R$

$$xSy \ \& \ x'Sy' \Longrightarrow (\alpha x + \beta x')\ S\ (\alpha y + \beta y') \qquad (2.1)$$

<u>Remark</u> This general definition encompasses the linear systems studied in Control Theory, Sequential Circuit Theory, and Automata Theory.

In particular, the triple $(\mathbb{R}, \mathbb{R}^m, \mathbb{R}^p)$ represents a finite dimensional LT-system studied in Control Theory. Linear Sequential Circuit Theory deals with discrete-time systems described by the triple (Z_r, Z_r^m, Z_r^p). Lastly, the triple (Z,A, B) defines a linear automaton.

Definition 2.1 not only recaptures the well known LT-systems but also leads to the study of new LT-systems. Consider the following example.

2.2 <u>Example</u> Let S be an LT-system over (F, F^m, F^p), where F is any field. Let S have the following properties:

1. S is a function, S: $\mathcal{D}S \longrightarrow \mathcal{R}S$, and $\mathcal{R}S[0] = \{0\}$
2. $\mathcal{D}S$ is zero pre-loadable, ie., for any $x \in \mathcal{D}S$ and any $t \in T$, $0^t_\circ x \in \mathcal{D}S$, and $0^t_\circ x = 0^t \bigcup \{(t+t'; x(t')) \mid t' \in T\}$
3. For any $t \in T$, $x \in \mathcal{D}S$

$$S(0^t_\circ x) = 0^t_\circ S(x)$$

Note that differential equations and difference equations with constant coefficients and zero initial conditions are members of this class of systems.

Additional algebraic structure can be induced on S by means of Property 3. For any $t, t' \in T$ and any $x \in \mathcal{D}S$, define the zero pre-loading operations.

$z \circ x = 0^t_\circ x$ and $z' \circ x = 0^{t'}_\circ x$.

Let $F[z]$ and $F[z']$ denote the polynomial rings under pointwise sum and convolution product with indeterminates z and z', respectively. Thus, for $p(z)\varepsilon F[z]$ and $p'(z')\varepsilon F[z']$, for any $x\varepsilon \mathcal{D}S$

$$S(p(z)\circ x) = p(z)\circ S(x) \text{ and } S(p'(z')\cdot x) = p'(z')\cdot S(x)$$

Thus, S is both an $F[z]$ and an $F[z']$-module of ordered pairs.

Moreover, since $(z'z)\circ x = (zz')\circ x = 0_\circ^{t+t'} x$, it is easy to show that S is an $F[z,z']$-module, where $F[z,z']$ denotes the polynomial ring over indeterminates z and z'. Clearly, if $\{z_1, z_2, \ldots, z_n\}$ is a set of zero pre-loading operations on $\mathcal{D}S$, then S is an $F[z_1, z_2, \ldots, z_n]$-module.

This class of LT-systems has not been studied in this more abstract formulation. Perhaps more efficient coding and decoding techniques could be developed by examining these modules. ■

Remark For notational simplicity the triple (R,A,B) will not be explicitly stated unless a particular LT-system is studied. ■

3. Structural Properties of LT-Systems

One can easily verify the

3.1 Theorem If S is an LT-system, then

1. $A^T \times B^T$ is an R-module of ordered pairs.
2. S is an R-module of ordered pairs.
3. S is a submodule of $A^T \times B^T$.
4. $\mathcal{D}S$ and $\mathcal{R}S$ are submodules of A^T and B^T, respectively.
5. For each $t\varepsilon T$, $\mathcal{D}S[t]$ and $\mathcal{R}S[t]$ are submodules of A and B, respectively.
6. For each $t,t'\varepsilon T$, $\tilde{\mathcal{D}S}^{t'}$, $\mathcal{D}S_t$, $\tilde{\mathcal{D}S}_t^{t'}$, and $\mathcal{R}S_t$ are R-modules,

Remark The sets $\mathcal{I}S$, $\mathcal{O}S$, and $\overline{\mathcal{D}S}$ are not necessarily R-modules because addition is defined pointwise at each time t, but not at two different times t and t'.

Fortunately, there is a simple way to endow $\mathcal{I}S$ and $\mathcal{O}S$ with R-module structure. It consists of constructing the smallest submodules of A and B which contain $\mathcal{I}S$ and $\mathcal{O}S$, respectively.

Recall that $\mathcal{I}S = A(\mathcal{D}S) = \bigcup_{t \in T} \mathcal{D}S[t]$ and $\mathcal{O}S = A(\mathcal{R}S) = \bigcup_{t \in T} \mathcal{R}S[t]$. Define the modules

$$\mathcal{I}S_m = \sum_T \mathcal{D}S[t] \quad \text{and} \quad \mathcal{O}S_m = \sum_T \mathcal{R}S[t] \tag{2.2}$$

such that for any $\hat{x} \in \mathcal{I}S_m$ and $\hat{y} \in \mathcal{O}S_m$

$$\hat{x} = \sum_{t_i \in T} x_i(t_i) \quad \text{and} \quad \hat{y} = \sum_{t_j \in T} y_j(t_j) \tag{2.3}$$

are finite sums.

The following theorem shows that the above construction is unnecessary when S is contracting.

3.2 Theorem Let S be a contracting LT-system. Then $\mathcal{I}S$, $\mathcal{O}S$, and $\overline{\mathcal{D}S}$ are R-modules.

Proof Since S is contracting, $\mathcal{I}S = \mathcal{D}S[0]$, $\mathcal{O}S = \mathcal{R}S[0]$, and $\mathcal{D}S = \overline{\mathcal{D}S}$. By Theorem 3.1, $\mathcal{D}S[0]$, $\mathcal{R}S[0]$, and $\mathcal{D}S$ have R-module structure. The conclusion follows. ∎

4. Morphisms, Linear Transition Systems, and Linear Static Systems.

Much of the previous section was devoted to the R-module structure induced on various sets by the LT-system S. This structure allows the notions of non-anticipation, transition systems, and static systems to be specialized to linear systems.

4.1 Definition Let A and A' be R-modules. A morphism of R-modules is a function $f: A \longrightarrow A'$ such that

$$f(a+b) = f(a) + f(b) \quad , \quad f(\lambda a) = \lambda f(a) \tag{2.4}$$

for all $a,b \in A$, $\lambda \in R$.

One can easily verify the

4.2 Theorem Let S be an LT-system. Then

1. S is a morphism, $S: \mathcal{D}S \longrightarrow \mathcal{R}S$, iff S is functional.

2. For each $t \in T$, S is non-anticipatory at time t iff the relation na^tS is a morphism.

$$na^tS : \mathcal{D}\tilde{S}^t \longrightarrow \mathcal{R}S[t] \tag{2.5}$$

3. For each $t' \in T$, S is weakly transitional iff tr_0^tS is a morphism.

$$tr_0{}^tS : \mathcal{R}S[0] \times \hat{\mathcal{D}S}^t \longrightarrow \mathcal{R}S[t] \tag{2.6}$$

4. For each $t,t' \in T$, S is transitional iff $tr_t^{t'}S$ is a morphism.

$$tr_t^{t'}S : \mathcal{R}S[t] \times \tilde{\mathcal{D}S}_t^{t'} \longrightarrow \mathcal{R}S[t+t'] \tag{2.7}$$

5. For each $t' \in T$, S is uniformly transitional iff $tr^{t'}S$ can be extended to a morphism.

$$tr^{t'}S : \mathcal{O}S_m \times \tilde{\mathcal{D}S}^{t'} \longrightarrow \mathcal{O}S_m \tag{2.8}$$

6. S is weakly static [static] iff $st_0 S : \mathcal{D}S[0] \longrightarrow \mathcal{R}S[0]$ $[st_t S : \mathcal{D}S[t] \longrightarrow \mathcal{R}S[t]\ ]$ is a morphism.

7. S is uniformly static iff st S can be extended to a morphism.

$$st\ S : \mathcal{I}S_m \longrightarrow \mathcal{O}S_m \qquad \blacksquare \qquad (2.9)$$

The connection between the notions of non-anticipatory and weakly transitional LT-systems is made by the

4.3 Theorem Let S be an LT-system. For each $t \in T$, S is non-anticipatory at time t iff S is weakly transitional and $\mathcal{R}S[0] = \{0\}$.

Proof $\Leftarrow$ If $\mathcal{R}S[0] = \{0\}$, then for each $t \in T$, $tr_0^t S = na^t S$ so that S is non-anticipatory at time t.

$\Longrightarrow$ The proof depends on a result for T-systems due to Windeknecht [2]: S is non-anticipatory iff S is weakly transitional and $\mathcal{R}S[0]$ has one element.

In the context of LT-systems, it must be shown that $\mathcal{R}S[0] = \{0\}$. Since $\mathcal{R}S[0]$ must be an R-module and contain only one element, that element must be zero, $\mathcal{R}S[0] = \{0\}$, a trivial module.

5. The Response Separation Property, Zero-State Linearity, Zero-Input Linearity.

This section is concerned with several well known linear system properties. The most important of these is the response separation property; zero-state linearity and zero-input linearity follow from the response separation property.

For example, the solution to a linear differential equation is expressed as the sum of two responses: 1) the homogeneous solution due to an initial condition and 2) the particular solution due to an input.

In most cases, the response separation property is postulated in the system's definition (see Arbib [5] or Zadeh and Deseor [3]), but in the General Time Systems framework it is derived from the linear relation. ■

There are several subsystems of S which play a role in the derivation of the response separation property. Consider the sets

$$F = \{(x,y) \mid xSy \;\; \& \; x = 0 \} \tag{2.10}$$

$$S^b = \{(x,y) \mid xSy \;\; \& \; y(0) = b \} \tag{2.11a}$$

$$S^b(x) = \{ y \mid xSy \;\; \& \; y(0) = b \} \tag{2.11b}$$

It is easy to verify the

5.1 <u>Theorem</u> Let S be an LT-system. Then

1. F is free and linear, S^0 is linear.

2. $S = F + S^0 \Longleftrightarrow \mathcal{D}S = \mathcal{D}S^0$

5.2 <u>Proposition</u> Let A_1 , A_2 be submodules of an R-module B. If $B = A_1 + A_2$ and $A_1 \cap A_2 = 0$ as sets, then B is isomorphic to the direct sum $A_1 \oplus A_2$,

$$B \cong A_1 \oplus A_2 \tag{2.12}$$

<u>Proof</u> See MacLane-Birkhoff [13], p. 213.

5.3 <u>Theorem</u> Let S be a weakly transitional LT-system. Then

1. F is weakly transitional; S^0 is both weakly transitional and non-anticipatory at time t, $t \in T$.

2. $S = F \oplus S^0 \Longleftrightarrow \mathcal{D}S^0 = \mathcal{D}S$ (2.13)

<u>Proof</u> 1. S is weakly transitional implies that both F and S^0 are also weakly transitional. Theorem 4.3 shows that for all $t \in T$, S^0 is non-anticipatory at time t.

2. $\Rightarrow$ Clearly, $S = F \oplus S^0 \Longrightarrow \mathcal{D}S^0 = \mathcal{D}S$.

$\Leftarrow$ Theorem 5.1 shows that $S = F + S^0 \Longleftrightarrow \mathcal{D}S^0 = \mathcal{D}S$.

It remains to show that $S = F \oplus S^0$. Now suppose that for $(0,y_1)$, $(0,y_2) \in F$ and (x,y), $(x',y') \in S^0$

$$(0,y_1) + (x,y) = (0,y_2) + (x',y')$$

Proposition 5.2 implies that $S = F \oplus S^0$ only if $y_1 = y_2$, $x = x'$, and $y = y'$

Now $y_1 = y_2$ because $y_1(0) = y_2(0)$ and F is weakly transitional. Furthermore, $y = y'$ because $y_1 + y = y_2 + y'$ and $y_1 = y_2 \Longrightarrow y = y'$. Finally, $x = x'$ because $y = y'$ and S^0 is non-anticipatory at time t, $t \in T$. Thus $S = F \oplus S^0$. ∎

<u>Remark</u> Theorem 5.1 and part 2 of Theorem 5.3 were communicated to the author by T. G. Windeknecht. The following lemma and theorem are also due to him.

5.4 <u>Lemma</u> Let S be a weakly transitional LT-system. For any $b \in RS[0]$, consider the T-system S^b.

If $\mathcal{D}S = \mathcal{D}S^0$, then for all $x \in \mathcal{D}S$, $x \in \mathcal{D}S^b \Longleftrightarrow 0 \in \mathcal{D}S^b$.

Proof See Windeknecht [2].

5.5 Theorem Let S be a weakly transitional LT-system. If $\mathcal{D}S^0 = \mathcal{D}S$ then

1. For all $b \in RS[0]$, $\mathcal{D}S^b = \mathcal{D}S$.

2. For all $b \in RS[0]$, $x \in \mathcal{D}S$, $\quad S^b(x) = S^b(0) + S^0(x) \qquad (2.14)$

Proof 1. By Theorem 5.3, $\mathcal{D}S^0 = \mathcal{D}S \Longrightarrow S = F \circledast S^0$. Moreover, $S^b = F^b \circledast S^0$ and $\mathcal{D}S^b = 0 \circledast \mathcal{D}S^0 = \mathcal{D}S^0$. Clearly, $\mathcal{D}S^b = \mathcal{D}S^0 = \mathcal{D}S \Longrightarrow \mathcal{D}S^b = \mathcal{D}S$.

2. Since S is weakly transitional, S^b is non-anticipatory. Thus, by pointwise extension on $t \in T$,

$$S^b(x) = F^b \circledast S^0(x) = S^b(0) + S^0(x) \qquad (2.15)$$

for all $b \in RS[0]$, and all $x \in \mathcal{D}S$.

A direct consequence of Theorem 5.5 is the

5.6 Corollary Let S be a weakly transitional LT-system. If $\mathcal{D}S^0 = \mathcal{D}S$ then

1. (Zero-input linearity) For all $b, b' \in RS[0]$, $k, k' \in R$,

$$S^{kb+k'b'}(0) = k\, S^b(0) + k'\, S^{b'}(0) \qquad (2.16)$$

2. (Zero-state linearity) For all $x, x' \in \mathcal{D}S$, $k, k' \in R$,

$$S^0(kx + k'x') = k\, S^0(x) + k'\, S^0(x') \qquad (2.17)$$

6. Index Modules and Morphism Representations for LT-Systems

Although S^0 is a morphism of R-modules, the free LT-system F is not a morphism. This section introduces the notion of an "index module" for RF so that S may be given a morphism representation.

6.1 Definition Let F be a free LT-system. An R-module Q is an "index module" for RF iff there exists an epimorphism

$$g : Q \longrightarrow RF \qquad (2.18)$$

of R-modules. If g is an isomorphism then Q is a reduced index module for RF.

6.2 Theorem Every free LT-system F admits an index module for RF.

The proof of this theorem is based on certain algebraic constructions found in MacLane-Birkhoff, pp. 205-209. They are stated here without proof.

Definition Let X be a subset of an R-module A and $j: X \longrightarrow A$ the injection of X into A. A is called a free module on the set of free generators X when for every function $f: X \longrightarrow B$, with B an R-module there is exactly one linear morphism $t: A \longrightarrow B$ with $f = t \circ j$.

Definition Let X be any set and let R^X be a function module, with R a ring. For each $x \in X$, define

$$E_x(y) = 1 \quad \text{if } x = y \qquad x,y \in X$$
$$E_x(y) = 0 \quad \text{if } x \neq y$$

Theorem For any set X, the module $R^{(X)}$, which is spanned by the E_x is a free R-module on the set $\{ E_x \mid x \in X \}$.

Proof of Theorem 6.2 Let Y be the set of non-zero elements of RF. The assignment $y \longrightarrow E_y$ is an injective function $k: Y \longrightarrow R^{(Y)}$.

Let I be the identity map $I : Y \longrightarrow RF$ such that $I(y) = y$.

By the above theorem, $R^{(Y)}$ is free so that there exists a morphism $g : R^{(Y)} \longrightarrow RF$ such that $I = gok$. Clearly g is an epimorphism because k is an injection while I is the identity mapping. Thus, $R^{(Y)}$ is the desired index module. $\blacksquare$

6.3 <u>Theorem</u> Let F be a free LT-system. If Q and Q' are reduced index modules for RF then $Q \cong Q'$.

<u>Proof:</u> Since Q and Q' are reduced index modules for RF, there exist isomorphisms $g: Q \longrightarrow RF$ and $g' : Q' \longrightarrow RF$. Clearly, $(g')^{-1}og$ and $g^{-1}\ og'$ are isomorphisms so that $Q \cong Q'$. $\blacksquare$

The next theorem exhibits a very special reduced index module for RF.

6.4 <u>Theorem</u> Let F be a free LT-system. Then F is weakly transitional iff RF[0] is a reduced index module for RF.

<u>Proof</u> $\Longrightarrow$ F weakly transitional implies that for each $t \in T$, $tr_0^t\ F : RF[0] \times 0^t \longrightarrow RF[t]$ is a morphism of R-modules. It must be shown that the assignment $b \longrightarrow F^b$ is an isomorphism $g : RF[0] \longrightarrow RF$.

Suppose that for $b, b' \in RF[0]$, $b = b'$ but $F^b \neq F^{b'}$. Then, for each $t \in T$ $(F^b - F^{b'})(t) = tr_0^t\ F(b,0^t) - tr_0^t\ F(b',0^t)$

$$= tr_0^t\ F(b-b',0^t) = tr_0^t\ F(0,0^t) = 0$$

Thus, $F^b = F^{b'}$ which contradicts the hypothesis. The morphism g is therefore 1:1 ; it is also onto because F is free and weakly transitional. Thus g is an isomorphism.

$\Longleftarrow$ If $\mathcal{R}F[0]$ is a reduced index module for $\mathcal{R}F$, then $g:\mathcal{R}F[0] \longrightarrow \mathcal{R}F$ is an isomorphism. For each $t \varepsilon T$, define the relation

$$\mathcal{tr}_0^t F = \{((b,0^t),\ g(b)(t)) \mid b \varepsilon \mathcal{R}F[0]\}$$

Clearly, $\mathcal{tr}_0^t F$ is a morphism and F is weakly transitional. ∎

The material presented in Theorems 5.3, 5.5, and 6.4 can be summarized as follows:

6.5 <u>Theorem</u> Let S be a weakly transitional LT-system with $S = F \oplus S^0$. Then

1. There exists a unique system morphism

$$\pi : \mathcal{R}S[0] \oplus \mathcal{D}S \longrightarrow \mathcal{R}S \qquad (2.19)$$

such that for all $b, b' \varepsilon \mathcal{R}S[0]$, $x,x' \varepsilon \mathcal{D}S$, $k,k' \varepsilon R$

$$\pi(b,x) = \pi(b,0) + \pi(0,x) \qquad (2.20)$$

$$\pi(kb+k'b', 0) = k\,\pi(b,0) + k'\,\pi(b',0) \qquad (2.21)$$

$$\pi(0,kx + k'x') = k\,\pi(0,x) + k'\,\pi(0,x') \qquad (2.22)$$

2. There exists a unique weakly transitional morphism

$$\pi_0^t S : \mathcal{R}S[0] \oplus \tilde{\mathcal{D}S}^t \longrightarrow \mathcal{R}S[t] \qquad (2.23)$$

such that for all $b \varepsilon \mathcal{R}S[0]$, $x^t \varepsilon \tilde{\mathcal{D}S}^t$, $t \varepsilon T$

$$\pi_0^t S(b,x^t) = \pi_0^t S(b,0^t) + \pi_0^t S(0,x^t) \qquad (2.24)$$

<u>Proof</u> The proof makes use of a theorem from MacLane-Birkhoff, p.212.

<u>Theorem A</u>: If C is any R-module and $m_i : A_i \longrightarrow C$ are two morphisms $(i = 1,2)$. There is a unique morphism $m: A_1 \oplus A_2 \longrightarrow C$ such that $m \circ e_1 = m_1$ and $m \circ e_2 = m_2$, where $e_i : A_i \longrightarrow A_1 \oplus A_2$ $(i=1,2)$.

1. $S = F \oplus S^0$ implies that $\mathcal{R}S[0] = \mathcal{R}F[0]$ and $\mathcal{R}S = \mathcal{R}F \oplus \mathcal{R}S^0$. Also $\mathcal{R}S[0]$ is a reduced index module for $\mathcal{R}F$, so that $g: \mathcal{R}S[0] \longrightarrow \mathcal{R}F$ is an isomorphism. Now g can be extended to a monomorphism g' : $\mathcal{R}S[0] \longrightarrow \mathcal{R}S$, where $g' = e_F \circ g$ and $e_F : \mathcal{R}F \longrightarrow \mathcal{R}S$.

Recall that $S^0: \mathcal{D}S \longrightarrow \mathcal{R}S$ is a morphism of R-modules, and Theorem A allows the construction of π in terms of g' and S^0. Thus, for all $b \in \mathcal{R}S[0]$, $x \in \mathcal{D}S$

$$\pi\ (b,x) = \pi\ (b,0) + \pi\ (0,x)$$
$$= g'(b) + S^0(x) \qquad (2.25$$

2. S is weakly transitional and $S = F \oplus S^0$. Thus, for each $t \in T$, both $tr_0^t F$ and $tr_0^t S^0$ are morphisms. By a construction similar to that for g', $tr_0^t F$ can be extended to the morphism $tr_0^t F'$: $\mathcal{R}S[0] \oplus 0^t \longrightarrow \mathcal{R}S[t]$, so that Theorem A allows $\pi_0^t S$ to be uniquely defined.

Thus, for all $b \in \mathcal{R}S[0]$, $x^t \in \tilde{\mathcal{D}}S^t$, $t \in T$,

$$\pi_0^t S\ (b,x^t) = \pi_0^t S\ (b,0^t) + \pi_0^t S\ (o,x^t)$$
$$= tr_0^t F'(b,0^t) + tr_0^t S^0(0,x^t) \qquad \blacksquare \qquad (2.26)$$

CHAPTER 3

THE OUTPUT-ALGEBRAIC LINEAR SYSTEM

1. Introduction

In this chapter the output-algebraic linear system (OALT-system) is introduced and studied. The notions of state attainability, controllability, and observability are used to study the OALT-system. It is shown that the notion of an observable state module is equivalent to the notion of a reduced index module for the free responses of S.

The Nerode equivalence relation for S in the zero state is used to construct an attainable and observable state module for S. A novel proof of the "minimality" of the state module is also presented.

2. The Output-Algebraic LT-system

This section introduces two systems which will be studied in the sequel. They are the algebraic and the output-algebraic LT-systems. The former is uniformly transitional, exhibits the response separation property, and its input set allows the concatenation of input segments. The latter is the series interconnection of an algebraic LT-system followed by a uniformly static LT-system. These concepts are now formalized.

2.1 Definition Let V be a T-object. If $v, v' \varepsilon V$ and $t \varepsilon T$, then

1. the juxtaposition of v and v' at time t is the set

$$v \overset{t}{\circ} v' = v^t \bigcup \{(t+t', v'(t')) \mid t' \varepsilon T \} \tag{3.1}$$

2. the <u>concatenation</u> of $v^t, v'^{t'} \epsilon \tilde{V}$ is defined as

$$v^t \circ v'^{t'} = (v \overset{t}{\circ} v')^{t+t'} \tag{3.2}$$

Thus concatenation is defined in terms of juxtaposition.

2.2 <u>Definition</u> Let V be a T-object. Then

1. V is <u>zero preloadable</u> iff for all $v \epsilon V$ and $t \epsilon T$, $0 \overset{t}{\circ} v \epsilon V$.

2. V is <u>zero postloadable</u> iff for all $v' \epsilon V$ and $t \epsilon T$, $v' \overset{t}{\circ} 0 \epsilon V$.

3. V is <u>zero loadable</u> iff ~~V~~ is both preloadable and post-loadable.

One can easily verify the

2.3 <u>Lemma</u> Let V be an LT-object. V is zero loadable iff for all $v, v' \epsilon V$ and $t \epsilon T$,

$$(v \overset{t}{\circ} v') = (v \overset{t}{\circ} 0) + (0 \overset{t}{\circ} v') \tag{3.3}$$

2.4 <u>Definition</u> Let S be an LT-system. S is <u>algebraic</u> iff

1. S is uniformly transitional.

2. $D(\pi^t S) = \mathcal{O}S \oplus \tilde{\mathcal{C}}_S^t$.

3. $\mathcal{D}S$ is zero loadable.

The next theorem relates Theorems 1.4.7 and 2.6.5 to the notion of the algebraic LT-system.

2.5 <u>Theorem</u> Let S be a weakly transitional LT-system with $S = F \oplus S^0$. If S is contracting and $\mathcal{D}S$ is zero loadable, then S is algebraic.

<u>Proof</u> It must be shown that S is uniformly transitional and exhibits the response separation property. Since S is contracting and weakly transitional, $S = \overline{S}$ and

$$\pi_0^t S = \pi_0^t \overline{S} : \mathcal{R}\overline{S}[0] \otimes \tilde{\mathcal{D}}\overline{S}^t \longrightarrow \mathcal{R}\overline{S}[t] \qquad (3.4)$$

But $\mathcal{R}S[0] = \mathcal{R}\overline{S}[0] = \mathcal{O}S$, $\mathcal{D}\overline{S} = \overline{\mathcal{D}S} = \mathcal{D}S$, and $\mathcal{R}\overline{S}[t] = \mathcal{R}\overline{S}[0] = \mathcal{O}S$. By substituting these equalities into (3.4) one obtains

$$\pi_0^t S = \pi_0^t \overline{S} = \pi^t S \ : \ \mathcal{O}S \otimes \tilde{\mathcal{D}}S^t \longrightarrow \mathcal{O}S \qquad (3.5)$$

Thus, S is uniformly transitional, exhibits the response separation property, and is therefore algebraic. ∎

Remark The fact that $\mathcal{D}S$ is zero loadable induces R-module structure on $\tilde{\mathcal{D}}S$. Two segments of unequal length may be summed by zero loading the shorter segment with an appropriate zero segment.

This allows a notational simplification of the morphism $\pi^t S$: $\mathcal{O}S \otimes \tilde{\mathcal{D}}S^t \longrightarrow \mathcal{O}S$. In particular, the final time t is implicitly given by the length of the initial segment. Thus, $\pi^t S$ is written as

$$\pi S \ : \ \mathcal{O}S \otimes \tilde{\mathcal{D}}S \longrightarrow \mathcal{O}S \qquad (3.6)$$

such that for all $b \in \mathcal{O}S$, $x^t, x'^{t'} \in \tilde{\mathcal{D}}S$

1. $\pi S\ (b, \Phi) = b$, Φ is the null sequence (3.7)
2. $\pi S\ (b, x^t) = \pi S\ (b, 0^t) + \pi S\ (0, x^t)$ (3.8)
3. $\pi S\ (0, (x^t \circ x')^{t+t'}) = \pi S\ (\pi S(0, x^t), x'^{t'})$ (3.9)
4. $\pi S\ (0, (x^t \circ x')^{t+t'}) = \pi S\ (0, (x^t \circ 0)^{t+t'}) + \pi S\ (0, (0^t \circ x')^{t+t'})$ (3.10)

2.6 Definition Let S be an LT-system. S is output-algebraic iff S is the series interconnection of a contracting, algebraic LT-system R, and a uniformly static LT-system P, such that

$$S = P \circ R$$

with

$$\mathcal{D}S = \mathcal{D}R, \quad \mathcal{R}R = \mathcal{D}P, \quad \mathcal{I}P = \mathcal{O}R, \text{ and } \mathcal{R}S = \mathcal{R}P.$$

The morphisms associated with S are the following:

$$\pi R : \mathcal{O}R \otimes \mathcal{D}\tilde{S} \longrightarrow \mathcal{O}R \quad ; \quad \text{st } P : \mathcal{O}R \longrightarrow \mathcal{O}S \tag{3.11}$$

and

$$\pi : \mathcal{O}R \otimes \mathcal{D}S \longrightarrow \mathcal{R}S \quad ; \quad \pi_R : \mathcal{O}R \otimes \mathcal{D}S \longrightarrow \mathcal{R}R \tag{3.12}$$

with

$$\pi = P \circ \pi_R \tag{3.13}$$

<u>Remark</u> The OALT-system is a generalization of those systems studied by Arbib, Gill and Kalman in that both continuous-time and discrete-time systems may be handled. Moreover, arithmetic operations are performed over an arbitrary ring R.

Arbib[4], for example, considered a discrete-time, time-invariant linear automaton defined by two functions

$$\lambda : Q \times X^* \longrightarrow Q \text{ and } \delta : Q \times X^* \longrightarrow Y$$

where Q, X, and Y are additive abelian groups and X* is the set of sequences on X. It is clear that λ represents an ALT-system, while δ is a mapping into the output space Y. Arbib shows that there is a reduction procedure which puts the system into output-algebraic form.

Both Gill [9] and Kalman [6] study discrete-time, linear time-invariant systems described by equations of the form

$$q(t+1) = Fq(t) + Gx(t) \text{ and } y(t) = Hq(t) \tag{3.14}$$

where K is a field, $q \in K^n$, $x \in K^m$, $y \in K^p$, $t \in N$ and F, G, H are nxn, nxm, and pxn constant matrices with elements from K.

In the literature, the equations in (3.14) are called the state transition and state-output equations, respectively.

It should be noted that the characterization of the OALT-system as an input-output relation is an unsolved problem.

3. Attainability, Controllability, and Observability

In this section the notions of attainability, controllability, and observability are used to study the properties of the state-transition morphism πR, and the state-output morphism st P of the OALT-system.

3.1 Definition Let S be an OALT-system.

1. A state $b \in \mathcal{O}R$ is attainable (from the zero state) iff there exists a $t \in T$, and an $x^t \in \widetilde{\mathcal{D}S}$ such that

$$\pi R\ (0, x^t) = b \tag{3.15}$$

2. A state $b' \in \mathcal{O}R$ is controllable (to the zero state) iff there exists a $t' \in T$, and an $x^{t'} \in \widetilde{\mathcal{D}S}$ such that

$$\pi R\ (b', x^{t'}) = 0 \tag{3.16}$$

3. The state module $\mathcal{O}R$ is completely controllable iff for any $b, b' \in \mathcal{O}R$, there exists a $t \in T$, an $x^t \in \widetilde{\mathcal{D}S}$ such that

$$\pi R\ (b, x^t) = b' \tag{3.17}$$

4. For each $t \in T$ define the sets of attainable and controllable states at time t as

$$A(t) = \{ b \mid (\exists\ x^t \in \widetilde{\mathcal{D}S^t}) : \pi R\ (0, x^t) = b \} \tag{3.18}$$

$$C(t) = \{ b' \mid (\exists\ x'^t \in \widetilde{\mathcal{D}S^t}) : \pi R(b', x'^t) = 0 \} \tag{3.19}$$

It is easy to verify the

3.2 <u>Theorem</u> For each $t \varepsilon T$, $A(t)$ and $C(t)$ are R-modules.

3.3 <u>Theorem</u> For all $t, t' \varepsilon T$ with $t \leq t'$,

1. $A(t) \subseteq A(t')$ and 2. $C(t) \subseteq C(t')$

<u>Proof</u> 1. If $b \varepsilon A(t)$ then there exists an $x^t \varepsilon \tilde{\mathcal{D}S}^t$ such that $\pi R(0, x^t) = b$. Also $\mathcal{D}S$ is zero preloadable $\Longrightarrow (0 \circ^{t'-t} x)^{t'} \varepsilon \tilde{\mathcal{D}S}^{t'} \Longrightarrow \pi R(0, (0 \circ^{t'-t} x)^{t'}) = b \varepsilon A(t')$. Thus, $A(t) \subseteq A(t')$ for $t \leq t'$.

2. If $b' \varepsilon C(t)$ then there exists an $x'^t \varepsilon \hat{\tilde{\mathcal{D}S}}^t$ such that $\pi R(b', x'^t) = 0$. Since $\mathcal{D}S$ is zero postloadable, $(x' \circ^t 0)^{t'} \varepsilon \tilde{\mathcal{D}S}^{t'}$ and $\pi R(b', (x' \circ^t 0)^{t'}) = 0$. Thus, $C(t) \subseteq C(t')$ for $t \leq t'$.

<u>Remark</u> The next series of statements help to relate $A(t)$ to $C(t)$. In particular, if $\mathcal{O}R$ is of <u>finite type</u>, then there exists a time $t^* \varepsilon T$ such that $A(t^*) \subseteq C(t^*)$.

3.4 <u>Definition</u> An R-module A satisfies the ascending chain condition (ACC) on submodules if for each ascending sequence

$$S_1 \subset S_2 \subset S_3 \quad \ldots \subset S_k \subset A$$

there is an index m with $S_m = S_{m+1} = S_{m+2}$, etc.

3.5 <u>Definition</u> An R-module A is of finite type if for some finite list $a_1, a_2, \ldots, a_k$, $a_i \varepsilon A$,

$$A = Ra_1 + Ra_2 + \ldots + Ra_k$$

and

$$Ra = \{ a' \mid a' = ra \ \& \ r \varepsilon R \}$$

3.6 Theorem An R-module A satisfies the ascending chain condition for submodules iff every submodule of A is finite type.

Proof See MacLane-Birkhoff [13], p 339.

3.7 Theorem Let S be an OALT-system. If $\mathcal{O}R$ is of finite type, then

1. The ascending sequences of submodules

$$A(t_1) \subset A(t_2) \quad . . . \subset A(t_k) \subset \mathcal{O}R \tag{3.20}$$

and

$$C(t_1') \subset C(t_2') \quad . . . \subset C(t_k') \subset \mathcal{O}R \tag{3.21}$$

satisfy the ACC for submodules.

2. There exists a time $t^* \varepsilon T$, such that

$$A(t^*) \subseteq C(t^*) \tag{3.22}$$

and $A(t^*)$ is completely controllable.

Proof 1. If $\mathcal{O}R$ is of finite type, then each of its submodules is of finite type. Hence sequences (3.20) and (3.21) satisfy the ACC for submodules. Furthermore, there exist $\tau_1, \tau_2 \varepsilon T$ such that $A(\tau_1)$ and $C(\tau_2)$ are maximal. Let $t^* = \tau_1 + \tau_2$.

2. $A(t^*) \subseteq C(t^*)$: Suppose $b \varepsilon A(t^*)$. Then there exists $x_b^{t^*} \varepsilon \tilde{\mathcal{D}S}$ such that $\pi R(0, x_b^{t^*}) = b$. For any $x^{t^*} \varepsilon \tilde{\mathcal{D}S}$

$$\pi R(b, x^{t^*}) = \pi R(b, 0^{t^*}) + \pi R(0, x^{t^*})$$

$$= \pi R(0, x_b^{t^*} \circ 0^{t^*}) + \pi R(0, 0^{t^*} \circ x^{t^*})$$

Now $\pi R(b, 0^{t^*}) = \pi R(0, x_b^{t^*} \circ 0^{t^*}) \varepsilon A(2t^*) = A(t^*)$. Choose x^{t^*} such that

$\pi R(0,x^{t^*}) = -\pi R(b,0^{t^*})$. Clearly $b \varepsilon C(t^*)$ which implies that $A(t^*)$ is completely controllable. ∎

It is of interest to know when $A(t^*) = C(t^*)$. The following result is due to Marino [14].

3.8 Theorem Let S be an OALT-system whose state module $\mathcal{O}R$ is of finite type. If the free system associated with R, F_R, is stationary, then $A(t^*) = C(t^*)$.

Proof Since F_R is stationary, $\mathcal{O}R = \mathcal{R}(F_R)[0] = \mathcal{R}(F_R)[t^*]$. Thus, if $b \varepsilon C(t^*)$ there exists $c \varepsilon \mathcal{R}(F_R)[0]$ such that $\pi R(c,0^{t^*}) = b$. Now $c \varepsilon C(t^*)$ because b is controllable. Clearly, $-c \varepsilon C(t^*)$ and there exists $x^{t^*} \varepsilon \mathcal{D}\tilde{S}$ such that

$$\pi R(-c,x^{t^*}) = \pi R(-c,0^{t^*}) + \pi R(0,x^{t^*}) = 0$$

Thus, $\pi R(0,x^{t^*}) = \pi R(c,0^{t^*}) = b$

so that $C(t^*) \subseteq A(t^*)$ and (3.22) $\Longrightarrow C(t^*) = A(t^*)$. ∎

Remark The assumption that $\mathcal{O}R$ is of finite type is strong, but it leads to the result that $A(t^*)$ is completely controllable.

While the notions of attainability and controllability pertain only to the ALT-system R, the notion of observability involves both R and the static system P. The next two definitions and the Remark follow Arbib [4]. ∎

3.9 Definition Let S be an OALT-system. Two states $b, b' \varepsilon \mathcal{O}R$ are equivalent iff for all $x \varepsilon \mathcal{D}S$

$$\pi(b,x) = \pi(b',x) \tag{3.23}$$

3.10 <u>Definition</u> The state module $\mathcal{O}R$ is <u>observable</u> iff no two states are equivalent.

<u>Remark</u> Definition 3.9 suggests an equivalence relation on $\mathcal{O}R$; namely for any b, $b' \varepsilon \mathcal{O}R$

$$b \underset{\sim}{\sim} b' \Longleftrightarrow \pi(b,x) = \pi(b',x) \text{ for all } x \varepsilon \mathcal{D}S \tag{3.24}$$

Since π exhibits the response separation property, (3.24) may be simplified to

$$b \underset{\sim}{\sim} b' \Longleftrightarrow \pi(b,0) = \pi(b',0) \tag{3.25}$$

Also, if two states are equivalent, then their difference is equivalent to the zero state. ∎

In some cases the state module is not completely observable, but it is possible to construct an observable submodule.

3.11 <u>Theorem</u> Let S be an OALT-system. Let $\overline{0} = \{b \mid b \varepsilon \mathcal{O}R \ \& \ \pi(b,0) = \pi(0,0) = 0\}$ denote the set of states equivalent to the zero state.

Then the quotient module $\mathcal{O}R/\overline{0}$ is observable.

<u>Proof</u> The set $\overline{0}$ is a submodule of $\mathcal{O}R$; it is also the kernal* of the morphism $(\pi|\mathcal{O}R \oplus 0)$. The conclusion is an immediate consequence of the Induced Morphism Theorem for modules (see Mac-Lane-Birkhoff, p.202 ; namely

$$Im\ (\pi|\mathcal{O}R \oplus 0) \cong \mathcal{O}R/\overline{0} \tag{3.26}$$

* See Appendix 1 for the definition.

Thus, $\mathcal{O}R/\overline{\mathcal{O}}$ is observable. ∎

It is easy to verify the

3.12 Theorem Let S be an OALT-system. The following statements are equivalent:

1. $\mathcal{O}R$ is a reduced index module for $Im(\pi|\mathcal{O}R \oplus 0)$,
2. $\mathcal{O}R$ is observable.
3. P is an isomorphism.

The concept of an invariant set is introduced below. It will be useful in the sequel.

3.13 Definition Let S be an OALT-system. A submodule $X \subset \mathcal{O}R$ is invariant under πR iff for any $b \varepsilon X$, and every $x^t \varepsilon \widetilde{\mathcal{D}S}$, $\pi R(b,x^t) \varepsilon X$.

3.14 Theorem Let S be an OALT-system whose state module $\mathcal{O}R$ is of finite type. Then the set of attainable states $A(t^*)$ is invariant under πR.

Proof Suppose $b \varepsilon A(t^*)$. Then there exists $x_b^{t^*} \varepsilon \widetilde{\mathcal{D}S}$ such that $\pi R(0,x_b^{t^*})=b$. For any $x^t \varepsilon \widetilde{\mathcal{D}S}$

$$\pi R(b,x^t) = \pi R(0,x_b^{t^*} \circ x^t) \varepsilon A(t^*+t) = A(t^*).$$

The conclusion follows. ∎

4. The State Reduction of OALT-Systems.

In this section the Nerode [15] equivalence relation for an OALT-system S in the zero-state is used to construct a subsystem S* whose state module is both attainable and observable. The transition and static system morphisms for S* are a natural consequence of the reduction process.

4.1 <u>Definition</u> Let S be an OALT-system. S is reduced iff its state module is attainable and observable.

4.2 <u>Definition</u> Let S be an OALT-system. Define the equivalence relation $\equiv_\pi \subset \widetilde{\mathcal{D}S} \times \widetilde{\mathcal{D}S}$ such that for any $x_1{}^t, x_2{}^{t'} \varepsilon \widetilde{\mathcal{D}S}$ and for all $x \varepsilon \mathcal{D}S$

$$x_1{}^t \equiv_\pi x_2{}^{t'} \Longleftrightarrow [\pi(0,x_1 \overset{t}{\cdot} x)]_t = [\pi(0,x_2 \overset{t'}{\cdot} x)]_{t'} \qquad (3.27)$$

where $[\pi]_t$ denotes the t-section of π (see Definition 1.4.1) and π is defined by (3.12) and (3.13).

<u>Remark</u> The relation $\equiv_\pi$ is the Nerode equivalence relation for S in the zero state. By using Lemma 2.3 and the fact that preloading by a zero segment when S is in the zero state yields the zero output segment, one obtains the following simplification of (3.27):

$$[\pi(0,x_1 \overset{t}{\cdot} x)]_t = [\pi(0,x_1 \overset{t}{\cdot} 0) + \pi(0,0 \overset{t}{\cdot} x)]_t$$

$$= [\pi(0,x_1 \overset{t}{\cdot} 0)]_t + [\pi(0,0 \overset{t}{\cdot} x)]_t$$

and $\qquad [\pi(0,0 \overset{t}{\cdot} x)]_t = [\pi(0,0 \overset{t'}{\cdot} x)]_{t'}$

Thus, $x_1{}^t \equiv_\pi x_2{}^{t'} \Longleftrightarrow [\pi(0,x_1 \overset{t}{\cdot} 0)]_t = [\pi(0,x_2 \overset{t'}{\cdot} 0)]_t \quad \blacksquare \qquad (3.28)$

4.3 <u>Theorem</u> Let S be an OALT-system. Define the sets

$$Y_t = \{ y_t \mid y_t = [\pi(0,x \overset{t}{\cdot} 0)]_t \ \& \ x^t \varepsilon \widetilde{\mathcal{D}S} \} \qquad (3.29)$$

$$Y = \{ y_t \mid y_t = [\pi(0,x \overset{t}{\cdot} 0)]_t \ \& \ x^t \varepsilon \widetilde{\mathcal{D}S} \ \& \ t \varepsilon T \} \qquad (3.30)$$

Then 1. For each $t\varepsilon T$, Y_t is an R-module.

2. For any $t,t'\varepsilon T$ with $t \leq t'$, then $Y_t \subseteq Y_{t'}$.

3. For each $y_t \varepsilon Y$, there exists a $b\varepsilon \mathcal{O}R$ such that

$\pi(b,0) = y_t$.

Proof 1. For $t = 0$, $Y_0 = \{0\}$, a trivial module. For any other $t\varepsilon T$, 0, y_t, and $-y_t \varepsilon Y_t \Longrightarrow Y_t$ is an R-module.

2. Since $\mathcal{D}S$ is zero preloadable,

$$y_t = [\pi(0,x^t_0)]_t = [\pi(0,0^{t'-t}x^t_0)]_{t'} = y_{t'} \Longrightarrow Y_t \subseteq Y_{t'} \text{ for } t \leq t' .$$

3. Recall that $S = P\circ R$ with R contracting. Thus $(0,y_t)\varepsilon S_t \Longrightarrow (0,y'_t)\varepsilon R_t$ for some $y'_t \varepsilon RR_t$. But $R_t \subset R \Longrightarrow (0,y'_t)\varepsilon R \Longrightarrow$ there exists $b\varepsilon \mathcal{O}R$ such that $\pi(b,0) = P\circ\pi_R(b,0) = y_t$. ∎

4.4 Theorem Let S be an OALT-system. If Y is of finite type, then every ascending sequence

$$Y_{t_1} \subseteq Y_{t_2} \subseteq \ldots\ldots \subseteq Y_{t_k} \subseteq Y$$

satisfies the ACC for submodules, and there exists a time $\tau\varepsilon T$ such that for all $t \geq \tau$

$$Y_\tau = Y_t = Y \qquad (3.31)$$

Proof Similar to that of Theorem 3.7.

Remark Since τ denotes the maximum input segment length required to generate Y, the assignment $x^t \longrightarrow y_t$ is an epimorphism

$\pi_\tau : \widetilde{\mathcal{D}S} \longrightarrow Y$ whose kernal is the module

$$\text{Ker } \pi_\tau = \{x^t \mid [\pi(0,x^t \circ 0)]_t = 0 \ \ \& \ x^t \varepsilon \widetilde{\mathcal{D}S}\} \tag{3.32}$$

Note that (3.31) is also the kernal of the natural projection, $p: \widetilde{\mathcal{D}S} \longrightarrow \widetilde{\mathcal{D}S}/\equiv\pi$. Thus, by the induced morphism theorem, the following diagram commutes and P* is an isomorphism.

$$\begin{array}{ccc} \widetilde{\mathcal{D}S} & \xrightarrow{\quad p \quad} & \widetilde{\mathcal{D}S}/\equiv\pi \ = \ X \\ & \searrow^{\pi_\tau} & \downarrow P^* \\ & & Y \end{array} \tag{3.33}$$

By the commutative diagram of (3.33), it is easy to prove the

4.5 Theorem Let S be an OALT-system. Suppose Y is of finite type, with $Y_\tau = Y$. Let $X = \widetilde{\mathcal{D}S}/\equiv\pi$ and let $[x] \varepsilon X$ denote the equivalence class

$$[x] = \{x'^t \mid x'^t \equiv\pi \ x^\tau \ \ \& \ x'^t \varepsilon \widetilde{\mathcal{D}S}\ \}$$

Then there exists a reduced subsystem S* of S whose state transition and static morphisms are

(i) $\pi R^* : X \oplus \widetilde{\mathcal{D}S} \longrightarrow X$ such that for all $[x] \varepsilon X$, $x'^t \varepsilon \widetilde{\mathcal{D}S}$

$$\begin{aligned} \pi R^*([x], x'^t) &= p(x^\tau \circ x'^t) = p(x^\tau \circ 0^t) + p(0^\tau \cdot x'^t) \\ &= [x^\tau \circ 0^t] + [0^\tau \circ x'^t] = [x^\tau \circ x'^t] \end{aligned} \tag{3.34}$$

(ii) st $P^* : X \longrightarrow \mathcal{O}S$ such that for any $[x] \varepsilon X$

$$\text{st } P^*([x]) = P^*([x])\,[0] = \pi S(0,x^{\tau}) \qquad (3.35)$$

<u>Remark</u> The Nerode equivalence relation for S in the zero state induces a partition, $\widetilde{\mathcal{D}S}/\equiv_\pi$, of $\widetilde{\mathcal{D}S}$ which serves as a reduced state module for S. Thus, input segments are associated with states, and states are in 1:1 correspondence with future outputs.

Kalman [8] probably had the Nerode equivalence relation in mind when he defined a linear, zero-state, input-output function on input segments to future outputs. The reduction presented here is a generalization of Kalman's work in that the results hold for continuous and discrete-time OALT-systems defined over an arbitrary ring. ∎

The remainder of this section is used to show that $\widetilde{\mathcal{D}S}/\equiv_\pi$ is the finest partition of $\widetilde{\mathcal{D}S}$ necessary to obtain an attainable and observable state module for S.

4.6 <u>Definition</u> A <u>partially ordered set</u> is a set S together with a binary relation $\leq$ which is

1. Reflexive : $x \leq x$, for all $x \in S$
2. Anti-symmetric : $x \leq y$ & $y \leq x \Longrightarrow x = y$ for all $x,y \in S$
3. Transitive : $x \leq y$ & $y \leq z \Longrightarrow x \leq z$, for all $x,y,z \in S$.

<u>Remark</u> Let X be a subset of a partially ordered set S. A lower bound of X is an element $b \in S$ such that $b \leq x$, for all $x \in X$. The element b is a greatest lower bound (g.l.b.) if it is greater than all other lower bounds.

An equivalence relation $\equiv_P$ induces a partition P of S. Denote the set of partitions of S by $P(S)$. A partial ordering is put on

$\mathcal{P}(S)$ by defining, for any $P_1, P_2 \in \mathcal{P}(S)$

$$P_1 \leq P_2 \iff \text{each equivalence class of } P_1 \text{ is contained in an equivalence class of } P_2.$$

Clearly, if $P_1 \leq P_2$ then $x \equiv_{P_1} y \Longrightarrow x \equiv_{P_2} y$, $x,y \in S$.

For any P_1, $P_2 \in \mathcal{P}(S)$ define $P_1 \circ P_2$ as the intersection of the equivalence classes of P_1 and P_2. Then

$$x \equiv_{P_1 \circ P_2} y \iff x \equiv_{P_1} y \text{ and } x \equiv_{P_2} y \tag{3.36}$$

and

$$P_1 \circ P_2 = \text{g.l.b. } (P_1, P_2) \quad ∎ \tag{3.37}$$

4.7 <u>Theorem</u> Let S be an OALT-system. Define the attainable and observable equivalence relations $\equiv_A \subset \tilde{\mathcal{D}S} \times \tilde{\mathcal{D}S}$ and $\equiv_{\Theta} \subset \widetilde{\mathcal{D}S} \times \widetilde{\mathcal{D}S}$ as follows

$$x_1{}^t \equiv_A x_2^{t'} \iff [\pi(0, x_1{}^t\cdot 0)]_t[0] = [\pi(0, x_2{}^{t'}\cdot 0)]_{t'}[0] \tag{3.38}$$

$$x_1{}^t \equiv_{\Theta} x_2^{t'} \iff (\forall\, t'' \in T) : [\pi(0, x_1{}^t\cdot 0)]_t[t''] = [\pi(0, x_2{}^{t'}\cdot 0)]_{t'}[t''] \tag{3.39}$$

Then 1. $\Theta \leq A$ and 2. $\widetilde{\mathcal{D}S}/\equiv_\pi = A \circ \Theta = \text{g.l.b. } (A, \Theta)$.

<u>Proof</u> 1. Let $t'' = 0$ in (3.39) so that $x_1{}^t \equiv_{\Theta} x_2^{t'} \Rightarrow x_1{}^t \equiv_A x_2^{t'} \Rightarrow A \leq \Theta$.

2. By pointwise extension on t'', it is easy to see that $x_1^t \equiv_{\Theta} x_2{}^{t'} \iff x_1^t \equiv_\pi x_2^{t'}$. Thus $\tilde{\mathcal{D}S}/\equiv_\pi = \Theta$ and $\Theta \leq A \Longrightarrow$

$$\tilde{\mathcal{D}S}/\equiv_\pi = A \circ \Theta = \text{g.l.b.}(A, \Theta) \quad ∎$$

Remark The above result shows that $\mathcal{D}S/{\equiv_\pi}$ is *maximal* in the sense that all non-equivalent attainable states are present, and *minimal* in the sense that only non-equivalent observable states are present. ∎

CHAPTER 4

RATIONAL CANONICAL FORMS AND MINIMAL SYSTEMS

1. Introduction

In this chapter discrete-time OALN-systems are studied. The triple for these systems is (K,K^m,K^p) where K is an arbitrary field. Linear sequential machines and linear discrete-time control systems are included in this class of systems.

Section 2 presents the vector-matrix representation of OALN-systems. Necessary and sufficient conditions for the complete controllability and observability of the state module are given in terms of the system matrices.

In section 3 the minimal OALN-system is introduced. This system provides a link between free and functional systems and has the minimum number of input and output terminals necessary for the complete controllability and observability of its state module. Moreover, the minimal system serves as a "template" for the design of control systems. An example is provided to illustrate these points.

2. Vector-Matrix Representations of OALN-Systems

Let S be an OALN-system with triple (K,K^m,K^p). The state transition and static system morphisms are

$$\pi R : \mathcal{O}R \oplus \tilde{\mathcal{D}}S^1 \longrightarrow \mathcal{O}R \quad ; \text{st } P : \mathcal{O}R \longrightarrow \mathcal{O}S \qquad (4.1)$$

where $\mathcal{O}R = K^n$, $\tilde{\mathcal{D}}S^1 = \mathcal{D}S[0] = K^m$, and $\mathcal{O}S = K^p$.

Suppose the unit vectors $\{e_i \mid i = 1,2, \ldots n\}$, $\{e_j \mid j = 1,2, \ldots,m\}$ and $\{e_k \mid k = 1,2, \ldots,p\}$ form bases for K^n, K^m, and K^p, respectively. Then the matrix representations for πR and st P are

$$(\pi R|\mathcal{O}R \oplus 0^1) = F : K^n \longrightarrow K^n \qquad (4.2)$$

$$(\pi R|0 \oplus \tilde{\mathcal{D}}S^1) = G : K^m \longrightarrow K^n \qquad (4.3)$$

$$\text{st } P = H : K^n \longrightarrow K^p \qquad (4.4)$$

where F, G, H are nxn, nxm, pxn dimensional matrices of K, respectively.

The vector-matrix representation for S is

$$S) \quad q(t+1) = F\,q(t) + G\,x(t) \qquad (4.5)$$

$$y(t) = H\,q(t) \qquad (4.6)$$

where $q(t)\in\mathcal{O}R$, $x(t)\in\mathcal{D}S[t]\subset\mathcal{D}S[0]$, $y(t)\in\mathcal{O}S$, and $t\in N$.

Let $\mathcal{S}$ denote the class of OALN-systems described by (4.1) through (4.6). In the sequel a system $S\in\mathcal{S}$ will be represented by its matrix triple (F,G,H) of compatible dimensions. At times, a system may be considered over several fields; in that case the system is described by the 4-tuple (F,G,H,K), where K is the field.

Many of the results obtained in previous chapters may be expressed in terms of (F,G,H). In particular, this section studies i) the response separation property, and ii) attainability, controllability, and observability conditions.

i) Response Separation - The values of the state and output at any time t may be obtained by recursion on (4.5) and (4.6). Thus for any $q(0) \in \mathcal{OR}$ and any $x^t \in \hat{\mathcal{DS}}$

$$q(1) = Fq(0) + Gx(0)$$

$$q(2) = Fq(1) + Gx(1) = F^2q(0) + FGx(0) + Gx(1)$$

$$\vdots$$

$$q(t) = F^t q(0) + \sum_{i=0}^{t-1} F^{t-i-1} Gx(i) \qquad (4.7)$$

and

$$y(t) = Hq(t) = HF^t q(0) + \sum_{i=0}^{t-1} F^{t-i-1} Gx(i) \qquad (4.8)$$

The response separation property is evident in (4.7) and (4.8). This is not surprizing when one considers the definition of F and G in (4.2) and (4.3).

ii) Attainability, Controllability, and Observability Theorem 3.3.7 shows that if $\mathcal{OR}$ is of finite type, then there exists a $t^* \in T$ such that for all $t \geq t^*$

$$A(t^*) = A(t) \text{ and } A(t^*) \subseteq C(t^*)$$

Since $\mathcal{OR}$ is an n-dimensional vector space, it is clearly of finite type. Moreover, it is easy to show that if $\mathcal{DS} = (K^m)^N$ then

$A(t^*) = A(n)$.

By setting $q(0) = 0$ and $t = n$ in (4.7), one obtains

$$q(n) = \sum_{i=0}^{n-1} F^{n-i-1} G\, x(i)$$

$$q(n) = [F^{n-1}G \mid F^{n-2}G \mid \ldots \; FG \mid G]\, x^n \qquad (4.9)$$

It is easy to prove the

2.1 <u>Theorem</u> Let $S \in \mathcal{S}$ with (F,G,H). Then $\mathcal{OR} = A(n)$ iff

$$\text{rank } [F^{n-1}G \mid F^{n-2}G \mid \ldots \; FG \mid G\,] = n \qquad (4.10)$$

A direct consequence of Theorem 3.3.8 is the

2.2 <u>Theorem</u> Let $S \in \mathcal{S}$ with (F,G,H). Let the free system associated with R be stationary. Then $\mathcal{OR} = A(n) = \mathcal{C}(n)$ iff

$$\text{rank } [F^{n-1}G \mid F^{n-2}G \mid \ldots \; FG \mid G\,] = n$$

$\blacksquare$

The observability conditions are obtained by recalling that if $\mathcal{OR}$ is a reduced state module then there is a 1:1 correspondence between states and future outputs (see the commutative diagram of (3.33)). Thus, for every free response there exists a unique initial state. Now $\mathcal{OR}$ is an n-dimensional vector space so that $Y_n = Y$ (see (3.31)). This implies that if $\mathcal{OR}$ is observable then the initial state q(0) can be determined from a free response of length n,

$$[y(n-1) \mid y(n-2) \mid \ldots \mid y(0)] = [HF^{n-1} \mid \ldots HF \mid H]q(0) \tag{4.11}$$

Now q(0) may be obtained by a least-squares fitting procedure in which the error function is zero. The observability condition can then be expressed as

2.3 <u>Theorem</u> Let $S \varepsilon \mathcal{S}$ with (F,G,H). Then $\mathcal{OR}$ is observable iff

$$\text{rank } [(F')^{n-1}H' \mid (F')^{n-2}H' \mid \ldots \ldots F'H' \mid H'] = n \tag{4.12}$$

where "'" denotes the transpose.

<u>Remark</u> The notions of controllability and observability, and the rank conditions of (4.10) and (4.12) insure that $\mathcal{OR}$ is spanned by n linearly independent vectors.

Now for each $g_i \varepsilon G$, $i = 1,2, \ldots, m$, and $h'_j \varepsilon H'$, $j = 1,2,\ldots,p$, define the vector subspaces

$$X_{g_i} = [F^{n-1}g_i \mid F^{n-2}g_i \mid \ldots \ldots Fg_i \mid g_i] \tag{4.13}$$

and

$$X_{h'_j} = [(F')^{n-1}h'_j \mid \ldots \ldots \ldots F'h'_j \mid h'_j] \tag{4.14}$$

spanned by the linearly independent column vectors of (4.13) and (4.14). Both X_{g_i} and $X_{h'_j}$ are said to be "generated" by g_i and h'_j, respectively. Clearly,

$$X_G = \bigcup_i X_{g_i} \quad \text{and} \quad X_{H'} = \bigcup_j X_{h'_j} \tag{4.15}$$

and if $\mathcal{OR}$ is completely controllable and observable then rank X_G = rank $X_{H'}$ = n. In section 3 the minimal system is introduced; it has the property that

$$X_G = X_{g_1} \oplus X_{g_2} \oplus \cdots \oplus X_{g_k}$$

and

$$X_{H'} = X_{h'_1} \oplus X_{h'_2} \oplus \cdots \oplus X_{h'_k}$$

where G and H' are both n x k dimensional matrices. ∎

3. <u>Rational Canonical Forms and Minimal Systems</u>

The previous section showed that the matrix F plays an important role in the controllability and observability conditions. Thus, F may be considered the "core" of an S&S with matrix triple (F,G,H). It will be shown that F has certain algebraic properties which allow it to be written as the direct sum of companion matrices (to be defined later). This direct sum decomposition motivates the definition of the minimal system which serves as a "template" for the design of control systems whose "core" matrix is F.

The discussion which follows uses concepts from Algebra such as ideals, principal ideal domains, torsion modules, and cyclic modules. Appendix 1 contains a review of this material, and a detailed account may be found in MacLane-Birkhoff.

3.1 <u>Proposition</u> Let K be a field. Every linear transformation $t : V \longrightarrow V$ from a vector space V over K into itself, induces a K[z]-module A, where A is the additive abelian group of V.

<u>Proof</u> See MacLane-Birkhoff [13].

<u>Remark</u> Many notions associated with vector spaces have module counterparts. For example, if V is finite dimensional, A is a torsion module of finite type. The minimal polynomial for t is the minimal annihilator for the K[z]-module A. Also, submodules of A are t-invariant subspaces of V. ∎

3.2 <u>Definition</u> A <u>monic</u> polynomial $f \in K[z]$ of degree n has the form

$$f = z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0 \qquad (4.20)$$

where the $c_i \in K$ and $c_n = 1$.

The <u>companion</u> <u>matrix</u> of f is the n x n matrix

$$M_f = \begin{bmatrix} 0 & . & . & . & 0 & -c_0 \\ 1 & 0 & . & . & . & -c_1 \\ 0 & 1 & 0 & . & . & . \\ 0 & 0 & 1 & 0 & . & . \\ 0 & 0 & 0 & 1 & 0 & -c_{n-2} \\ 0 & 0 & 0 & 0 & 1 & -c_{n-1} \end{bmatrix} \qquad (4.21)$$

Remark The minimal polynomial of M_f is f and its characteristic polynomial is $(-1)^n f$. ∎

The next theorem is a special case of the Rational Canonical Decomposition Theorem for torsion modules over a principal ideal domain.

3.3 Theorem Let $t : V \longrightarrow V$ be a morphism of a finite-dimensional vector space V over K. Then there exists exactly one list of non-constant monic polynomials of K[z], $\{f_1, f_2, \ldots, f_k\}$ with f_{i+1} dividing f_i, such that V has at least one basis for which the matrix of t is

$$M_t = M_{f_1} \oplus M_{f_2} \oplus \ldots \ldots \oplus M_{f_k} \qquad (4.22)$$

The K[z]-module A associated with (V,t) is the direct sum of k cyclic submodules of order f_i , i = 1,2, ..., k, of the form

$$A \cong K[z]/(f_1) \oplus K[z]/(f_2) \oplus \ldots \oplus K[z]/(f_k) \qquad (4.23)$$

where $(f_i) = \{r \cdot f_i \mid r \in K[z]\}$ is a principal ideal.

Proof See MacLane-Birkhoff [13].

Remark The list $\{f_1, f_2, \ldots, f_k\}$ are the invariant factors of M_t. In particular, f_1 is the minimal polynomial for t, and the product $f_1 \cdot f_2 \cdot \ldots \cdot f_k$ is the characteristic polynomial for t. ∎

An immediate consequence of the theorem is the

3.4 Corollary Any square matrix F over K is similar* over K to one matrix of the form (4.22).

*Two n x n matrices F and $\hat{F}$ (over K) are similar (over K) iff there is an invertible n x n matrix P (over K) for which $\hat{F} = PFP^{-1}$. A direct consequence of statements 3.1 through 3.4 is the

3.5 <u>Theorem</u> Let $S \in \mathcal{S}$ with representation (F,G,H,K). Then F is similar (over K) to a matrix

$$\hat{F} = M_{f_1} \oplus M_{f_2} \oplus \cdots \oplus M_{f_k} \qquad (4.24)$$

where the f_i, i = 1,2, ..., k, are defined in Theorem 3.3.

Furthermore, the K[z]-module X associated with $[\mathcal{O}R,(\pi R|\mathcal{O}R \oplus 0^1)]$ is the direct sum of cyclic submodules

$$X \cong K[z]/(f_1) \oplus K[z]/(f_2) \oplus \cdots \oplus K[z]/(f_k) \qquad (4.25)$$

<u>Remark</u> The similarity transformation which puts F into rational canonical form may be interpreted as a coordinate transformation of the state vector in (4.5) and (4.6). Let P be the nonsingular transformation such that $\hat{F} = PFP^{-1}$. Also, let $\hat{q} = Pq$ so that (4.5) and (4.6) become

$$P^{-1}\hat{q}(t+1) = FP^{-1}\hat{q}(t) + G\,x(t) \;;\quad y(t) = HP^{-1}\hat{q}(t)$$

or

$$\hat{q}(t+1) = PFP^{-1}\hat{q}(t) + PG\,x(t) \;;\quad y(t) = HP^{-1}\hat{q}(t)$$

Define $\hat{F} = PFP^{-1}$, $\hat{G} = PG$, and $\hat{H} = HP^{-1}$ so that

$$S) \qquad \hat{q}(t+1) = \hat{F}\hat{q}(t) + \hat{G}\,x(t) \qquad (4.26)$$

$$y(t) = \hat{H}\hat{q}(t) \qquad (4.27)$$

Note that S has the same input-output representation but now $\hat{F}$ is in rational canonical form. Henceforth, it is assumed that $S \in \mathcal{S}$ has matrix triple (F,G,H) with F already in rational canonical form. This avoids the cumbersome notation of (4.26) and (4.27).

Both Gill [10] and MacLane-Birkhoff [13] give algorithms to calculate the invariant factors and the corresponding similarity transformation. ∎

Kalman [8] views $\tilde{\mathcal{D}}S$ as a K[z]-module on m free generators. By defining an equivalence relation similar to $\equiv_\pi$ (3.27) he obtains a reduced state module $\tilde{\mathcal{D}}S/\equiv_\pi$ of finite type, which is also K[z]-module. He then uses the Rational Canonical Decomposition Theorem for torsion modules over a principal ideal domain to decompose $\tilde{\mathcal{D}}S/\equiv_\pi$ into the direct sum of cyclic and free modules, i.e.

$$\tilde{\mathcal{D}}S/\equiv_\pi \;\cong\; {}^{K[z]}/_{(f_1)} \oplus \ldots \oplus {}^{K[z]}/_{(f_k)} \qquad (4.28)$$

where the $\{f_1, \ldots f_k\}$ are the invariant factors.

In order to obtain this decomposition, Kalman sacrifices the physical significance of the input and output terminals. The approach taken here not only yields the same decomposition (Theorem 3.5) but also preserves the physical interpretation of the input and output terminals. The notion of the <u>minimal</u> system provides the link between F and the matrices G, H. Consider the

3.6 <u>Definition</u> Let $S \in \mathcal{S}$ with (F,G,H). Suppose the invariant factors of F are $\{f_1, f_2, \ldots, f_k\}$. Then S is <u>minimal</u> iff

(i) G and H' are n x k dimensional matrices.

(ii) For $i = 1,2, \ldots, k$, there exist column vectors $g_i[h_i']$ with zero entries except for a 1 in the position corresponding to the first [last] row of M_{f_i}.

3.7 Example Let F be a 5 x 5 matrix with invariant factors $f_1 = x^3+4x^2+5x+2$, $f_2 = x^2+3x+2$. The minimal system has the form

$$F=\begin{bmatrix} 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & -5 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \quad G=\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad H'=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \blacksquare \qquad (4.29)$$

Minimal systems have interesting structural properties which provide insight into the notions of controllability and observability.

3.8 Theorem If S is minimal then S is completely controllable and observable.

Proof Suppose $S \varepsilon \mathcal{S}$ with (F,G,H) is minimal and F has invariant factors $\{f_1, f_2, \ldots, f_k\}$. Now for each $i = 1,2, \ldots, k$, g_i and h_i' generate cyclic subspaces of order f_i, where $f_i = z^{\nu_i} + c_{\nu_i - 1} z^{\nu_i - 1} + \ldots c_0$ and ν_i is the degree of f_i.

Thus the subspaces generated by g_i and h_i' are spanned by the linearly independent vectors

$$X_{g_i} = [F^{\nu_i-1} g_i \mid F^{\nu_i-2} g_i \mid \ldots\ldots F g_i \mid g_i] \quad (4.30)$$

and

$$X_{h_i'} = [(F')^{\nu_i-1} h_i' \mid (F')^{\nu_i-2} h_i' \ldots\ldots F'h_i' \mid h_i'] \quad (4.31)$$

Moreover, $X_{g_i} = X_{h_i'}$, $i = 1,2, \ldots , k$, with

$$X_G = X_{g_1} \oplus X_{g_2} \oplus \cdots\cdots \oplus X_{g_k} \quad (4.32)$$

and

$$X_{H'} = X_{h_1'} \oplus X_{h_2'} \oplus \cdots\cdots \oplus X_{h'_k} \quad (4.33)$$

Hence rank $X_G = \sum_{i=1}^{k} \text{rank } X_{g_i} = \sum_{i=1}^{k} \nu_i = n \quad (4.34)$

$$\text{rank } X_{H'} = \sum_{i=1}^{k} \text{rank } X_{h'_i} = \sum_{i=1}^{k} \nu_i = n \quad (4.35)$$

so that *O*R (and S) is completely controllable and observable. ∎

The following statements are a direct consequence of Theorem 3.8 and equations (4.30) through (4.35).

3.9 Theorem Let $S \in \mathcal{S}$. If S is minimal then S is the direct sum of k single-input, single-output minimal systems,

$$S = S_1 \oplus S_2 \oplus \cdots\cdots \oplus S_k \quad (4.32)$$

3.10 Theorem (i) Every free ALN-system has an associated minimal OALN-system.

(ii) Every OALN-system has an associated minimal system.

3.11 Theorem Let $S \epsilon \mathcal{S}$ with (F,G,H). Suppose F has invariant factors $\{f_1, f_2, \ldots, f_k\}$.

If S is completely controllable [observable], then S has at least k input[output] terminals.

3.12 Corollary Let $S \epsilon \mathcal{S}$ with (F,G,H) as in Theorem 3.11. If S is completely controllable [observable] then rank G $\geq$ k [rank H $\geq$ k]. Moreover, if S has exactly k input [output] terminals, then rank G = k [rank H = k].

Remark There are two conclusions to be drawn from the above statements. First, the minimal system links free and functional systems. More importantly the minimal system specifies the minimum number of input and output terminals necessary for complete controllability and observability.

Second, the minimal system of (4.32) can be written as the direct sum of single-input, single-output minimal systems. Conversely, minimal systems can be constructed from component minimal systems. To be completely general, the concept could be extended to the study of OALT-systems with arithmetic operations over an arbitrary ring R, or a set of arbitrary rings.

The rational canonical form for F is not necessarily the most elementary canonical form for F. For example, the invariant factors $f_1 = x^3+4x^2+5x+2$, $f_2 = x^2+3x+2$ of Example 3.7 may be factored into the product of linear, monic polynomials, i.e.,

$$f_1 = (x+1)^2(x+2) \quad ; \quad f_2 = (x+1)(x+2)$$

As will be shown in the sequel, F is similar to the direct sum of Jordan elementary matrices. This brief discussion provides the motivation for the statements which follow.

3.13 Definition Let D be a principal ideal domain.

(i) Elements $a,b \in D$ are associate to one another when $a|b$ (a divides b) and $b|a$. If $d|1$ then $d^{-1} \in D$ and d is said to be invertible.

For example, $f,g \in K[z]$ are associates in K[z] iff $g = c\,f$ where c is an invertible constant in K[z].

(ii) A prime $p \in D$ is not invertible in D and has no proper divisors. A polynomial which is prime in D[z] is called irreducible.

For example, the linear monic polynomials $x+b$ and primes $p \in D$ are irreducible in D[z].

(iii) Let p be a prime in D. A p-module is a D-module P in which every element has order some power of p. Any D-module is primary if it is a p-module.

3.14 Theorem (The Primary Decomposition Theorem) Any torsion module A of finite type over a principal ideal domain D is the biproduct (direct sum) of primary modules. Thus, if A has minimal annihilator

$$\nu = p_1^{e_1} \, p_2^{e_2} \, \ldots \ldots \, p_\ell^{e_\ell} \tag{4.33}$$

where $p_1, p_2, \ldots, p_\ell$ are primes in D, no two associates to one another, then

$$A \cong T_{p_1}(A) \oplus \ldots \ldots \oplus T_{p_\ell}(A) \tag{4.34}$$

where $T_{p_i}(A)$ is the largest p_i - submodule of A.

Proof See MacLane-Birkhoff [13].

A direct consequence of the theorem is the

3.15 Corollary Each n x n matric F over a field has a list $\{p_1^{e_1}, p_2^{e_2}, \ldots p_\ell^{e_\ell}\}$ of monic irreducible polynomials $p_i \varepsilon K[z]$ such that F is similar to the direct sum of companion matrices.

$$F \cong M_{p_i^{e_i}} \oplus \ldots\ldots \oplus M_{p_\ell^{e_\ell}} \tag{4.35}$$

Each $p_i^{e_i}$ is called an elementary divisor of F

Remark An important case to study is the linear monic polynomial $p^e = (z-\lambda)^e$, of power e. Suppose e = 3 so that $p^3 = (z-\lambda)^3$. Corresponding to this polynomial is the cyclic p-module B of order p^3. Suppose b_0 generates B, then $(t-\lambda)^3 b_0 = 0$ while $(t-\lambda)^2 b_0 \neq 0$ and $(t-\lambda)b_0 \neq 0$. Thus the vectors $g_1 = b_0$, $g_2 = (t-\lambda)b_0$, and $g_3 = (t-\lambda)^2 b_0$ form a basis for the vector space corresponding to B. In particular

$$(t-\lambda)g_1 = g_2 \Longrightarrow t_{g_1} = \lambda g_1 + g_2$$

$$(t-\lambda)g_2 = g_3 \Longrightarrow t_{g_2} = \lambda g_2 + g_3$$

$$(t-\lambda)g_3 = 0 \Longrightarrow t_{g_3} = \lambda g_3$$

The matrix corresponding to t for this basis is

$$J(\lambda,3) = \begin{bmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{bmatrix}$$

and $J(\lambda,3)$ is called a Jordan matrix.

It is easy to show the

3.16 Lemma If $p = z-\lambda$ is a monic linear polynomial in K[z] and B is a cyclic p-module of order $(z-\lambda)^e$, with (V,t) its correponding vector space and morphism then the matrix of t relative to a suitable basis of V is $J(\lambda, e)$.

The lemma can be used to prove the

3.17 Theorem Let F be a square matrix over a field K. If the elementary divisors of F are monic linear factors to some power, then F is similar to the direct sum of Jordan matrices, one for each elementary divisor.

The material in this section is applied in the following example.

3.18 Example

Let $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the rational, real and complex number fields, respectively. Let S be a free ALN-system with matrix representation

$$q(t+1) = Fq(t) \qquad (4.36)$$

where F is an 8 x 8 matrix over $\mathbb{Q}$. The invariant factors of F over $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are

$$f_1 = z^6+2z^4-4z^2-8 \text{ and } f_2 = z^2-2 \tag{4.37}$$

The objectives of this example are the following:

1. To obtain the rational canonical form for F.
2. To construct the <u>minimal</u> system for S.
3. To use the minimal system as a "template" to check controllability and/or observability of an $\underline{S} \in \mathcal{S}$ with $(F,\underline{G},\underline{H})$.
4. To list the elementary divisors of F over $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$.
5. To decompose F into primary form and to discuss the corresponding changes in the minimal system.

1-2 Rational Canonical Form for F and its Minimal System

The rational canonical form for F is simply the direct sum of companion matrices

$$F \cong M_{f_1} \oplus M_{f_2}$$

where f_1, f_2 are defined by (4.37). Let $\hat{S} \in \mathcal{S}$ with $(F,\hat{G},\hat{H})$ denote the minimal system for S. The three matrices are given below.

$$F=\left[\begin{array}{cccccc|cc} 0 & 0 & 0 & 0 & 0 & 8 & & \\ 1 & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 1 & 0 & 0 & 0 & 4 & & \\ 0 & 0 & 1 & 0 & 0 & 0 & \multicolumn{2}{c}{\bigcirc} \\ 0 & 0 & 0 & 1 & 0 & -2 & & \\ 0 & 0 & 0 & 0 & 1 & 0 & & \\ \hline & & & & & & 0 & 2 \\ \multicolumn{6}{c|}{\bigcirc} & 1 & 0 \end{array}\right] \quad \hat{G}=\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \hat{H}'=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.38}$$

3. Controllability and Observability of arbitrary $\underline{S} \in S$

Let $\underline{S} \in S$ with $(F, \underline{G}, \underline{H})$. The matrix F is that of (4.38), and $\underline{G}$ and $\underline{H}$ are given below

$$\underline{G} = \begin{bmatrix} 0 & 0 & 0 \\ 3/8 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \underline{H}' = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Controllability - The system S is completely controllable. In fact, any two of the three columns (input terminals) of $\underline{G}$ may be used to control $\underline{S}$. However, columns 2 and 3 provide "non-interacting" control, i.e., column 2 does not affect the states described by M_{f_2} while column 3 does not affect the states described by M_{f_1}.

Observability S is unobservable because $\underline{H}'$ has only one column and the minimal system dictates that at least 2 columns are necessary.

4. The Elementary Divisors of F

(a) Over $\mathbb{Q}$: $(z^2+2)^2(z^2-2)$; (z^2-2)

(b) Over $\mathbb{R}$: $(z^2+2)^2, (z+\sqrt{2}), (z-\sqrt{2})$; $(z\_\sqrt{2}), (z-\sqrt{2})$

(c) Over $\mathbb{C}$: $(z+i\sqrt{2})^2$, $(z-i\sqrt{2})^2, (z+\sqrt{2}), (z-\sqrt{2}); (z+\sqrt{2}), (z-\sqrt{2})$

The factors to the left [right] of the semicolon are associated with $f_1[f_2]$.

5. Primary Forms for F

The primary form for F consists of a companion matrix for each elementary divisor.

(a) Over $\mathbb{Q}$, $F \cong M_{(z^2+2)^2} \oplus M_{(z^2-2)} \oplus M_{(z^2-2)}$

(b) Over $\mathbb{R}$, $F \cong M_{(z^2+2)^2} \oplus M_{(z+\sqrt{2})} \oplus M_{(z-\sqrt{2})} \oplus M_{(z+\sqrt{2})} \oplus M_{(z-\sqrt{2})}$

(b) Over $\mathbb{C}$, $F \cong M_{(z+i\sqrt{2})^2} \oplus M_{(z-i\sqrt{2})^2} \oplus M_{(z+\sqrt{2})} \oplus M_{(z-\sqrt{2})} \oplus M_{(z+\sqrt{2})} \oplus M_{(z-\sqrt{2})}$

The reduction of F to primary form is the matrix analog of the partial fraction expansion for transfer functions. Recall that by Theorem 3.9, $\hat{S}$ is actually the direct sum of two single input-single output minimal systems. Thus, each column of $\hat{G}$ and $\hat{H}'$ will reflect the transformation of its invariant factor to elementary divisor form.

The minimal systems are written in primary form below

(a) $\hat{S}$ over $\mathbb{Q}$

$$
F=\left[\begin{array}{cccc|cc|cc}
0 & 0 & 0 & -4 & & & & \\
1 & 0 & 0 & 0 & & \multicolumn{2}{c}{\bigcirc} & \\
0 & 1 & 0 & -4 & & & & \\
0 & 0 & 1 & 0 & & & & \\
\hline
 & & & & 0 & 2 & \multicolumn{2}{c}{\bigcirc} \\
 & \multicolumn{2}{c}{\bigcirc} & & 1 & 0 & & \\
\cline{5-8}
 & & & & \multicolumn{2}{c|}{\bigcirc} & 0 & 2 \\
 & & & & & & 1 & 0
\end{array}\right]
\quad
\hat{G}=\begin{bmatrix}
1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0
\end{bmatrix}
\quad
\hat{H}'=\begin{bmatrix}
0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1/4 & 0 \\ 0 & 0 \\ 1/16 & 0 \\ 0 & 0 \\ 0 & 1
\end{bmatrix}
$$

(4.39)

(b) $\hat{S}$ over $\mathbb{R}$

$$
F=\left[\begin{array}{cccc|cccc}
0 & 0 & 0 & -4 & & & & \\
1 & 0 & 0 & 0 & & \multicolumn{2}{c}{\bigcirc} & \\
0 & 1 & 0 & -4 & & & & \\
0 & 0 & 1 & 0 & & & & \\
\hline
 & & & & -\sqrt{2} & & \bigcirc & \\
 & \multicolumn{2}{c}{\bigcirc} & & & +\sqrt{2} & \bigcirc & \\
 & & & & \bigcirc & & -\sqrt{2} & \bigcirc \\
 & & & & & & & +\sqrt{2}
\end{array}\right]
\quad
\hat{G}=\begin{bmatrix}
1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1
\end{bmatrix}
\quad
\hat{H}'=\begin{bmatrix}
0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -\sqrt{2}/64 & 0 \\ +\sqrt{2}/64 & 0 \\ 0 & -\sqrt{2}/4 \\ 0 & +\sqrt{2}/4
\end{bmatrix}
$$

(4.40)

(c) $\hat{S}$ over $\mathbb{C}$

$$F=\begin{bmatrix} -i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & i\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{bmatrix} \quad \hat{G}=\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \hat{H}'=\begin{bmatrix} 0 & 0 \\ 1/32 & 0 \\ 0 & 0 \\ 1/32 & 0 \\ -\sqrt{2}/64 & 0 \\ \sqrt{2}/64 & 0 \\ 0 & -\sqrt{2}/4 \\ 0 & \sqrt{2}/4 \end{bmatrix} \qquad (4.41)$$

Equations (4.39), (4.40), and (4.41) show that as larger fields are considered, the matrix F can be put into primary form. The new non-zero elements found in the $\hat{G}$ and $\hat{H}'$ matrices are generators of the primary cyclic submodules whose orders are the elementary divisors of F over each field.

4. Summary

This chapter has specialized the OALT-system to discrete-time, $T = N$, and arithmetic operations over an arbitrary field K. The vector-matrix representation of $S \in \mathcal{S}$ is given by the 4-tuple (F,G,H,K) where F,G, and H are matrices defined by (4.2) through (4.6).

The most important topics of this chapter are the rational canonical form for F and the minimal system based on this canonical form. The minimal system yields necessary conditions for the minimum number of input and output terminals required for controllability and observability. Moreover, it serves as a template for control system design.

Luenberger [16] has developed a canonical form for ALT-systems which consists of a set of coupled single-input subsystems. The subsystems are coupled because the submodules generated by the input terminals (column vectors of G) interact, i.e. $X_{g_i} \cap X_{g_j} \neq \Phi$.

Now suppose the matrix F is put into rational canonical form and the G matrix is appropriately transformed. The advantage of this form is that 1) superfluous input terminals may be eliminated without sacrificing complete controllability and 2) one can visually choose those input terminals which will yield the least amount of interaction between subsystems.

The concept of the minimal system may possibly be used in the reduction of the dimensionality of a system, and it seems that a measure of subsystem interaction could be used as a control system design criterion.

CHAPTER 5

MULTILINEAR SYSTEMS THEORY

1. Introduction

This chapter presents an exposition of Multilinear Systems Theory. The multilinear system is defined as an input-output relation, and it is shown that the linear system is a special case of this new system. Theorem 2.3 presents a multilinear version of the response separation property. Two examples are provided.

Section 3 presents two characterization theorems for the internal structure of a class of multilinear systems. The second of these theorems uses the tensor product map to linearize certain multilinear maps; the resulting characterization contains linear component systems which are interconnected by tensor product maps.

In section 4, the tensor product of linear systems is presented. Systems constructed in this manner are shown to be linear as well as multilinear. The tensor product of minimal systems has interesting structural properties which are used to define a "unit" system for the tensor product operation. An example concludes the chapter.

2. The Multilinear T-System

In this section the multilinear T-system is introduced, and some of its properties are studied. This system is a direct generalization of the concept of a multilinear function. A review of multilinear function theory is presented in Appendix 1.

In this section, K will denote a commutative ring. Let $[x]_r A^T$ denote the cartesian product of r function modules, $[x]_r A^T = A_1^T \times A_2^T \times \ldots \times A_r^T$, where for i = 1,2..., r, each A_i^T is the cartesian product of m_i K-modules.

2.1 <u>Definition</u> Let the triple $(K,[x]_r A,B)$ be K-modules. A multi-linear T-system (MT-system) M is a relation

$$M \subseteq [x]_r A^T \times B^T \qquad (5.1)$$

such that for i = 1,2 ..., r, each partial relation

$$M_{\underline{x},i} = M_{x_1,x_2,\ldots x_{i-1},x_{i+1},\ldots x_r} \subseteq A_i^T \times B^T \qquad (5.2)$$

is a linear relation, i.e., for all $\underline{x} \varepsilon \mathcal{D}M$ with x_j arbitrary but fixed, $j \neq i$, and x_i variable, and all y, $y' \varepsilon \mathcal{R}M$, $\alpha,\beta \varepsilon K$

$$x_i M_{\underline{x},i} y \;\&\; x_i' M_{\underline{x},i} y' \Longrightarrow (\alpha x_i + \beta x_i') M_{\underline{x},i} (\alpha y + \beta y') \qquad (5.3)$$

<u>Remark</u> The system has r input <u>lines</u>, and the signal on line i, for example, is connected to m_i <u>input terminals</u>. The K-module B is the cartesian product of p K-modules so that the system has p output terminals.

It is assumed that the input signal on line i is completely independent from the input signal on line j. This assumption will be important in the sequel. ∎

It is easy to verify the

2.2 <u>Theorem</u> Let M be an MT-system. If M has only one input line ($r = 1$), then M is a linear T-system. ∎

<u>Notation</u> Consider an MT-system M with three input lines ($r=3$). Since M is a relation, there are 2^3 input signal configurations which are of special importance; namely

$$\{(0,0,0);\ (x_1,0,0),\ (0,x_2,0),\ (0,0,x_3);\ (x_1,x_2,0),$$

$$(x_1,0,x_3),\ (0,x_2,x_3)\ ;\ (x_1,x_2,x_3)\} \tag{5.4}$$

Let $X_{(i)}$ denote the set of $\binom{r}{i} = \frac{r!}{i!(r-i)!}$

input configurations with non-zero T-functions on i input lines and zero T-functions on the remaining $(r-i)$ lines.

For example, the inputs in (5.4) can be divided into four subsets:

$$X_{(0)} = \{(0,0,0)\} \tag{5.5}$$

$$X_{(1)} = \{(x_1,0,0),\ (0,x_2,0),\ (0,0,x_3)\} \tag{5.6}$$

$$X_{(2)} = \{(x_1,x_2,0),\ (x_1,0,x_3),(0,x_2,x_3)\} \tag{5.7}$$

$$X_{(3)} = \{(x_1,x_2,x_3)\} \tag{5.8}$$

To be completely general, one might distinguish the input signal x_1 of $X_{(1)}$ from x_1 of $X_{(2)}$, since they need not be equal. For the case in which they are equal, the multilinear property of M can be used to combine input-output pairs. This point will be examined shortly.

It is clear that $\mathcal{D}M = \bigcup_i X_{(i)}$ since $x_{(i)} \varepsilon X_{(i)}$ implies there exists $y \varepsilon \mathcal{R}M$ such that $x_{(i)} My$. ∎

The next two easily proved theorems are the multilinear version of Theorem 2.5.1.

2.3 Theorem Let M be an MT-system. Define the relation

$$M_{(i)} = \{(x_{(i)}, y) \mid x_{(i)} \, My \} \qquad (5.9)$$

Then $M \subseteq \sum_{i=0}^{r} M_{(i)} = M_{(0)} + M_{(1)} + \dots + M_{(r)}$ (5.10)

2.4 Theorem Let M be an MT-system. If M is functional, then

1. M is a multilinear function.
2. For $i = 0, 1, \dots, r-1$, $\mathcal{R}(M_{(i)}) = \{0\}$ and $M = M_{(r)}$ (5.11)

Remark Equation (5.10) shows that M can be expressed as the sum of multilinear subsystems. The linearity condition on the partial relations $M_{\underline{x},i}$ of (5.2) can be used to combine input-output pairs.

Suppose M is bilinear, then

$$(0,0) M_{(0)} \, y + (x_1, 0) \, M_{(1)} \, y_1 + (0, x_2) \, M_{(1)} \, y_2 + (x_1, x_2) \, M_{(2)} \, y_{12}$$

$$\Longrightarrow (x_1, x_2) \, M \, (y_0 + y_1 + y_2 + y_{12}) \qquad (5.12)$$

The summation procedure for a trilinear system is more complicated because of the larger number of input configurations (see (5.5) through (5.8)).

Theorem 2.4 shows that when M is functional, a zero T-function applied to any input line yields a zero output T-function. ∎

The notions of non-anticipation, transition systems, and static systems are also applicable to MT-systems. The relations mentioned in the next two theorems are defined in Definition 1.4.3.
It is easy to prove the

2.5 Theorem Let M be an MT-system. M is weakly static [static; uniformly static] iff st_0 M [$\forall$ tεT, st_t M; st M] can be extended to a multilinear function. $\blacksquare$

The next theorem shows when na^tM can be extended* to a multilinear function.

2.6 Theorem Let M be an MT-system and define

$$M^0 = \{(x,y) \mid xMy \;\&\; y(0) = 0\} \qquad (5.13)$$

If M^0 is non-anticipatory at time t, then

$$na^tM^0 : \tilde{\mathcal{D}M^0}^t \longrightarrow \mathcal{R}M^0[t] \qquad (5.14)$$

can be extended* to a multilinear function.

Proof Since M^0 is non-anticipatory at time t, then na^tM^0 is a function. Moreover, for i = 0,1, ..., r-1, $\mathcal{R}(M^0{}_{(i)}) = \{0\}$. It remains to show that na^tM^0 is a multilinear function. For convenience and without loss of generality, assume that r = 2. Then for all α, $\beta \in K$, (x_1,x_2), $(x_1', x_2) \in \mathcal{D}M^0$, $y,y' \in \mathcal{R}M^0$

$$(x_1,x_2)M^0y \;\&\; (x_1',x_2)M^0y' \Longrightarrow (\alpha x_1 + \beta x_1', x_2)\; M^0(\alpha y+\beta y')$$

and

$$na^tM^0[(\alpha x_1 + \beta x_1', x_2)^t] = \alpha\, na^tM^0[(x_1,x_2)^t] + \beta\, na^tM^0[(x_1', x_2)^t] \qquad (5.15)$$

Thus, na^tM^0 is K-linear in x_1 when x_2 is held fixed. Similarly, na^tM^0 is K-linear in x_2 when x_1 is held fixed. Hence, for each $t\varepsilon T$, na^tM^0 can be extended to a multilinear function. ∎

<u>Remark*</u> The sets $RM[t]$, IM, and OM are not K-modules, but addition and scalar multiplication can be performed in the underlying K-modules B, $[x]_r$ A, and B, respectively. Thus the extension to a multilinear function is not difficult. ∎

2.7 <u>Example</u> For $i = 1,2$ let S_i be LT-systems with triple (K,A_i,K). Assume that $RS_i[0] = \{0\}$ and that for each $t\varepsilon T$, S_i is non-anticipatory at time t.

Define the relation

$$na^tM = \{((x_1^t, x_2^t), y_1(t)\cdot y_2(t)) \mid x_1S_1y_1 \ \& \ x_2S_2y_2\} \qquad (5.16)$$

Since for each $t\varepsilon T$, na^tS_1 and na^tS_2 are morphisms of K-modules, it follows that na^tM is a function. In fact, it is a bilinear function

$$na^tM : \tilde{\mathcal{D}}S_1^t \times \tilde{\mathcal{D}}S_2^t \longrightarrow K \qquad (5.17)$$

Thus, $M \subset (A_1^T \times A_2^T) \times K^T$ is a non-anticipatory, bilinear T-system with $RM[0] = \{0\}$.

2.8 <u>Example</u> Let M be a uniformly static, bilinear T-system with triple $(K,A_1 \times A_2,K)$ where K is a field and A_1,A_2 are n_1 and n_2 dimensional vector spaces over K. Furthermore, let $IM = A_1 \times A_2$

and OM K. Then st M can be extended to the bilinear function

$$\text{st } M : A_1 \times A_2 \longrightarrow K \qquad (5.18)$$

Let $\underline{b} = \{b_1, b_2, \ldots, b_{n_1}\}$, $\underline{c} = \{c_1, c_2, \ldots, c_{n_2}\}$ be basis vectors for A_1 and A_2, respectively. Then the matrix of stM relative to $\underline{b}$ and $\underline{c}$ is the $n_1 \times n_2$ matrix Q over K with entries

$$Q_{ij} = \text{st } M(b_i, c_j) \quad i = 1,2, \ldots, n_1 \qquad (5.19)$$
$$j = 1,2, \ldots, n_2$$

Thus, for any $t \varepsilon T$ and any $(x,x^*)(t) \ \varepsilon \ IM$,

$$\text{stM}[(x,x^*)(t)] = x'(t)\, Q\, x^*(t) = [x_1, \ldots, x_{n_1}] \begin{bmatrix} Q_{11} & \cdots & Q_{1n_2} \\ \vdots & & \\ Q_{n_1 1} & & Q_{n_1 n_2} \end{bmatrix} \begin{bmatrix} X_1^* \\ \vdots \\ X_{n_2}^* \end{bmatrix} \qquad (5.20)$$

The foregoing examples bring this section to a close. One of the major unsolved problems is the specification of the conditions under which M admits a direct sum decomposition similar to Theorem 2.5.3 for linear systems. In section 4 it is shown that such a decomposition does exist for multilinear systems which are constructed from component systems from the class S.

3. Characterization Theorems for Multilinear Systems

In this section a special class M of MT-systems is defined, and the Nerode minimization technique is used to obtain two characterizations of the internal structure of such systems.

Let M be an MT-system with the following properties:

1. $T = N$ - discrete-time ; K is a field.
2. M is contracting.
3. M is weakly transitional.
4. $\mathcal{D}M = \mathcal{D}M^0$ (see (5.13)).
5. $\mathcal{D}M$ is zero-loadable.
6. $\mathcal{D}M = A_1^N \times A_2^N \times \ldots \times A_r^N$ where $A_i = K^{m_i}$, $i = 1,2,\ldots,r$.

Denote by $\mathcal{M}$ the class of MT-systems which satisfy the above properties. It is easy to prove the

3.1 Theorem Let $M \in \mathcal{M}$. Then the following statements hold.

1. M is uniformly transitional.
2. M^0 is non-anticipatory and naM^0 can be extended to a multilinear function.
3. $\widetilde{\mathcal{D}M}$ is a K-module of intitial segments.
4. $\mathcal{D}M$ is the set of sequences on N to $[x]_r\, A$. $\blacksquare$

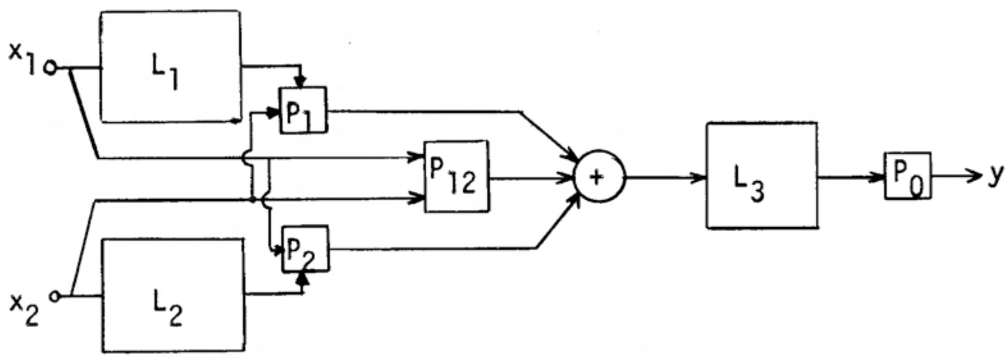
The next theorem gives a characterization of the internal structure of the subsystem M^0 of $M \in \mathcal{M}$.

3.2 Theorem (Arbib[12]) Let $M \in \mathcal{M}$ be a bilinear system ($r=2$). The bilinear function

$$f = naM^0 : \widetilde{\mathcal{D}M} \longrightarrow \mathcal{O}M \qquad (5.21)$$

can be realized by means of three ALN-systems L_1, L_2, L_3, three bilinear uniformly static systems P_1, P_2, P_{12}, and a uniformly static LN-system P_0, such that f is the non-anticipation function of the

(0,0,0)-state system shown in Figure 5.1.



Internal Realization of M^0

Figure 5.1

If $M \epsilon \mathcal{M}$ has r input lines then the realization has r layers*, and the j th layer has $\binom{r}{j}$ ALN-systems, one for each j-element subset of $\{1,2, \ldots, r\}$, with any ALT-system receiving inputs from those ALT-systems, and those system inputs, corresponding to subsets of its own subset.

* The bilinear system in Figure 5.1 has two layers: L_3 comprizes the first layer, while L_1 and L_2 form the second layer.

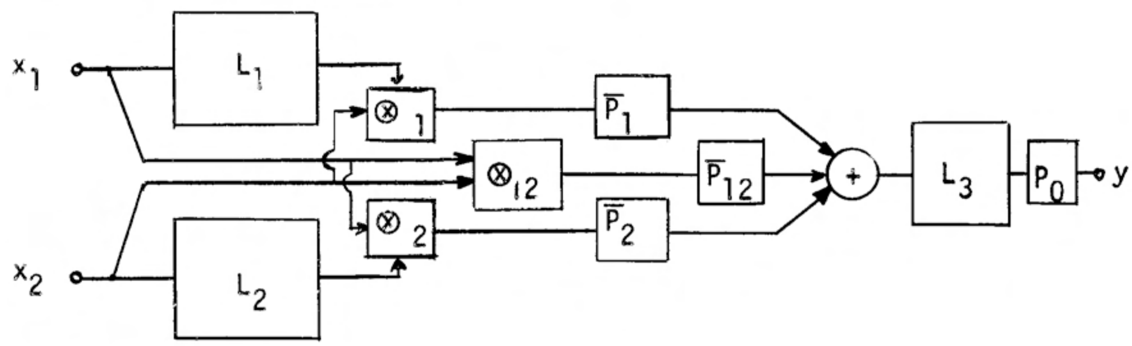
The next theorem is a "linearization" of Theorem 3.2.

3.3 <u>Theorem</u> Let $M \epsilon \mathcal{M}$ be a bilinear system. The bilinear function

$$f = naM^0 : \widetilde{\mathcal{D}M} \longrightarrow \mathcal{O}M$$

can be realized by means of three ALN-systems L_1, L_2, L_3, three tensor product maps $\otimes_1$, $\otimes_2$, $\otimes_{12}$, and four uniformly static LN-systems P_0, P_1, P_2, P_{12}, such that f is the non-anticipation function of the

(0,0,0)-state system shown in Figure 5.2.



Linearized Realization of M^0

Figure 5.2

<u>Proof</u> (of Theorem 3.2) The proof presented here paraphrases Arbib[12] For notational simplicity let

$$\begin{pmatrix}\theta & & \psi\\ \phi & \circ & \omega\end{pmatrix} = \begin{pmatrix}\theta & t & \psi\\ \phi & \circ & \omega\end{pmatrix}^{t+t'} \quad \text{where} \begin{pmatrix}\theta\\ \phi\end{pmatrix}^t, \begin{pmatrix}\psi\\ \omega\end{pmatrix}^{t'} \varepsilon \tilde{\mathcal{D}M} \quad \text{and}$$

$\theta^t, \psi^{t'} \varepsilon \widetilde{\mathcal{D}M}_1 = A_1^N$, $\phi^t, \omega^{t'} \varepsilon \widetilde{\mathcal{D}M}_2 = A_2^N$. Thus the time variable will not be written, unless it is needed for clarity.

For the bilinear system M^0 define the Nerode equivalence relation

$$\begin{matrix}\theta\\ \phi\end{matrix} E_f \begin{matrix}\theta'\\ \phi'\end{matrix} \text{ iff } f\begin{pmatrix}\theta & & \psi\\ \phi & \circ & \omega\end{pmatrix} = f\begin{pmatrix}\theta' & & \psi\\ \phi' & \circ & \omega\end{pmatrix} \text{ for all} \begin{pmatrix}\psi\\ \omega\end{pmatrix} \varepsilon \tilde{\mathcal{D}M} \tag{5.22}$$

The bilinearity of f, and the zero-loadability of $\widetilde{\mathcal{D}M}$ implies that

$$f\begin{pmatrix}\theta & & \psi\\ \phi & \circ & \omega\end{pmatrix} = f\begin{pmatrix}\theta & & 0\\ \phi & \circ & \omega\end{pmatrix} + f\begin{pmatrix}0 & & \psi\\ \phi & \circ & \omega\end{pmatrix}$$

$$= f\begin{pmatrix}\theta & & 0\\ \phi & \circ & 0\end{pmatrix} + f\begin{pmatrix}\theta & & 0\\ 0 & \circ & \omega\end{pmatrix} + f\begin{pmatrix}0 & & \psi\\ \phi & \circ & 0\end{pmatrix} + f\begin{pmatrix}0 & & \psi\\ 0 & \circ & \omega\end{pmatrix} \tag{5.23}$$

Equation (5.23) suggest three new equivalence relations:

$$\theta \sim_1 \theta' \text{ iff } f\begin{pmatrix} \theta & \circ & 0 \\ 0 & & \omega \end{pmatrix} = f\begin{pmatrix} \theta' & \circ & 0 \\ 0 & & \omega \end{pmatrix} \text{ for all} \begin{pmatrix} 0 \\ \omega \end{pmatrix} \varepsilon\ \tilde{\mathcal{D}}M \tag{5.24}$$

$$\phi \sim_2 \phi \text{ iff } f\begin{pmatrix} 0 & \circ & \psi \\ \phi & & 0 \end{pmatrix} = f\begin{pmatrix} 0 & \circ & \psi \\ \phi' & & 0 \end{pmatrix} \text{ for all} \begin{pmatrix} \psi \\ 0 \end{pmatrix} \varepsilon \tilde{\mathcal{D}}M \tag{5.25}$$

$$\begin{matrix} \theta \\ \phi \end{matrix} \sim_3 \begin{matrix} \theta' \\ \phi' \end{matrix} \text{ iff } f\begin{pmatrix} \theta & \circ & 0^t \\ \phi & & 0^t \end{pmatrix} = f\begin{pmatrix} \theta' & \circ & 0^t \\ \phi' & & 0^t \end{pmatrix} \text{ for all } \begin{pmatrix} 0 \\ 0 \end{pmatrix}^t \varepsilon \tilde{\mathcal{D}}M \tag{5.26}$$

Now, $\begin{matrix} \theta \\ \phi \end{matrix} E_f \begin{matrix} \theta' \\ \phi' \end{matrix} \Longleftrightarrow \theta \sim_1 \theta'$, $\phi \sim_2 \phi'$, and $\begin{matrix} \theta \\ \phi \end{matrix} \sim_3 \begin{matrix} \theta' \\ \phi' \end{matrix}$. From (5.23) it is easy to see that $\theta \sim_1 \theta'$, $\phi \sim_2 \phi'$, and $\begin{matrix} \theta \\ \phi \end{matrix} \sim_3 \begin{matrix} \theta' \\ \phi' \end{matrix} \Longrightarrow \begin{matrix} \theta \\ \phi \end{matrix} E_f \begin{matrix} \theta' \\ \phi' \end{matrix}$.

To see the converse, note that

$$\begin{matrix} \theta \\ \phi \end{matrix} E_f \begin{matrix} \theta' \\ \phi' \end{matrix} \Longrightarrow \begin{matrix} \theta \\ \phi \end{matrix} \sim_3 \begin{matrix} \theta' \\ \phi' \end{matrix} \text{ when } \begin{pmatrix} \psi \\ \omega \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}^t \Longrightarrow$$

$$f\begin{pmatrix} \theta & \circ & 0 \\ 0 & & \omega \end{pmatrix} + f\begin{pmatrix} 0 & \circ & \psi \\ \phi & & 0 \end{pmatrix} = f\begin{pmatrix} \theta' & \circ & 0 \\ 0 & & \omega \end{pmatrix} + f\begin{pmatrix} 0 & \cdot & \psi \\ \phi' & & 0 \end{pmatrix}$$

Thus, if $\omega = 0$ then $f\begin{pmatrix} \theta & \circ & 0 \\ 0 & & 0 \end{pmatrix} = f\begin{pmatrix} \theta' & \circ & 0 \\ 0 & & 0 \end{pmatrix} = 0$ so that $\phi \sim_2 \phi'$ for all $\psi \ \varepsilon\ \tilde{\mathcal{D}}M$. Similarly, if $\psi = 0$ then $\theta \sim_1 \theta'$ for all $\omega\ \varepsilon\ \tilde{\mathcal{D}}M_2$.

In order to obtain the structure of Figure 5.1, it is necessary to study the updating of the equivalence classes $[\theta]_1$ of θ under $\sim_1$, $[\phi]_2$ of ϕ under $\sim_2$ and $\begin{bmatrix} \theta \\ \phi \end{bmatrix}_3$ of $\begin{pmatrix} \theta \\ \phi \end{pmatrix}$ under $\sim_3$.

Let $\begin{pmatrix} \theta \\ \phi \end{pmatrix}$, $\begin{pmatrix} \psi \\ \omega \end{pmatrix} \varepsilon\ \tilde{\mathcal{D}}M$ and $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \varepsilon\ \tilde{\mathcal{D}}M^1$, then

$$f\begin{pmatrix} \theta & \circ & x_1 & \circ & 0 \\ 0 & & 0 & & \omega \end{pmatrix} = f\begin{pmatrix} \theta & \circ & 0 & \circ & 0 \\ 0 & & 0 & & \omega \end{pmatrix} + f\begin{pmatrix} 0 & \circ & x_1 & \circ & 0 \\ 0 & & 0 & & \omega \end{pmatrix}$$

and

$$f\begin{pmatrix} 0 & \circ & 0 & \circ & \psi \\ \phi & & x_2 & & 0 \end{pmatrix} = f\begin{pmatrix} 0 & \circ & 0 & \circ & \psi \\ \phi & & 0 & & 0 \end{pmatrix} + f\begin{pmatrix} 0 & \cdot & 0 & \circ & \psi \\ 0 & & x_2 & & 0 \end{pmatrix}$$

$$\text{Thus}\quad [\theta]_1 \circ x_1 = [\theta \circ 0^1]_1 + [0 \circ x_1]_1 = [\theta \circ x_1]_1 \qquad (5.27)$$

$$[\phi]_2 \circ x_2 = [\phi \circ 0^1]_2 + [0 \circ x_2]_2 = [\phi \circ x_2]_2 \qquad (5.28)$$

showing that the updating under $\sim_1$ and $\sim_2$ is linear. Let L_1 and L_2 represent the ALN-systems corresponding to $\sim_1$ and $\sim_2$, respectively. Also, let $X_1 = \tilde{\mathcal{D}M}_1 \times \tilde{O}_2 / \sim_1$ and $X_2 = \tilde{O}_1 \times \tilde{\mathcal{D}M}_2 / \sim_2$ denote their state modules.

The updating of the $\sim_3$ equivalence class follows (5.23):

$$f\begin{pmatrix} \theta & \circ & x_1 & \circ & 0 \\ \phi & & x_2 & & 0 \end{pmatrix} = f\begin{pmatrix} \theta & \circ & 0 & \circ & 0 \\ \phi & & 0 & & 0 \end{pmatrix} + f\begin{pmatrix} \theta & \circ & 0 & \circ & 0 \\ 0 & & x_2 & & 0 \end{pmatrix}$$
$$+ f\begin{pmatrix} 0 & \circ & x_1 & \circ & 0 \\ \phi & & 0^1 & & 0 \end{pmatrix} + f\begin{pmatrix} 0 & \circ & x_1 & \circ & 0 \\ 0 & & x_2 & & 0 \end{pmatrix}$$

Define three bilinear, uniformly static systems P_1, P_2, P_{12} such that

$$\text{st } P_1 : X_1 \times \mathcal{D}M_2^{\sim_1} \longrightarrow X_3 : \left(\begin{bmatrix} \theta \\ 0 \end{bmatrix}_1, x_2\right) \longmapsto \begin{bmatrix} \theta & \circ & 0 \\ 0 & & x_2 \end{bmatrix}_3$$

$$\text{st } P_2 : X_2 \times \mathcal{D}M_1^{\sim_1} \longrightarrow X_3 : \left(\begin{bmatrix} 0 \\ \phi \end{bmatrix}_2, x_1\right) \longmapsto \begin{bmatrix} 0 & \circ & x_1 \\ \phi & & 0^1 \end{bmatrix}_3$$

$$\text{st } P_{12} : \mathcal{D}M_1^{\sim_1} \times \mathcal{D}M_2^{\sim_1} \longrightarrow X_3 : (x_1, x_2) \longmapsto \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_3$$

Also define a map $F_3 : X_3 \longrightarrow X_3 : \begin{bmatrix} \theta \\ \phi \end{bmatrix}_3 \longrightarrow \begin{bmatrix} \theta & \circ & 0^1 \\ \phi & & 0^1 \end{bmatrix}_3$

$$\text{Thus, } \begin{bmatrix} \theta & \circ & x_1 \\ \phi & & x_2 \end{bmatrix}_3 = F_3 \begin{bmatrix} \theta \\ \phi \end{bmatrix}_3 + \text{st } P_1([\theta]_1, x_2) + \text{st } P_2([\phi]_2, x_1)$$
$$+ \text{st } P_{12}(x_1, x_2)$$

Note that $\begin{matrix}\theta\\ \phi\end{matrix} \sim_3 \begin{matrix}\theta'\\ \phi'\end{matrix} \Longrightarrow f\begin{pmatrix}\theta\\ \phi\end{pmatrix} = f\begin{pmatrix}\theta'\\ \phi'\end{pmatrix}$ so that the map

$$H_3 : X_3 \longrightarrow \mathcal{O}M \quad : \begin{bmatrix}\theta\\ \phi\end{bmatrix}_3 \longrightarrow f\begin{pmatrix}\theta\\ \phi\end{pmatrix}$$

defines the state-output map which completes the system specification.

However, Arbib aptly points out that the maps F_3 and H_3 are not linear since

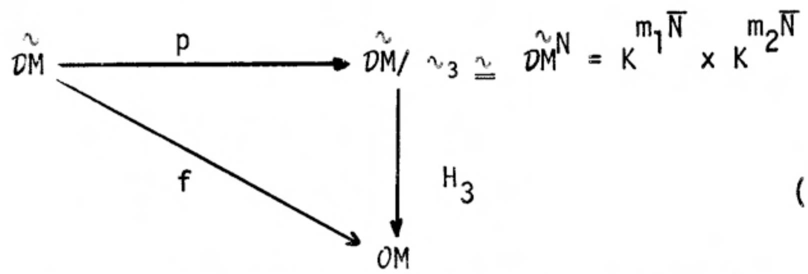
$$f\begin{pmatrix}\theta + \theta'\\ \phi + \phi'\end{pmatrix} \text{ need not equal } f\begin{pmatrix}\theta\\ \phi\end{pmatrix} + f\begin{pmatrix}\theta'\\ \phi'\end{pmatrix}$$

He resolves this problem by imbedding the minimal state space X_3 into a non-minimal linear space $\hat{X}_3$. Suppose all states of X_f are attainable in $\bar{N}$ steps, then all states of X_3 are attainable in at most $\bar{N}$ steps. Thus, only input segments of maximum length $\bar{N}$ need be applied to the system. Since $\overset{\sim_1}{\mathcal{D}M_1} = K^{m_1}$ and $\overset{\sim_1}{\mathcal{D}M_2} = K^{m_2}$, $\tilde{\mathcal{D}M}^{\bar{N}} = K^{m_1\bar{N}} \times K^{m_2\bar{N}}$. Hence any input sequence is $\sim_3$ - equivalent to a sequence which can be expressed as an $m_1 m_2 \bar{N}^2$ - tuple. Let $\hat{X}_3 = K^{m_1 m_2 \bar{N}^2}$.

With X_3 as the new state space, st P_1, st P_2, st P_{12}, F_3 and H_3 = st P_0 can be redefined. Now F_3 and st P_0 are linear maps. ∎

<u>Proof</u> (of Theorem 3.3) Although he does not state it explicitly, Arbib uses the tensor product map $\otimes$, to construct the linear state space $\hat{X}_3$. This fact will be demonstrated in the discussion which follows.

Consider the commutative diagram below which represents the factorization of f by means of the relation $\sim_3$. Note that p is the natural projection $p : \tilde{\mathcal{D}}M \longrightarrow \tilde{\mathcal{D}}M / \sim_3$



(5.29)

Now H_3 is still bilinear, and the objective is to linearize this map. This is done by employing two well known results about the tensor product map (see MacLane-Birkhoff).

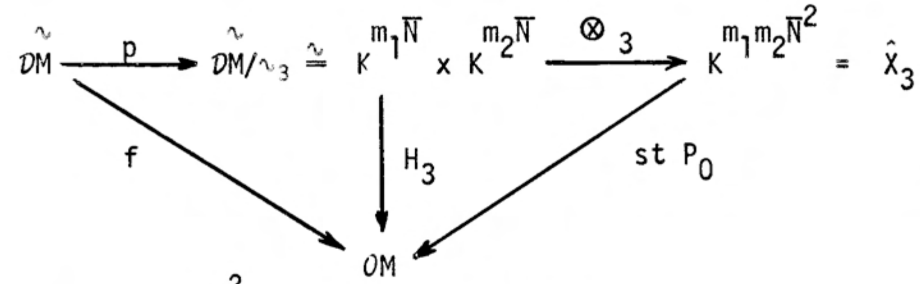
1. <u>Proposition</u> If A and B are K-modules on finite sets $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_n\}$, respectively, then $A \otimes B$ is a free K-module on the set $\{a_i \otimes b_j \mid i = 1, , m; j = 1,2, \ldots, n\}$ of m n elements.

2. <u>Theorem</u> To each K-bilinear function $h: A \times B \longrightarrow C$ there is <u>exactly</u> one K-linear transformation $t : A \otimes B \longrightarrow C$ with $t(a \otimes b) = h(a,b)$.

Let $A = \tilde{\mathcal{D}}M_1{}^{\bar{N}} = K^{m_1\bar{N}}$ and $B = \tilde{\mathcal{D}}M_2{}^{\bar{N}} = K^{m_2\bar{N}}$. Then by the above proposition

$$\tilde{\mathcal{D}}M^{\bar{N}} = \tilde{\mathcal{D}}M_1{}^{\bar{N}} \times \tilde{\mathcal{D}}M_2{}^{\bar{N}} \xrightarrow{\ \otimes_3\ } \tilde{\mathcal{D}}M_1{}^{\bar{N}} \otimes_3 \tilde{\mathcal{D}}M_2{}^{\bar{N}} \cong K^{m_1 m_2 \bar{N}^2}$$

The use of the above theorem makes the following diagram commute.

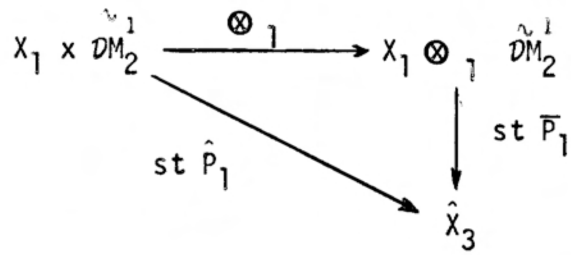


(5.30)

and st $P_0 : K^{m_1 m_2 \bar{N}^2} \longrightarrow \mathcal{O}M$ is linear.

The static bilinear maps are easily redefined, e.g., st P_1 = $\otimes_3 \circ$ st $\hat{P}_1 : X_1 \times \tilde{\mathcal{D}}M_2^1 \longrightarrow \hat{X}_3$.

However, st $\hat{P}_1$ is still bilinear, as are the maps st $\hat{P}_2$ and st $\hat{P}_{12}$. But they can be "linearized" by factoring them through appropriate tensor product maps. For example,



(5.31)

The above diagram commutes and st $\bar{P}_1$ is linear. Thus st $\hat{P}_2$ = st $\bar{P}_2 \circ \otimes_2$ and st $\hat{P}_{12}$ = st $\bar{P}_{12} \circ \otimes_{12}$, and this establishes the linearized realization of M^0 as in Figure 5.2. ∎

Remark The linearized realization of M^0 is canonical in the sense that the tensor product map $\otimes$, is a "universal" among multilinear maps. The major drawback of this form is the increased dimensionality of the tensored spaces.

Notice that multilinear systems can be constructed by specifying the appropriate ALT-systems L_i , the tensor product maps $\otimes_i$, and the uniformly static systems P_i . $\blacksquare$

3.4 Example - Nonlinear System Modelling

Barrett [17] has shown that the response of a class of non-linear systems at time t may be expressed as a functional power series expansion of the form

$$y(t) = \sum_{n=1}^{\infty} \int_0^t \cdots\cdots \int_0^t \Phi_n (t;\tau_1, \ldots \tau_n)\, x(\tau_1) \ldots\ldots x(\tau_n)\, d\tau_1 \ldots\ldots d\tau_n \tag{5.32}$$

where Φ_n is a symmetric kernal.

x is the input function.

y is the output function.

Consider a contracting system which can be modelled by the expression

$$y(t) = \int_0^t \varphi_1 (t-\tau)x(\tau)d\tau + \int_0^t \int_0^t \Phi_2 (t-\tau_1-\tau_2)x(\tau_1)x(\tau_2)d\tau,d\tau_2 \tag{5.33}$$

The discrete-time analog of (5.33) is

$$y(t) = naM_1^0 (x^t) + naM_2^0 ([x]_2\, x^t) \tag{5.34}$$

where M_1^0 , $M_2^0 \in M$. Now, Theorem 3.3 suggest that M_2^0 can be appropriately "linearized" such that

$$y(t) = naM_1^0 (x^t) + naM_2^0 ([\, \otimes\,]_2\, x^t) \qquad 5.35)$$

The general form corresponding to (5.32) is then

$$y(t) = \sum_{n=1}^{\infty} na\, M_n^0 ([\otimes]_n\, x^t) \qquad (5.36)$$

This reformulation of the problem suggests that multilinear systems can be modelled and analyzed by means of component ALN-systems tensor product maps, and uniformly static LN-systems. ∎

4. The Linear Tensor Product System

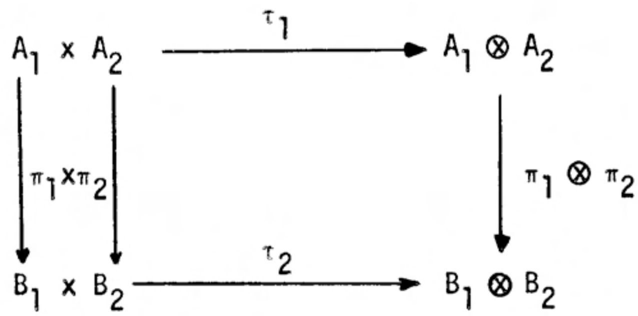
In this section the tensor product of linear systems is studied. It results that the new system is not only multilinear, but also linear. Moreover, it exhibits a multilinear response separation, hasstructure similar to Figure 5.2, and satisfies modified rank conditions for attainability and observability.

The proofs for the next three theorems are found in MacLane-Birkhoff [13].

4.1 Theorem Let K be a commutative ring. For i = 1,2, let π_i : $A_i \longrightarrow B_i$ be morphisms of K-modules. Define their cartesian product, $\pi_1 \times \pi_2\,(a_1, a_2) = (\pi_1(a_1), \pi_2(a_2))$.

There exist two (universal) tensor product maps $\tau_1 : A_1 \times A_2 \longrightarrow A_1 \otimes A_2$ and $\tau_2 : B_1 \times B_2 \longrightarrow B_1 \otimes B_2$ such that the diagram in Figure 5.3 commutes and

$$\pi_1 \otimes \pi_2 \; : \; A_1 \otimes A_2 \longrightarrow B_1 \otimes B_2 \; .$$



The Tensor Product, $\pi_1 \otimes \pi_2$

Figure 5.3

4.2 Theorem Let A and B be K-modules. If A and B may be written as direct sums

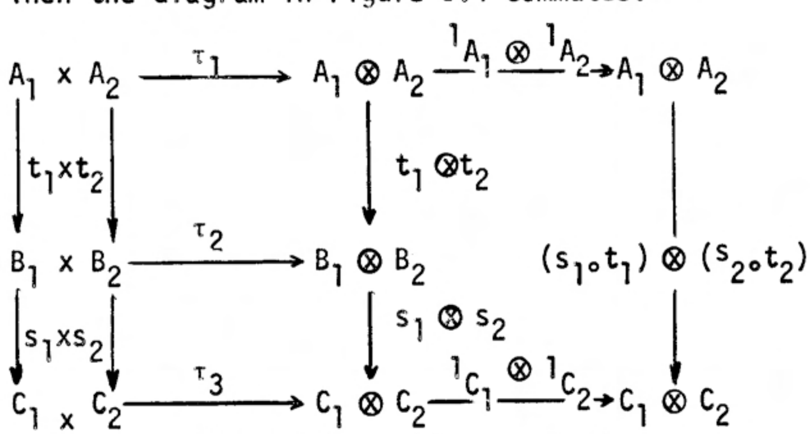
$$A = \bigoplus_{\mu \varepsilon U} A_\mu \quad \text{and} \quad B = \bigoplus_{\nu \varepsilon V} B_\nu$$

then

$$A \otimes B = \bigoplus_{(\mu,\nu)} [A_\mu \otimes B_\nu] \tag{5.37}$$

4.3 Theorem For $i = 1,2$ let $t_i : A_i \longrightarrow B_i$ and $s_i : B_i \longrightarrow C_i$ be morphisms of K-modules such that $s_i \circ t_i : A_i \longrightarrow C_i$ denotes their composition.

Then the diagram in Figure 5.4 commutes.



Composition of Tensor Products

Figure 5.4

System Theoretical Interpretation - For $i = 1,2$ let $S_i = P_i \circ R_i$ represent OALT-systems (see Definition 3.2.6) with morphism representation

$$\pi_i \ : \ \mathcal{O}R_i \oplus \mathcal{D}S_i \longrightarrow RS_i \tag{5.38}$$

and state transition and state-output morphisms

$$\pi R_i \ : \ \mathcal{O}R_i \oplus \tilde{\mathcal{D}S}_i \longrightarrow \mathcal{O}R_i \quad \text{and st } P_i : \mathcal{O}R_i \longrightarrow \mathcal{O}S_i \tag{5.39}$$

4.4 Theorem The tensor product system, $S_1 \oplus S_2$, has the following linear maps:

1. $\pi_1 \otimes \pi_2 : (\mathcal{O}R_1 \oplus \mathcal{D}S_1) \otimes (\mathcal{O}R_2 \oplus \mathcal{D}S_2) \longrightarrow RS_1 \otimes RS_2$ (5.40)

2. $\pi R_1 \otimes \pi R_2 : (\mathcal{O}R_1 \oplus \tilde{\mathcal{D}S}) \otimes (\mathcal{O}R_2 \oplus \tilde{\mathcal{D}S}_2) \longrightarrow \mathcal{O}R_1 \otimes \mathcal{O}R_2$ (5.41)

3. $stP_1 \otimes stP_2 : \mathcal{O}R_1 \otimes \mathcal{O}R_2 \longrightarrow \mathcal{O}S_1 \otimes \mathcal{O}S_2$ (5.42)

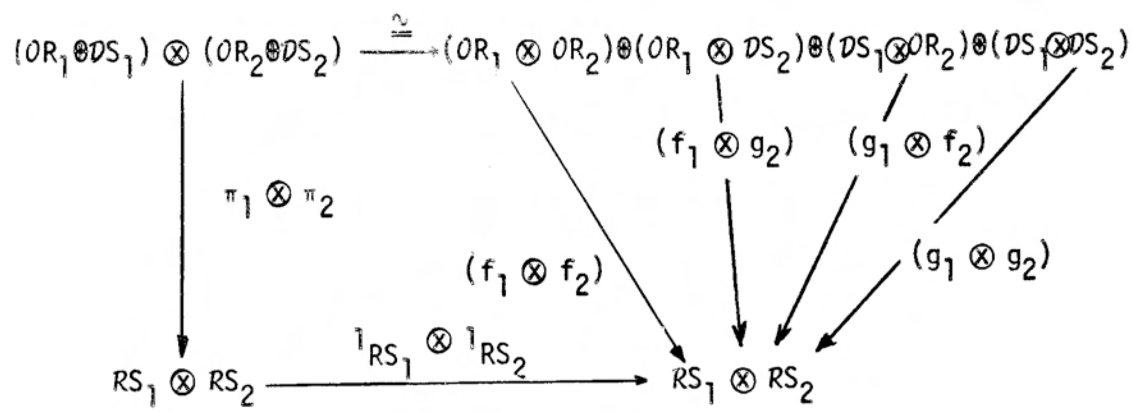
4. $\pi S_1 \otimes \pi S_2 = (stP_1 \otimes stP_2) \circ (\pi R_1 \otimes \pi R_2)$ (5.43)

Proof Follows directly from Theorems 4.1 and 4.3.

4.5 Theorem For $i = 1,2$ consider the following restrictions of (5.38)

$$f_i = (\pi_i \mid \mathcal{O}R_i \oplus 0) \quad \text{and} \quad g_i = (\pi_i \mid 0 \oplus \mathcal{D}S_i) \tag{5.44}$$

Then the K-linear function $\pi_1 \otimes \pi_2$ of (5.40) admits a four part response decomposition which is represented by the commutative diagram in Figure 5.5:



Four Part Response Decomposition

Figure 5.5

Proof Follows directly from Theorem 4.2 and the response separation property of π_i, $i = 1,2$.

Remark The four part response decomposition of Figure 5.5 is valid for $r = 2$, the tensor product of two linear T-systems. Consider now the tensor product of three linear T-system morphisms $\pi_1 \times \pi_2 \times \pi_3 \cong (\pi_1 \otimes \pi_2) \otimes \pi_3$ where the decomposed response maps are

$$\{f_1 \otimes f_2 \otimes f_3 \,;\, f_1 \otimes g_2 \otimes f_3,\ g_1 \otimes f_2 \otimes f_3,\ f_1 \otimes f_2 \otimes g_3;$$
$$f_1 \otimes g_2 \otimes g_3,\ g_1 \otimes f_2 \otimes g_3,\ g_1 \otimes g_2 \otimes f_3;\ g_1 \otimes g_2 \otimes g_3\} \tag{5.45}$$

over appropriately defined domains and ranges.

Notice the similarity between the response separation of (5.45) and the allowable input configuration of (5.4). Thus, the sub-systems $M_{(0)}$ $M_{(1)}$, and $M_{(2)}$ of (5.10) are input-output pairs of M which correspond to responses due to non-zero initial states, and zero input signals on various input lines. ∎

Vector-Matrix Equations for the Linear Tensor Product System

Let $S_1, S_2 \in \mathcal{S}$ so that K is a field, T = N, and each S_i has matrix triple (F_i, F_i, H_i). The vector-matrix representation for S_i, $i = 1,2$, is given by (4.5) and (4.6). Thus for $i = 1,2$

$$S_i) \quad q_i(t+1) = F_i q_i(t) + G_i\, x_i\,(t)$$

$$y_i(t) = H_i\, q_i(t)$$

The vector-matrix representations of $\pi R_1 \otimes \pi R_2$ (5.41) and $stP_1 \otimes stP_2$ (5.42) are given below:

$$S_1 \otimes S_2)\ (q_1 \otimes q_2)\ (t+1) = (F_1 \otimes F_2)\ (q_1 \otimes q_2)\ (t)$$
$$+ (F_1 \otimes G_2)(q_1 \otimes x_2)(t) + G_1 \otimes F_2)(x_1 \otimes q_2)(t)$$
$$+ (G_1 \otimes G_2)\ (x_1 \otimes x_2)\ (t) \qquad (5.46)$$

$$(y_1 \otimes y_2)\ (t) = (H_1 \otimes H_2)\ (q_1 \otimes q_2)\ (t) \qquad (5.47)$$

Remark A mnemonic for writing (5.46) is to let $f_i = F_i$ and $g_i = G_i$, $i = 1,2$, in Figure 5.5. Appendix 1 has a section dealing with bases for tensor products of vector spaces, and tensor products of matrices. ∎

The response of the system at time $t = n$ may be calculated by recalling that by (4.7)

$$q_i(n) = F_i^{\,n}\, q_i(0) + \sum_{j=0}^{n-1} F_i^{\,n-j-1}\, G_i\, x_i(j) \qquad (i = 1,2)$$

$$= F_i^{\,n}\, q_i(0) + [F_i^{\,n-1}\, G_i : \ldots\ldots : F_i G_i : G_i]\; x_i^n$$

where $(x_i^n)' = [x_i'\,(0) : x_i'\,(1) : \ldots\ldots : x_i'\,(n-1)\,]$

Let $C_i^n = [F_i^{\,n-1} G_i : \ldots\ldots\; F_i G_i : G_i]$ so that

$$q_i(n) = F_i^{\,n}\, q_i(0) + C_i^n\; x_i^n \tag{5.48}$$

The response of the tensor product $S_1 \otimes S_2$ at $t = n$ is

$$(q_1 \otimes q_2)(n) = (F_1^{\,n} \otimes F_2^{\,n})(q_1 \otimes q_2)\,(0) + (F_1^{\,n} \otimes C_2^n)\,(q_1(0) \otimes x_2^n) + (C_1^n \otimes F_2^{\,n})(x_1^{\,n} \otimes q_2(0)\,) + (C_1^n \otimes C_2^{\,n})(x_1^{\,n} \otimes x_2^{\,n}) \tag{5.49}$$

$$(y_1 \otimes y_2)\,(n) = (H_1 \otimes H_2)\,(q_1 \otimes q_2)\,(n) \tag{5.50}$$

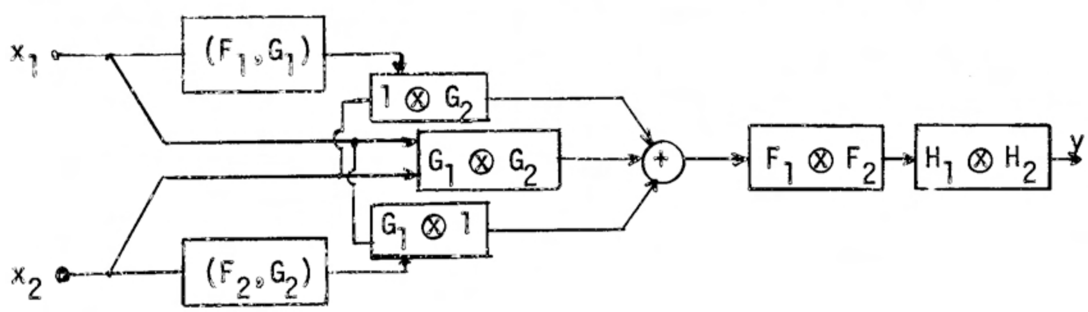
The state may also be expressed in terms of summations,

$$(q_1 \otimes q_2)(n) = (F_1^{\,n} \otimes F_2^{\,n})(q_1 \otimes q_2)(0) + \sum_{j=0}^{n-1} (F_1^{\,n} \otimes F_2^{\,n-j-1} G_2)\,(q_1(0) \otimes x_2(j)\,) + \sum_{i=0}^{n-1} (F_1^{\,n-i-1} G_1 \otimes F_2^{\,n})\,(x_1(i) \otimes q_2(0)\,) + \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (F_1^{\,n-i-1} G_1 \otimes F_2^{\,n-j-1} G_2)\,(x_1(i) \otimes x_2(j)\,) \tag{5.51}$$

Remark Equations (5.46), (5.49), and (5.51) suggest that the tensor product system $S_1 \otimes S_2$, may be viewed in either of two ways: (1) as a new linear system, or (2) as an algebraic construction in which the component linear systems are handled independently, with tensor product operations being performed whenever necessary.

The first approach may be useful when a tensor product system is a component of a larger system simulation. The second approach is more convenient when the output of the tensor product system is required only at certain instants of time. Moreover, the dimension of the state space of $S_1 \otimes S_2$ is the product of the dimension of the individual state spaces. Thus, by considering each system separately one reduces the dimensionality of the problem.

Notice that the internal structure of $S_1 \otimes S_2$ which is shown in Figure 5.6 is essentially that of Figure 5.2 with $F_3 = F_1 \otimes F_2$.



Internal Structure of $S_1 \otimes S_2$

Figure 5.6

Lastly, notice that (5.51) has a double summation in the last term. Thus for $S_1 \otimes S_1$, with $q_1(0) = 0$, (5.51) has the form

$$(q_1 \otimes q_1)(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (F_1^{n-i-1} G_1 \otimes F_1^{n-j-1} G_1)(x_1(i) \otimes x_1(j)) \tag{5.52}$$

Equation (5.52 is analogous to the term $naM_2^0 ([\otimes]_2 x^t)$ in (5.35).

Thus, it appears that the tensor product $S_1 \otimes S_1$ can be used for non-linear system modelling as discussed in Example 3.4. ■

<u>Attainability, Controllability, and Observability</u>

Since the tensor product of two linear systems is also a linear system, one would expect to obtain controllability and observability conditions similar to (4.10) and (4.12). The conditions for attainability and observability are presented below, but the controllability condition is still an unsolved problem.

Let $S_i \in S$ with (F_i, G_i, H_i, K) for $i = 1,2$. Suppose that S_1 and S_2 have state spaces of dimension n_1 and n_2, respectively. Using (5.48) through (5.51), one can easily prove the

4.6 <u>Theorem</u> All states of $S_1 \otimes S_2$ are <u>attainable</u> from the zero-state iff

$$\text{rank } (C_1^{n^*} \otimes C_2^{n^*}) = n_1 \cdot n_2, \; n^* = \max (n_1, n_2) \qquad (5.53)$$

4.7 <u>Theorem</u> The system $S_1 \times S_2$ is (zero-input) <u>observable</u> iff

$$\text{rank } [(F_1 \otimes F_2)'^{n_1 n_2 - 1} (H_1 \otimes H_2)' \; \ldots\ldots\ldots \; (H_1 \otimes H_2)'] = n_1 \cdot n_2 \qquad (5.54)$$

<u>Remark</u> Notice the large dimensionality of the matrix multiplications in (5.53) and (5.54). An interesting problem would be to characterize the attainability, controllability, and observability of $S_1 \otimes S_2$ in terms those conditions for S_1 and S_2. ■

The tensor product of free systems and minimal systems yields special structural properties and several interesting results. Consider the

4.8 Theorem Let $S_1, S_2 \in \mathcal{S}$ be free ALN-systems. Suppose F_1 and F_2 have invariant factors $\{f_1, f_2, \ldots, f_k\}$ and $\{p_1, p_2, \ldots, p_\ell\}$, respectively. Let the K[z]-modules X_i (i=1,2) denote their state modules such that

$$X_1 = K[z]/(f_1) \oplus K[z]/(f_2) \oplus \ldots\ldots \oplus K[z]/(f_k)$$

and

$$X_2 = K[z]/(p_1) \oplus K[z]/(p_2) \oplus \ldots\ldots \oplus K[z]/(p_\ell)$$

Then the state module for $S_1 \otimes S_2$ is

$$X_1 \times X_2 \cong \bigoplus_{i,j} K[z]/(f_i) \otimes K[z]/(p_j) \quad \begin{matrix} i=1,2,\ldots, k \\ j=1,2,\ldots, \ell \end{matrix} \qquad (5.55)$$

Proof The conclusion follows directly from Theorem 4.2.

Remark Since (5.55) is the direct sum of tensor products, it is sufficient to consider one such product, call it $K[z]/(f) \times K[z]/(p)$ Assume that $f = z^\nu + a_{\nu-1} z^{\nu-1} + \ldots + a_0$ and $p = z^\rho + b_{\rho-1} z^{\rho-1} + \ldots + b_0$. Then the companion matrix for this tensor product is

$$\bar{M} = M_f \otimes M_p = \begin{bmatrix} 0 & & & & & -a_0 & M_p \\ M_p & 0 & & & & -a_1 & M_p \\ 0 & M_p & 0 & & & & \\ & 0 & M_p & 0 & & & \\ & & & M_p & 0 & -a_{\nu-2} & M_p \\ 0 & 0 & & & M_p & -a_{\nu-1} & M_p \end{bmatrix} \qquad (5.56)$$

where $\bar{M}$ is an $\nu \cdot p$ x $\nu \cdot p$ dimensional matrix.

It is easy to see that if $p = z-\lambda$, then $\bar{M} = \lambda M_f$. Moreover, if $\lambda = 1$, then $\bar{M} = M_f$. $\blacksquare$

The next result is easy to prove.

4.9 <u>Theorem</u> Let $S_\mu \in \mathcal{S}$ be a free, single-output minimal system with invariant factor $p = z-1$ and initial state $q_\mu(0) = 1$. Then for any $S \in \mathcal{S}$

$$S \otimes S_\mu = S_\mu \otimes S = S \qquad (5.57)$$

Thus, S_μ serves as a unit system for the tensor product of systems.

<u>Remark</u> The existence of the unit system S_μ implies that a theory of interconnections for tensor product systems is both feasible and desirable. In this way both linear and tensor product systems could be analyzed within a common framework.

This chapter closes with an illustrative example of the tensor product of linear systems.

4.10 Example - Simulation of a Sampling Device

Let $S \varepsilon \mathcal{S}$ with (F,G,H). The basic time interval for this problem is 5 units long. During each such interval, the output y of S is to be sampled such that

$$\{ y(0), y(1), 0, y(3), 0 \} \tag{5.58}$$

is the output of the sampler. For example, the output sequence over the first three operating intervals would be

$$\{y(0), y(1), 0, Y(3), 0; y(5), y(6), 0, y(8), 0; y(10), y(11), 0, y(13), 0 \} \tag{5 59}$$

Solution The basic time interval can be obtained by means of a 5-stage ring counter which can be simulated by a free single-output minimal system $S_1 \varepsilon \mathcal{S}$ with (F_1, h_1). The minimal polynomial for F_1 is $p = z^5 - 1$ and the vector-matrix equations for the counter are

$$S_1) \quad q_1(t+1) = F_1 q_1(t) \qquad q_1' (0) = [1,1,0,1,0]$$

$$y_1 (t) = h_1 q_1 (t)$$

where

$$F_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad h'_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.60}$$

It is readily apparent that the tensor product $S \otimes S_1$ yields the desired result. Note that the same result can be obtained by considering each system independently, and then taking the tensor product of their outputs. This significantly reduces the complexity of the computations.

This same approach can be extended to simulate a multiplexing system. It seems that the tensor product approach might be used to simulate a measuring device's effect on the measured variable.

CHAPTER 6

SUMMARY AND SUGGESTIONS FOR FUTURE RESEARCH

1. Summary

This thesis presents an algebraic approach to the study of linear and multilinear systems within the framework of Windeknecht's General Time Systems formalism. By defining linear and multilinear systems over arbitrary rings, one unifies the study of systems found in Automata Theory, Sequential Circuit Theory, and Control Theory.

Of equal importance is the fact that an axiomatic method has been employed in this study. Beginning with general notions and definitions, one adds axioms when needed, and studies their effect. Examples of the method are found in the response separation property for linear systems, the specialization of the notions of non-anticipation and transition systems to a time t so as to preserve R-module structure, and the notion of zero loadability inducing R-module structure on $\tilde{\mathcal{D}S}$.

2. Suggestions for Future Research

The systems studied in this thesis are contracting, so that transition systems and static systems are time-invariant. One possible area of future research is the study of time-varying linear and multilinear systems.

The Rational Canonical Form for a free ALN-system, and its associated minimal system may provide insight into the state decoupling of large scale linear control systems. It appears that the problem of non-interacting control reduces to a compatability requirement between the system's invariant factors and its input terminals. Perhaps the theory of modal control developed by Simon and Mitter [18] could be used to fulfill this compatability requirement.

The theory of multilinear systems can be extended along several lines. First, modelling techniques should be developed which take advantage of the tensor canonical form of these systems. Thus, systems can be constructed from "linear component systems" which are connected by tensor product maps. The converse problem is the decomposition of large scale multilinear systems into smaller linear subsystems.

Second, the existence of a unit system, $S_\mu \epsilon \mathcal{S}$, for the tensor product operation, suggests that an interconnection theory for tensor product systems is feasible. The theories of graded modules and exterior algebra may prove useful in this effort.

Third, there is a need for a control theory of multilinear systems. The basic property of these systems is the multilinear relationship between the state and control (input) variables, which does not exist in linear systems.

APPENDIX 1

R-MODULE THEORY

1. Rings, R-modules, and Function Modules.

1.1 Definition A ring $R = (R,+,\circ, 1)$ is a set R together with two binary operations, addition and multiplication, and a unit "1" such that

(i) $(R,+)$ is an additive abelian group.

(ii) $(R,\circ, 1)$ is a monoid under multiplication.

(iii) Multiplication is distributive over addition, $a(b+c) = ab + ac$, for all $a, b, c \in R$.

A commutative ring is one in which multiplication is commutative.

A non-trivial commutative ring in which each non-zero element has a multiplicative inverse is called a field.

Examples: N, the natural numbers, is both an additive and multiplicative monoid.

Z, the integers, is a ring while Z_r, the intergers modulo a prime is a field. The real numbers $\mathbb{R}$, is also a field.

1.2 Definition Let R be a ring. An R-module A is an additive abelian group together with a function $R \times A \longrightarrow A$, written $(\alpha,a) \longrightarrow \alpha a$ and subject to the following axioms: for all $\alpha,\beta \in R$, $a,b \in A$

$$\alpha(a+b) = \alpha a + \alpha b$$

$$(\alpha + \beta)a = \alpha a + \beta a$$

$$(\alpha\beta)a = \alpha(\beta a)$$
$$1\, a = a$$

A submodule A' of A is a subset of A which is closed under addition and multiplication.

Remark If R is a field, then the R-module A is a vector space, and its submodules are vector subspaces.
It is easy to verify the

1.3 Lemma Let A^T be T-object. If A is an R-module, then A^T is an R-module under the following pointwise sum and scalar product:
fr $x, x' \in A^T$, $\alpha \in R$

$$(x + x')(t) = x(t) + x'(t) \text{ and } (\alpha x)(t) = \alpha\, x(t)$$

2. Morphisms, Ideals, and Torsion Modules

Definition 2.4.1 defines a morphism of R-modules as a linear function. Here, some special morphisms and associated modules are defined.

2.1 Definition Let $f : A \longrightarrow A'$ be a morphism of R-modules.

1. f is a monomorphism if it is 1:1.
2. f is an epimorphism if it is onto.
3. f is an isomorphism if it is both 1:1 and onto.
4. The kernal of f is defined by the set

   $$\text{Ker } f = \{\, a \mid f(a) = 0 \ \& \ a \in A \,\}$$

5. The image of f is defined by the set

   $$\text{Im } f = \{\, a' \mid f(a) = a' \text{ for some } a \in A \,\}$$

Remark Both Ker f and Im f are R-modules.

2.2 Definition 1. A (two-sided) ideal A in a ring R is a non-empty subset A of R with

(i) a_1 and $a_2 \in A \Longrightarrow a_1 - a_2 \in A$

(ii) $r \in R$ and $a \in A \Longrightarrow ra$ and $ar \in A$

2. An integral domain is a non-trivial commutative ring with no zero divisors, i.e., there exist no $a \neq 0$, $b \neq 0$ such that $ab = 0$.

3. Let K be a commutative ring. Let B be an ideal of K, then the ideal (b) of all multiples $k\,b$, $k \in K$, $b \in B$, is called a principal ideal of K. An integral domain D is called a principal ideal domain if all ideals are principal.

Remark 1. An example of an ideal is the kernal of a morphism of rings, $m : R \longrightarrow R'$.

2. Every non-zero element of a field has a multiplicative inverse, hence a field is an integral domain.

3. Let K be a field, then the polynomial ring K[z] is a principal ideal domain (P.I.D.)

2.3 Definition An element a of a D-module A is a torsion element when $a\kappa = 0$ for some $\kappa \neq 0$ in D, and D is P.I.D. The set

$A_a = \{ \kappa \mid \kappa \in D \ \& \ a\kappa = 0 \}$ is an ideal in D.

Remark Since D is P.I.D., then $(\mu) = A_a$ and μ is called the order of a. Thus $\mu a = 0$ and $\kappa a = 0 \Longrightarrow \mu \mid \kappa$. Note that μ is unique up to an invertible factor.

A <u>torsion module</u> A is a D-module in which every element is a torsion element. Let A be a torsion module of finite type. If the elements $a_1, a_2, \ldots, a_k$ spanning A have respective orders $\mu_1, \mu_2, \ldots, \mu_k$, then the product $\mu = \mu_1\mu_2\ldots\mu_k$ is non zero and $a\mu = 0$, for all $a \in A$. Thus, μ is said to <u>annihilate</u> A. In fact, the set of $\mu \in D$ with $A\mu = 0$ is also an ideal in D, hence a principal ideal (ν), and ν is the <u>minimal annihilator</u> of A.

2.4 <u>Definition</u> A D-module C is said to be <u>cyclic</u> if it is spanned (generated) by one element $c_0 \in C$, and the assignment $k \longmapsto c_0 k$ is an epimorphism $D \longrightarrow C$ of D-modules.

<u>Remark</u> The kernal of the morphism $D \longrightarrow C$ is not only a submodule of D but also a principal ideal (μ). Thus c_0 is of <u>order</u> μ and $C \cong D/(\mu)$. If $\mu = 0$ then $C \cong D$ and is a free module on c_0.

3. <u>Multilinear Functions and the Tensor Product Map</u>

This section discusses multilinear functions and the tensor product map. Of particular interest is the fact that every multilinear function can be written as the composition of the tensor product map followed by a linear map. Unless otherwise specified, K denotes a commutative ring.

3.1 <u>Definition</u> Let A, B, C be K-modules. A K-bilinear function f on A x B to C is a function $f: A \times B \longrightarrow C$ such that for all $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$, $\alpha, \beta \in K$

$$f(a_1 \alpha + a_2 \beta,\ b) = f(a_1,b)\,\alpha + f(a_2,b)\beta \qquad (A.1)$$

$$f(a,b_1 \alpha + b_2 \beta) = f(a,b_1)\alpha + f(a,b_2)\beta \qquad (A.2)$$

Remark For each $b \varepsilon B$ the assignment $a \longrightarrow f(a,b)$ defines a K-linear partial function $f_b : A \longrightarrow C$ such that

$$f_b\,(a_1\alpha + a_2\beta) = f(a_1\alpha + a_2\beta, b) = f(a_1,b)\alpha + f(a_2,b)\beta$$

$$= f_b(a_1)\alpha + f_b(a_2)\beta \qquad (A.3)$$

Similarly, for each $a \varepsilon A$, $f_a : B \longrightarrow C$ is a K-linear function.

Although f_a and f_b are K-linear, f itself is non-linear, e.g.,

$$f(a_1\alpha + a_2\beta, b_1\gamma + b_2\delta) = f(a_1\alpha + a_2\beta, b_1)\,\gamma + f(a_1\alpha + a_2\beta, b_2)\delta$$

$$= f(a_1,b_1)\alpha\gamma + f(a_2,b_1)\beta\gamma + f(a_1,b_2)\alpha\delta + f(a_2,b_2)\beta\delta \qquad (A.4)$$

The multilinear function definition is a direct generalization of Definition 3.1.

3.2 Definition Denote the cartesian product of r K-modules A_i, $i = 1,2, \ldots, r$ as

$$[x]_r\, A = A_1 \times A_2 \times \cdots \times A_r \qquad (A.5)$$

A multilinear function $h: [x]_r A \longrightarrow C$ has the property that for $i = 1,2, \ldots, r$, each partial function.

$$h_{a_1,a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_r} : A_i \longrightarrow C \qquad (A.6)$$

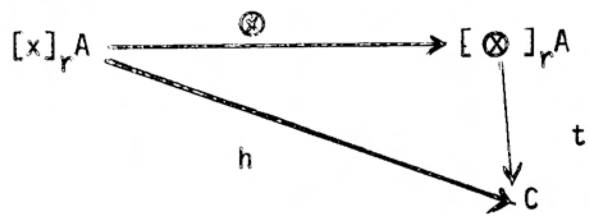
is a linear map.

3.3 Theorem To each K-multilinear function $h: [x]_r A \longrightarrow C$, there is exactly one K-linear transformation $t: [\otimes]_r A \longrightarrow C$ with

$$t(a_1 \otimes a_2 \otimes \ldots\ldots \otimes a_r) = h(a_1, a_2, \ldots\ldots, a_r) \qquad (A.7)$$

where $\otimes$ denotes the tensor product map.

Remark The proof of the theorem may be found in MacLane-Birkhoff. Theorem 3.3 implies that the following diagram commutes



(A.8)

Since t is a K-linear map, the tensor product map, $\otimes : [x]_r A \longrightarrow [\otimes]_r A$, must be multilinear. This is indeed the case; in fact, $\otimes$ is called the universal multilinear function because any multilinear function can be "factored" as in (A.8). Moreover, the tensor product module $[\otimes]_r A$ is the largest module necessary to make the map $\otimes$ multilinear.

4. The Tensor Product of Matrices

Let P and Q be finite dimensional vector spaces over a field K with respective dimensions p and q. Let $\{e_i \; ; \; i = 1, \ldots, p\}$ and $\{e_j^* \; ; \; j = 1, \ldots\ldots, q\}$ denote the unit vectors which span P and Q, respectively. Thus, any vectors $x \in P$ and $y \in Q$ can be expressed as

$$x = \sum_{i=1}^{p} \xi_i e_i \quad \text{and} \quad y = \sum_{j=1}^{q} \eta_j e_j^* , \xi_i , \eta_j \in K$$

The tensor product, x x y is expressed as

$$x \otimes y = \sum_{i=1}^{p} \xi_i \, e_i \otimes \sum_{j=1}^{q} \eta_j \, e_j^* = \sum_{i=1}^{p} \sum_{j=1}^{q} \xi_i \, \eta_j \, (e_i \otimes e_j^*) \qquad \text{(A.9)}$$

$$\text{where} \quad (e_i \otimes e_j^*)' = [0, 0 \ldots, 0, \underset{\uparrow}{1}, 0, \ldots, 0] \qquad \text{(A.10)}$$

$[(i-1)p + j]^{th}$ position.

Let $t_A : P \longrightarrow P$ and $t_C : Q \longrightarrow Q$ be two linear transformations. The matrices associated with t_A and t_C are the p x p and q x q dimensional matrices A and C, respectively. Now A and C are determined by the equations

$$t_A(e_i) = \sum_{k=1}^{p} \alpha_{ik} \, e_k \quad \text{and} \quad t_C(e_j^*) = \sum_{\ell=1}^{q} \beta_{j\ell} \, e_\ell^* \qquad \text{(A.11)}$$

$$\text{such that} \quad A = (\alpha_{ik}) \quad \text{and} \quad C = (\beta_{j\ell}) \qquad \text{(A.12)}$$

The tensor product of t_A and t_C is $t_A \otimes t_C$ and the associated matrix $A \otimes C$ is calculated as follows

$$(t_A \otimes t_C)(e_i \otimes e_j^*) = \sum_{k=1}^{p} \sum_{\ell=1}^{q} (\alpha_{ik} \cdot \beta_{j\ell})(e_k \otimes e_\ell^*) \qquad \text{(A.13)}$$

Thus

$$A \otimes C = \begin{bmatrix} \alpha_{11} C & \alpha_{12} C & \cdots & \alpha_{1p} C \\ \vdots & & & \vdots \\ \alpha_{p1} C & \cdots & \cdots & \alpha_{pp} C \end{bmatrix} \qquad \text{(A.14)}$$

$$\text{and} \quad \alpha_{ik}\, C = \begin{bmatrix} \alpha_{ik}\beta_{11} & \circ \circ \circ & \alpha_{ik}\beta_{1q} \\ \circ & & \\ \circ & & \\ \circ & & \\ \alpha_{ik}\beta_{p1} & \circ \circ \circ & \alpha_{ik}\beta_{qq} \end{bmatrix} \qquad \text{(A.15)}$$

It is easy to show the following identities.

Let A, A_1, A_2 and C, C_1, C_2 be matrices of compatible dimensions over K, then

$$A \otimes (C_1 + C_2) = A \otimes C_1 + A \otimes C_2$$

$$(A_1 + A_2) \otimes C = A_1 \otimes C + A_2 \otimes C$$

$$(\alpha A \otimes C) = \alpha(A \otimes C) = (A \otimes \alpha C) \quad \alpha \in K$$

$$(A_1 \otimes C_1)(A_2 \otimes C_2) = (A_1 A_2 \otimes C_1 C_2)$$

$$(A \otimes C) = (A \otimes I\ )(I\ \otimes C) = (I \otimes C)(A \otimes I\ ) \qquad \blacksquare$$

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