GENERALIZED CONTROLLABILITY FOR PERTURBED LINEAR SYSTEMS

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ABSTRACT

Although the term controllability has been more or less restricted to non-perturbed systems, it is readily extended to the perturbed case if the following definition is adopted:

> A system is controllable in the presence of disturbances belonging to a predetermined set of disturbances W, if for all the disturbances in the set W, the system remains controllable.

Several sets of necessary and sufficient conditions can be obtained to satisfy the above definition. However, the introduction of two geometrical concepts, "the unperturbed attainable set" and "the reduced target set", leads to interesting results for linear systems with additive disturbances.

For perturbed linear systems, in addition to geometrical and analytical necessary and sufficient conditions for controllability, some other significant results were obtained:

(1) impossibility of strict controllability when disturbances are present,

(2) existence of a minimum bound on the size of the elements of the control set to insure controllability,

(3) existence of a maximum bound on the size of the elements of the disturbance set to insure controllability,

(4) existence of a minimal "target size" to insure controllability,

(5) analytical equations to find the above parameters,

(6) method of decomposition of the analytical equations arising from the previous problems to simplify their numerical solution.

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CHAPTER I

INTRODUCTION

The problem of controllability has been treated mostly for unperturbed systems ([1],[2],[3],[4],[5],[6]), and perturbed situations were approached by statistical techniques except for a few isolated cases ([2],[7]). The definition of controllability adopted here is the natural extension of the concept of controllability in the unperturbed case: a system can be considered as controllable in the presence of disturbances if for all disturbances it remains controllable in the original sense of the word controllability for the unperturbed system.

The approach to this problem is not unique ([7]), but geometrical techniques proved to be more fruitful than purely analytical methods. For instance, the use of the satisfaction approach suggested by Takahara [7], although conceptually simple, does not yield explicit results. The difficulty lies in a min-max problem for a functional which is convex in both variables; this makes it impossible to change the order of the two operations which would be the only way to obtain some decent results.

The basic problem is then to obtain for a general abstract linear system ($\mathcal{L}$) the geometrical necessary and sufficient condition for controllability; this is done by introducing the concepts of the unperturbed attainable set and the reduced target set. The next step is to translate the geometrical condition into analytical terms by using the classical separation theorem for convex sets.

Among the interesting problems that naturally arise is that of characterization. Roughly speaking it shows the impossibility of strict controllability in the presence of disturbances and the existence of lower and upper bounds on the sizes of the control set, disturbance set and the target set.

Also one of the peripheral problems is how to solve the analytical equations arising from the controllability conditions.

The purpose of this research is to find a solution to these problems.

CHAPTER II

BASIC DEFINITIONS AND PROBLEM FORMULATION

This chapter includes:

a. a definition of an abstract linear system ($\mathcal{L}$)

b. an extension of the notion of controllability to perturbed systems

c. a formulation of the problem of controllability

d. a brief description of the nature of the results.

2.1 Mathematical Description of the System

Given

1) three reflexive Banach spaces X_1, X_2, X_3 with norms $| \ |_1$, $| \ |_2$, $| \ |_3$, respectively,

2) the closed bounded and convex subsets U and W of X_2 and X_3, respectively,

3) a fixed element s in X_1,

4) and the linear continuous maps, S and T,

$$S : U \to X_1$$
$$T : W \to X_1,$$

The abstract linear system ($\mathcal{L}$) is defined by a linear operator L

$$L : \{s\} \times U \times W \to X_1 ,$$

such that

$$L \quad (s,u,w) = s + Su + Tw$$

where

X_1 is the state space of $(\mathcal{L})$,

U is the control set of $(\mathcal{L})$

W is the disturbance set of $(\mathcal{L})$

and s is the initial state of $(\mathcal{L})$.

By extension, the elements of U will be called controls, those of W, disturbances and the image of (s,u,w), L (s,u,w), a state of $(\mathcal{L})$.

It will be shown later, if not obvious now, that the linear differential and difference equation systems have the same structures as the system $(\mathcal{L})$ and that all the results obtained for $(\mathcal{L})$ are true for the two systems above.

2.2 Definition of Controllability

Due to the unfortunate fact that the term controllability is defined in several ways in the literature, a definition is required to avoid any confusion.

Definition 2.1: The system $(\mathcal{L})$ is controllable under disturbances with respect to $(U, W, |\ |_1, x_d, \varepsilon)$ if there exists a control $\bar{u}$ in U such that

$$|s + Su + Tw - x_d|_1 \leq \varepsilon$$

for all disturbances w in W; if the disturbance set W contains only the null element $(W = \{o\})$, then the system is not perturbed and controllability with respect to $(U,|\ |_1,x_d,\varepsilon)$ is obtained

if there exists $\bar{u}$ in U such that

$$|s + Su - x_d|_1 \leq \varepsilon ;$$

if ε is zero in either one of the above definitions, the system is said to be strictly controllable.

2.3 Problem Formulation

The basic problem may be formulated in four parts:

1. to obtain geometrical and analytical necessary and sufficient conditions for controllability in the presence of disturbances of the abstract linear system $(\mathcal{L})$,

2. to apply the results to the linear difference equation system for both point and functional controllability,

3. to solve some aspects of the characterization problem (existence of a maximum bound on the elements of the disturbance set, a minimum bound on the elements of the control set and a minimal target for which the controlability property is retained),

4. to construct an approximation scheme in order to solve the equations arising from the analytical necessary and sufficient conditions and the different aspects of the characterization problem.

2.4 Nature of the Results

The results obtained constitute a complete answer to the four parts of the problem (see section 2.3). More specifically, generalized geometrical and analytical necessary and sufficient conditions for controllability in the absence or in the presence of disturbances have been obtained by the introduction of the concepts of the unperturbed attainable set and the reduced target set (Chapter 4). One of the applications of the previous results was the solution of the problem of point and functional controllability for linear difference equation systems (Chapter 5).

Almost all the aspects of the characterization problem have been solved for the abstract linear system ($\mathcal{L}$), (Chapter 6). Finally a decomposition scheme was found to break the equations arising from the controllability conditions into several subproblems for which successive approximation techniques already exist (Chapter 7). Numerical results can be obtained by using Goldstein's iterative method [12] which finds extrema of continuous functions with a continuous gradient. This continuous gradient requirement is not necessarily satisfied by all the functions to be maximized although their gradient is well-defined when the spaces are combinations of Euclidean spaces.

The possible extensions of the results presented here are numerous and some of the possibilities will be discussed in the conclusion.

CHAPTER III

MATHEMATICAL PRELIMINARIES

This chapter will present the necessary and sufficient conditions for the separability and embeddability of convex subsets of real linear spaces; in all cases the spaces appearing in this chapter are Hansdorff unless the contrary is clearly specified.

3.1 Separability of Convex Sets

Definition 3.1: If f is a non-identically-zero linear functional on a real linear space X and t is any real number, each of the sets $\{x : f(x) \geq t\}$ and $\{x : f(x) \leq t\}$ is a half-space, and the pair are complementary half-spaces; their intersection, the set $f^{-1}[t]$ is a linear manifold called a hyperplane (see [8], page 19).

Definition 3.2: Two subsets, A and B, of a real linear space can be separated if there are complementary half-spaces which contain A and B respectively (see [8], page 22).

Definition 3.3: A linear functional f is said to separate A and B if and only if f is not identically zero and

$$\sup\{f(x) : x \in A\} \leq \inf\{f(y) : y \in B\} .$$

The linear functional f strongly separates A and B if the inequality above is strong; that is, if

$$\sup\{f(x) : x \in A\} < \inf\{f(y) : y \in B\}$$

(see [8], page 119, Corollary 14.4).

Theorem 3.1

Let A and B be non-void disjoint convex subsets of a locally convex linear topological space, and suppose that A is compact and B is closed. Then there is a continuous linear functional strongly separating A and B; that is, there exists f linear and continuous such that

$$\sup\{f(x) : x \in A\} < \inf\{f(y) : y \in B\}$$

(see [8] page 119 , Corollary 14.4).

Theorem 3.2: Theorem on Strong Separation

Two convex subsets, A and B, of a real linear space E can be strongly separated by a linear functional if and only if there is a convex set U which is radial at the origin 0 such that $(A+U) \cap B$ is void (see [8] page 23, Theorem 3.9); note that a subset U of a linear space is called radial at a point x if and only if U contains a line segment through x in each direction.

Definition 3.4: The conjugate space X^* of a linear topological space X is the space of all linear continuous functionals on X (see [8] page 119); the conjugate, X^*, of a normed linear space X is a Banach space (see [9] page 62 , Corollary 9).

Theorem 3.3

Under the premises of Theorem 3.1,

$$(A \cap B \neq \emptyset) \Leftrightarrow (\inf\{f(y) : y \in B\} - \sup\{f(x) : x \in A\} \leq 0, \ \forall f \in X_1^* \sim \{o\}$$

Proof:

1) ($\Leftarrow$)

This is the contrapositive of Theorem 3.1.

2) ($\Rightarrow$)

Assume that for some non-identically zero f in X_1^*

$$\inf\{f(y) : y \epsilon B\} - \sup\{f(x) : x \epsilon A\} > o,$$

then f strongly separates A and B and by Theorem 3.2 there is a convex set U which is radial at the origin 0 such that $(A+U) \cap B$ is void. In particular A is a subset of A+U; hence

$$A \cap B \subset (A+U) \cap B = \emptyset$$

contradicts the fact that $A \cap B$ is non-void. This completes the proof of Theorem 3.3.

Definition 3.5: The Weak and Weak* Topologies

Let X be a linear space with a vector topology τ, and X^* the space of all τ-continuous linear functionals, then the weak topology $w(X,X^*)$ for the linear space (X,τ) is the topology having as base the family of sets of the form

$$\{x : |f_i(x)| \leq 1 \qquad \text{for } i = 1,\ldots,n\}$$

where $\{f_1,\ldots,f_n\}$ is an arbitrary finite subset of X^*; the weak* topology $w(X^*,X)$ for the conjugate space (X^*,τ^*) (where τ^* is the strong topology induced by τ on X^*) is the topology having as base the family of sets of the form

$$\{f: |f(x_i)| \leq 1 \quad \text{for } i = 1,\dots,n\}$$

where $\{x_1,\dots,x_n\}$ is an arbitrary finite subset of X (see [8] page 155).

Definition 3.6: The functional f is called weakly lower semi-continuous at the point x_o if, for any $\{x_n\}$ which converges weakly to x_o,

$$f(x_o) \leq \underline{\lim}_{n\to\infty} f(x_n)$$

(see [11] page 73, Definition 8.1).

Theorem 3.4

In a reflexive Banach space X, a convex and strongly lower semi-continuous functional is weakly lower semi-continuous.

Proof:

Although this theorem exists for some particular cases in the literature, it has not yet been given in the above general form. The proof, due to S.K. Mitter, will be reproduced here without any major modifications.

Under the premises of the theorem let us assume that the functional f is not weakly semi-continuous. Hence for some x_o in X

Definition 3.6 is not satisfied; that is there exists a weakly convergent sequence, $\{x_n\}$,

$$\{x_n\} \rightharpoonup x_o$$

such that

$$f(x_o) > \underline{\lim}_{n\to\infty} f(x_n).$$

This implies that for some positive real ε, we can find a subsequence, also denoted by $\{x_n\}$, for which

$$f(x_n) \leq f(x_o) - \varepsilon \quad , \text{ for all } n. \tag{3.1}$$

Now since X is a reflexive Banach space, for every weakly convergent sequence and any positive real δ, we can find elements x_i, $i = 1,\ldots,N(\delta)$ such that

$$|x_o - \sum_{i=1}^{N} \alpha_i^N \; x_i|_X < \delta, \text{where } \alpha_i^N \geq 0 \text{ and } \sum_{i=1}^{N} \alpha_i^N = 1$$

(see [9] page 422, Corollary 14).

Since f is strongly lower semi-continuous, we can choose δ sufficiently small such that

$$f(\sum_{i=1}^{N} \alpha_i^N x_i) > f(x_o) - \frac{\varepsilon}{2} \tag{3.2}$$

However, the convexity of the functional f and Equation 3.1 and 3.2 lead to a contradiction:

$$f(x_o) - \frac{\varepsilon}{2} < f(\sum_{i=1}^{N} \alpha_i^N x_i) \leq \sum_{i=1}^{N} \alpha_i^N f(x_i) \leq \sum_{i=1}^{N} \alpha_i^N (f(x_o)-\varepsilon) = f(x_o)-\varepsilon$$

and

$$\frac{\varepsilon}{2} > \varepsilon > 0.$$

This completes the proof of Theorem 3.4.

Theorem 3.5

Let X be a reflexive Banach space and B a closed, convex and bounded subset of X, then the map

$$\lambda : X^* \to R$$
$$x^* \to \lambda(x^*)$$

defined by

$$\lambda(x^*) = \sup\{x^*(y) : y \varepsilon B\}$$

is weak* lower semi-continuous on X^*.

Corollary 3.5

For X and B as described in Theorem 3.5, the functional λ on X^* defined by

$$\lambda(x^*) = \inf\{x^*(y) : y \varepsilon B\}$$

is weak* upper semi-continuous on X^*.

<u>Proof of Theorem 3.5</u>:

It suffices to prove that $\lambda(x^*)$ is a strongly continuous convex functional on X^* to show that it is weakly lower semi-continuous on X^* (see Theorem 3.4).

That $\lambda(x^*)$ is a convex function is obvious from the definition:

$$(\mu x_1^* + (1-\mu)x_2^*)(y) = \mu x_1^*(y) + (1-\mu)x_2^*(y), \ \forall \mu \varepsilon [0,1]$$

and

$$\lambda(\mu x_1^* + (1-\mu)x_2^*) \leq \mu\lambda(x_1^*) + (1-\mu)\lambda(x_2^*), \ \forall \mu \varepsilon [0,1]$$

To show that λ is strongly continuous, simply note that since B is bounded in the norm topology of X, there exists a positive real N such that

$$|x|_X \leq N, \ \forall x \varepsilon B.$$

Now given x_1^* and x_2^* in X^*,

$$x_2^*(y) = (x_2^* - x_1^*)(y) + x_1^*(y), \ \forall y \varepsilon B$$

and

$$\lambda(x_2^*)(y) \leq \sup\{(x_2^* - x_1^*)(y) : x \varepsilon B\} + \lambda(x_1^*);$$

but

$$\sup\{(x_2^*-x_1^*)(y) \ : \ y\varepsilon B\} \ \leq \ |x_2^*-x_1^*|_{X^*} \sup\{|y|_X \ : \ y\varepsilon B\}$$

$$\leq N|x_2^*-x_1^*|_{X^*}$$

and by symmetry,

$$|\lambda(x_2^*)-\lambda(x_1^*)| \ \leq \ N|x_2^*-x_1^*|_{X^*}$$

From the last equation it is clear that λ satisfies the ε-δ definition of continuity and that λ is strongly continuous on X^* completing the proof of Theorem 3.5.

Definition 3.7: Let T be a linear transformation,

$$T \ : \ E \to F,$$

where E and F are linear topological spaces with the natural pairings $\langle E,E^*\rangle$ and $\langle F,F^*\rangle$, then the adjoint of T, denoted by T^*,

$$T^* \ : \ F^* \to E^*$$

defined by

$$T^*(h)(x) = h(T(x)) \ , \forall \, x\varepsilon E, \ \forall \, h\varepsilon F.$$

(see [8] page 204).

Theorem 3.6

Let E and F be locally convex, linear topological spaces, and let T be a linear transofmration of E into F; then

(i) if T is continuous, then T is weakly continuous, and T has an adjoint T^* if and only if T is weakly continuous. The adjoint, if it exists, is unique and is necessarily weak* continuous (and linear) (see [8] page 204, Theorem 21.5(i)),

(ii) if E and F are also normed spaces, then the adjoint linear operator T^* on F^* into E^* is continuous with respect to the weak* topologies if and only if it is continuous with respect to the norm topologies (see [9] page 422, Theorem 15); the norm, or strong, topology for E^* (or F^*) is the topology induced by the natural norm on E^* (or F^*) (see [9] page 61, Lemma 8).

Theorem 3.7

Let X be a reflexive Banach space, let A and B be closed and convex subsets of X and suppose that A is bounded; then the following four statements are equivalent:

(i) $A \cap B \neq \emptyset$

(ii) $\inf\{f(y) : y \in B\} - \sup\{f(x) : x \in A\} \leq 0,\ \forall f \in X^* \sim \{0\}$

(iii) $\sup\{\inf[f(y) : y \in B] - \sup[f(x) : x \in A] : f \in X^*,\ |f|_{X^*} = 1\} \leq 0$

(iv) $\max\{\inf[f(y) : y \in B] - \sup[f(x) : x \in A] : f \in X^*,\ |f|_{X^*} \leq 1\} \leq 0$

Proof:

a) (i) $\Longleftrightarrow$ (ii)

The space X with the $w(X,X^*)$-topology or weak topology of X is a locally convex Hansdorff linear topological space; moreover, the convex sets A and B are respectively weakly compact and weakly closed. Then Theorem 3.3 can be applied and (i) is equivalent to (ii).

b) (ii) $\Longleftrightarrow$ (iii)

It is clear that (ii) implies (iii). Conversely if (iii) is true then for any non-identically zero f in X*,

$$\inf\{f(y):y\varepsilon B\}-\sup\{f(x):x\varepsilon A\}=|f|_{X*}\{\inf[\frac{f}{|f|_{X*}}(y):y\varepsilon B]$$

$$-\sup[\frac{f}{|f|_{X*}}(x):x\varepsilon A]\} \leq |f|_{X*}\sup\{\inf[g(y):y\varepsilon B]$$

$$-\sup[g(x):x\ A] : g\varepsilon X^*, |g|_{X*} = 1\} \quad \leq 0$$

c) (ii) $\Longleftrightarrow$ (iv)

This result follows by an argument identical to the one given in part b. The "max" appearing in condition (iv) is easily justified by the fact that the expression over which the supremum is to be taken is weak* upper semi-continuous on the weak* conpact unit ball in X* (see [11]page 78, Theorem 9.2).

This completes the proof of Theorem 3.5.

3.2 Imbeddability of Convex Sets

Theorem 3.8

Let X be a reflexive Banach space and

a) $B = \{x \varepsilon X : |x|_X \leq \varepsilon\}$, $0 \leq \varepsilon < \infty$

b) A, a convex, closed and bounded subset of X; then the following three statements are equivalent

(i) $A \subset B$

(ii) $\sup\{|a|_X : a \varepsilon A\} \leq \varepsilon$

(iii) $\sup\{\sup[f(a) : a \varepsilon A] : f \varepsilon X^*, |f|_{X^*} = 1\} \leq \varepsilon$

Proof:

The equivalence of (i) and (ii) is obvious. The equivalence between the last two is a consequence of the Hahn-Banach Theorem (see [9], page 65, Corollary 15):

$$|a|_X = \sup\{f(a) : f \varepsilon X^*, |f|_{X^*} = 1\}$$

and

$$\sup\{|a|_X : a \varepsilon A\} = \sup\{\sup[f(a) : f \varepsilon X^*, |f|_{X^*} = 1] : a \varepsilon A\};$$

but since the function f(a) is real valued the order of the two sup can be changed (see [10],page 352, Proposition 9).

This completes the proof of Proposition 3.8.

CHAPTER IV

NECESSARY AND SUFFICIENT CONDITIONS FOR CONTROLLABILITY

This chapter is devoted to the proof and discussion of the necessary and sufficient condition for controllability. In the first section the concepts of the unperturbed attainable set, the zone of disturbances around a point and the reduced target set will be defined and their geometric properties investigated; from the latter it will then be proven that a necessary and sufficient condition for controllability is the non-empty intersection of the reduced target set and the unperturbed attainable set. The second section will translate the geometrical necessary and sufficient condition into analytical terms.

All the results will be presented for the abstract linear system ($\mathcal{L}$) defined in section 2.1 and the Definition 2.1 of controllability. It is worth noting that the linear difference equation and linear differential equation systems have the same structure as the abstract linear system ($\mathcal{L}$); hence to solve the controllability problem for the system ($\mathcal{L}$) is to solve it for them also.

4.1 The Geometrical Condition for Controllability

Definition 4.1: Given the system ($\mathcal{L}$), the unperturbed attainable set A is defined as

$$A = \{s + Su : u \in U\}$$

Definition 4.2: Given the system ($\mathcal{L}$), the zone of disturbances D_x at a point in X_1 is defined as

$$D_x = \{x + Tw : w \in W\}$$

Definition 4.3: Given the system ($\mathcal{L}$), a target B is a subset of X_1 defined as

$$B = \{x \in X_1 : |x - x_d|_1 \leq \varepsilon\}$$

where x_d is a desired point in the state space X_1 and ε is a finite non-negative real number.

Definition 4.4: Given the system ($\mathcal{L}$), the reduced target set T_D for the target B is defined as

$$T_D = \{x \varepsilon X_1 : D_x \subset B\}$$

Proposition 4.1:

The unperturbed attainable set A and the zone of disturbance D_x at any point are convex and weakly compact in X_1.

Proof:

A and D_x are translations in the X_1 space of the sets S(U) and T(W); hence it suffices to study the properties of the latter two. Moreover, since S(U) and T(W) have the same structure, what will be true for S(U) will be true for T(W).

1) convexity

A linear map on a convex set has a convex image (see [9], page 410, Lemma 5).

2) weak compactness

Since U is a closed and bounded convex subset of a reflexive Banach space, it is weakly compact in X_2; but, by Theorem 3.6(i), the strong continuity of the linear operator S implies that S is also continuous if X_1 and X_2 have the weak topologies. Hence S(U) is weakly compact in X_1. This completes the proof of Proposition 4.1.

Proposition 4.2:

The target set B and the reduced target set T_D are convex and weakly compact in X_1.

Proof:

By definition B is clearly convex and weakly compact; showing the same for T_D requires a somewhat longer proof.

1) convexity

Given x_1, x_2 in T_D, then D_{x_1} and D_{x_2} are subsets of B and for any w in W,

$$x_1 + Tw \in B \quad , \quad x_2 + Tw \in B.$$

The convexity of B implies that

$$\lambda x_1 + (1-\lambda)x_2 + Tw \in B \quad , \forall \lambda \in [0,1], \forall w \in W,$$

and

$$D_{\lambda x_1 + (1-\lambda)x_2} \subset B \quad , \forall \lambda \in [0,1].$$

2) <u>weak compactness</u>

Note that in the reflexive Banach space X_1, a closed bounded convex set is weakly compact (see [9], page 422, Theorem 13, and page 425, Corollary 8). T_D is clearly bounded as a subset of B; to prove it is closed in the strong topology of X_1, consider a Cauchy sequence $\{x_n\}$ in T_D (X_1 with the strong topology). Since T_D is a subset of the complete space X_1, $\{x_n\}$ converges to some point x in X_1. Now consider for any w in W the translated sequence $\{x_n + Tw\}$; this sequence is Cauchy and converges to x + Tw in X_1. However as points of T_D, the x_n's are such that

$$x_n + Tw \in B \qquad \text{for all } w \text{ in } W.$$

Hence $\{x_n + Tw\}$ is a Cauchy sequence whose points are all in B; but B being closed in the strong topology contains all limit points of B. In particular x + Tw is in B for all w in W:

$$D_x \subset B \implies x \in T_D$$

This completes the proof of Proposition 4.2.

Theorem 4.1

Given the system ($\mathcal{L}$), it is controllable in the presence of disturbances (Definition 2.1) if and only if the unperturbed attainable set A (Definition 4.1) and the reduced target set T_D (Definition 4.4) have a non-empty intersection.

Proof:

Mathematically the above statement is equivalent to

$$\bar{u} \in U \quad \text{such that}$$

$$|s + Su + Tw - x_d|_1 \leq \varepsilon \quad \Longleftrightarrow \quad A \cap T_D \neq \emptyset$$

$$\text{for all } w \in W$$

1) ($\Longrightarrow$)

Clearly

$$s + S\bar{u} + Tw \in B \quad , \forall w \in W,$$

and hence,

$$s + S\bar{u} \in T_D \ ;$$

However, $\bar{u}$ in U implies that

$$s + S\bar{u} \in A$$

and then $A \cap T_D$ is non-empty.

2) ($\Leftarrow$)

If $A \cap T_D$ is non-empty, then there exists x in X_1 such that

$$x \in A \qquad \text{and} \qquad x \in T_D.$$

Since x is in A, there exists u in U such that

$$x = s + Su; \tag{4.1}$$

similarly x being an element of T_D,

$$x + Tw \in B \;, \forall\, w \in W,$$

and

$$|x + Tw - x_d|_1 \leq \varepsilon \;, \forall\, w \in W. \tag{4.2}$$

The substitution of Equation 4.1 in Equation 4.2 now shows the existence of an u in U for which the system is controllable. This completes the proof of Theorem 4.1.

4.2 Analytical Necessary and Sufficient Condition for Controllability

Theorem 4.2

Given the system ($\mathcal{L}$) and

$$U = \{u \in X_2 : |u|_2 \leq \rho\},\ 0 \leq \rho < \infty \tag{4.3}$$

$$W = \{w \in X_3 : |w|_3 \leq \beta\},\ 0 \leq \beta < \infty \tag{4.4}$$

the system ($\mathcal{L}$) is controllable in the presence of disturbances with respect to $(s, U, W, |\ |_1, x_d, \varepsilon)$ if and only if

$$\sup\{\beta |T^*g|_{3*} : |g|_{1*} = 1\} \leq \varepsilon \tag{4.5}$$

$$h(s-x_d) - \rho|S^*h|_{2*} - \max\{h(x) : x \quad \partial T_D^{x_d}\} \leq 0,\ \forall\, h \in X_1^* \sim \{0\} \tag{4.6}$$

where the set $T_D^{x_d}$ is defined as

$$\partial T_D^{x_d} = \{x \in X_1 : \sup[g(x) + \beta|T^*g|_{3*} : g \quad X_1^*, |g|_{1*} = 1] \leq \varepsilon\}, \tag{4.7}$$

S^* and T^* are the adjoint operators of S and T, and, $|\ |_{1*}$, $|\ |_{2*}$ and $|\ |_{3*}$ are the natural norms induced on X_1^*, X_2^* and X_3^* respectively; furthermore condition 4.7 is equivalent to

$$\sup\{h(s-x_d) - \rho|S^*h|_{2*} - \max\{h(x) : x \in \partial T_D^{x_d} : h \in X^*_1, |h|_{1*} = 1\} \leq 0. \tag{4.8}$$

Corollary 4.2

Under the premises of Theorem 4.2 and if $W = \{0\}$ ($\beta=0$), then ($\mathcal{L}$) is controllable with respect to $(s,U,|\ |_1,x_d,\varepsilon)$ if and only if

$$\sup\{h(s-x_d)-\rho|S^*h|_{2*} : h\in X_1^*,\ |h|_{1*} = 1\}-\ \varepsilon \leq 0. \quad (4.9)$$

The proof of the Theorem necessitates several propositions which will be proved after the theorem itself. At this juncture, it is interesting to note that Corollary 4.2, the necessary and sufficient condition for controllability of an unperturbed linear system is of the same nature as previous results obtained in the literature ([1], [2],[4],[5]). Results on complete controllability and functional controllability using algebraic methods have also been obtained ([3],[6]).

Proof of Theorem 4.2:

By Theorems 4.1 and 3.7, the system ($\mathcal{L}$) is controllable in the presence of disturbances if and only if either

$$\inf\{h(y):y\in A\}-\sup\{h(x):x\in T_D\} \leq 0,\quad h\in X_1^*\sim\{0\} \quad (4.10)$$

or

$$\sup\{\inf[h(y):y\in A]-\sup[h(x):x\in T_D]:h\in X_1^*,\ |h|_{1*}=1\}\leq 0. \quad (4.11)$$

To insure that the above expressions have a meaning it is necessary that the sets A and T_D non-empty. The set A is not empty by definition; the set T_D is not empty if and only if

$$\sup\{\beta|T^*g|_{3*}: |g|_{1*} = 1\} \leq \varepsilon$$

(see Proposition 4.5) justifying the condition expressed in Equation 4.5.

All that remains is to compute the above expressions and verify that they agree with Equations 4.6, 4.7 and 4.8.

(1) $\underline{\inf[h(y):y\in A]}$

$$\inf[h(y):y\in A]=\inf[h(s+Su):u\in U]$$

$$=h(s) + \inf[S^*h(u):|u|_2 \leq \rho]$$

since, S being continuous, its adjoint S* exists and is unique (Theorem 3.6(i)); but S*h is an element of X_2^* and

$$\inf[\overset{*}{S}h(u):|u|_2 \leq \rho]= - \sup[\overset{*}{S}h(u):|u|_2 \leq 1]= -\rho|\overset{*}{S}h|_{2*} .$$

(2) $\underline{\sup[h(x):x\in T_D]}$

First of all, note that the set T_D can be translated to the origin in X_1; let $T_D^{x_d}$ be the translation of T_D,

$$T_D^{x_d} = \{x\in X_1 : D_x \subset \{x\in X_1 : |x|_1 \leq \varepsilon \}\}$$

then

$$\sup[h(x) : x \in T_D] = \sup[h(x - x_d) : x \in T_D^{x_d}]$$

$$= h(x_d) + \sup[h(x) : x \in T_D^{x_d}].$$

The set $T_D^{x_d}$, having the same properties as T_D, is weakly compact; then the weak continuity of h implies the existence of an $\bar{x}$ in $T_D^{x_d}$ such that $h(\bar{x})$ is equal to

$$\sup[h(x) : x \in T_D^{x_d}] ;$$

moreover, since h is linear, h(x) assumes its maximum on the boundary of $T_D^{x_d}$ (see Proposition 4.3):

$$\sup[h(x) : x \in T_D^{x_d}] = \sup[h(x) : x \in \partial T_D^{x_d}].$$

All that remains is to obtain an analytical definition of the elements of the set $T_D^{x_d}$. Since D_x is closed convex and bounded in X_1, then by Theorem 3.8

$$D_x \subset \{x : |x|_1 \leq \varepsilon\}$$

is equivalent to

$$\sup\{\sup[g(y) : y \in D_x] : g \in X_1^*, \ |g|_{X_1^*} = 1\} \leq \varepsilon$$

and since

$$\sup[g(y):y\in D_x] = \sup[g(x+Tw):w \in W]$$

$$= g(x) + \beta|T^*g|_{3*}$$

then

$$T_D^{x_d}=\{x\in X_1:\sup[g(x)+\beta|T^*g|_{3*}:g\in X_1^*, |g|_{1*}=1]\leq \varepsilon\}$$

and by Proposition 4.4,

$$\partial T_D^{x_d}=\{x\in X_1:\sup[g(x)+\beta|T^*g|_{3*}:g\in X_1^*, |g|_{1*}=1]=\varepsilon\}.$$

The substitution of the above results in Equations 4.10 and 4.11 gives Equations 4.6, 4.7 and 4.8 and completes the proof of Theorem 4.2.

Proposition 4.3: Given any element h on the unit sphere in X_1^* and a weakly compact subset C of X_1, then the element $\tilde{x}$ in C which maximizes the functional h(x) is a boundary point of C.

Proof:

The weak continuity of h on C guarantees the existence of an element in C which maximizes h(x).

If it is assumed that none of the boundary points maximizes h(x), then one can find an interior point $\tilde{x}$ for which

$$h(\tilde{x}) = \max\{h(x) : x \in C\}.$$

As an interior point of the compact set C, $\tilde{x}$ has an open neighborhood $N_\delta(\tilde{x})$ entirely contained in C:

$$N_\delta(\tilde{x}) = \{x \in X_1 : |x - \tilde{x}|_{X_1} < \delta\}.$$

Let $B_{\delta/2}(\tilde{x})$ be the closed ball centered at $\tilde{x}$, then $B_{\delta/2}(\tilde{x})$ is weakly compact; but for any x in $B_{\delta/2}(\tilde{x})$

$$h(x) = h(\tilde{x}) + h(x - \tilde{x})$$

and

$$\max\{h(x) : x \in B_{\delta/2}(x)\} = h(\tilde{x}) + \max\{h(x-\tilde{x}) : x \in B_{\delta/2}(x)\}$$

so that

$$\max\{h(x) : x \in C\} \geq h(\tilde{x}) + \max\{h(x-\tilde{x}) : |x-\tilde{x}| \leq \delta/2\} = h(\tilde{x}) + \delta/2$$

since h has unit norm in X_1^*. This contradicts the maximality of $h(\tilde{x})$. and hence $\tilde{x}$ cannot be an interior point of the set C. This establishes Proposition 4.3.

<u>Proposition 4.4</u>: If the set T_D is not empty, its boundary, ∂T_D, can be defined as

$$\{x \in X_1 : \sup[g(x-x_d)+\beta|T^*g|_{3*} : |g|_{1*}=1]=\varepsilon\}. \tag{4.12}$$

Proof:

The definition of the set T_D (Definition 4.4) can be translated into analytical terms by the application of Proposition 4.2 and Theorem 3.8:

$$T_D = \{x \in X_1 : \sup[g(x-x_d)+\beta|T^*g|_{3*} : |g|_{1*}=1] \leq \varepsilon\}.$$

a) Let x_o be a point of the set defined by Equation 4.11 and assume it is an interior point of T_D; then one can find an open neighborhood $N_\delta(x_o)$ of x_o entirely contained in T_D. If $B(x_o)$ is the closed ball

$$B(x_o) = \{x \in X_1 : |x-x_o|_1 \leq \delta/2\} \subset N_\delta(x_o) \subset T_D ,$$

then for all y in $B(x_o)$

$$g(y-x_d)+\beta|T^*g|_{3*}=g(y-x_o)+g(x_o-x_d)+\beta|T^*g|_{3*} . \tag{4.13}$$

Let us now verify that all elements of $B(x_o)$ still satisfy the definition of Equation 4.12; let us consider the expression

$$\sup\{\sup[g(y-x_o)+g(x_o-x_d)+\beta|T^*g|_{3*} : |g|_{1*}=1] : y \in B(x_o)\}; \tag{4.14}$$

then the order of the sup can be changed (see [10], page 352, Proposition 9). Now $B(x_o)$ is weakly compact in X_1^*, so for any g on the unit sphere in X_1^*,

$$\sup\{g(y-x_o)+g(x_o-x_d)+\beta|T^*g|_{3*}:y\in B_{x_o}\}$$

is equal to

$$\delta/2 + g(x_o-x_d) + \beta|T^*g|_{3*}$$

and finally Equation 4.14 is equal to

$$\delta/2 + \sup\{g(x_o-x_d) + \beta|T^*g|_{3*}:|g|_{1*}=1\}=\delta/2 +\varepsilon .$$

Consequently there exists a sequence $\{y_i\}$ in $B(x_o)$ such that

$$\{\sup[g(y_i-x_d)+\beta|T^*g|_{3*}:|g|_{1*} = 1\}\to \varepsilon + \delta/2 ;$$

thus given an arbitrary real $\eta < \delta/2$, there exists N (and hence y_N in $B(x_o)$) such that

$$\sup[g(y_N-x_d)+\beta|T^*g|_{3*}:|g|_{1*}=1\}\geq \varepsilon + \delta/2 - \eta > \varepsilon$$

or there exists y_N in $B(x_o)$ such that $y \notin T_D$.

The last contradiction establishes that all the elements of the set defined in Equation 4.12 are boundary points of T_D.

b) Let x_o be a boundary point of T_D and assume that

$$\alpha = \sup\{g(x_o - x_d) + \beta |T^*g|_{3*} : |g|_{1*} = 1\} < \varepsilon \; ; \qquad (4.15)$$

let $\delta = (\varepsilon - \alpha)/2$ and note that δ is strictly positive. Let $N_\delta(x_o)$ be an open neighborhood of x_o defined as

$$N_\delta(x_o) = \{x \in X_1 : |x - x_o|_1 < \delta\}$$

then for all y in $N_\delta(x_o)$,

$$\begin{aligned} &\sup\{g(y - x_d) + \beta |T^*g|_{3*} : |g|_{1*} = 1\} \\ &\quad \leq \sup\{g(x_o - x_d) + \beta |T^*g|_{3*} : |g|_{1*} = 1\} + \sup\{g(y - x_o) : |g|_{1*} = 1\} \\ &\quad \leq \alpha + |y - x_o|_1 \leq \alpha + \frac{\varepsilon - \alpha}{2} < \varepsilon \; ; \end{aligned}$$

that is

$$N_\delta(x_o) \subset T_D \; .$$

This last contradiction establishes that a boundary point of T_D must satisfy the definition of Equation 4.12. This completes the proof of Proposition 4.4.

Proposition 4.5: The set T_D is not empty if and only if

$$\sup\{\beta|T^*g|_{3*}: |g|_{1*} = 1\} \leq \varepsilon \ .$$

Corollary 4.5

The set T_D is not empty if and only if x_d belongs to T_D.

Proof of Proposition 4.5:

a) if $\sup\{\beta|T^*g|_{3*}: |g|_{1*} = 1\} \leq \varepsilon$

then clearly x_d is in T_D and T_D is not empty.

b) Let T_D be not empty and assume that

$$\sup\{\beta|T^*g|_{3*}: |g|_{1*} = 1\} > \varepsilon. \tag{4.16}$$

Now if it can be shown that for any x in X_1,

$$\sup\{g(x-x_d)+\beta|T^*g|_{3*}: |g|_{1*}=1\} \geq \sup\{\beta|T^*g|_{3*}: |g|_{1*}=1\}; \tag{4.17}$$

then Equations (4.17) and (4.16) imply that T_D is empty, contradicting the initial hypothesis.

To verify Equation 4.17, note that for any g in X_1^* such that

$$g(x-x_d) < 0$$

there exists (-g) in X_1^* such that

$$(-g)(x-x_d) = -g(x-x_d) > 0$$

and hence

$$\sup\{g(x-x_d)+\beta|T^*g|_{3*}:|g|_{1*}=1\}=\sup\{|g(x-x_d)|+\beta|T^*g|_{3*}:|g|_{1*}=1\}.$$

Thus for any x in X_1^*,

$$|g(x-x_d)|+\beta|T^*g|_{3*} \geq \beta|T^*g|_{3*} \quad , \forall g \in X_1^* ,$$

and

$$\sup\{g(x-x_d)+\beta|T^*g|_{3*}:|g|_{1*} = 1\} \geq \sup\{\beta|T^*g|_{3*}:|g|_{1*} = 1\}$$

This completes the proof of Proposition 4.5.

CHAPTER V

APPLICATION: THE LINEAR DIFFERENCE EQUATION SYSTEM ($\mathcal{S}$)

The theory which has been developed in Chapter IV can be applied without any difficulties to linear difference and differential equations systems and certain types of linear partial differential equations; for simplicity it was then decided to present the application of the theory only for the first class of systems.

Let R^n be the n-dimentional Euclidean space and I be a discrete time set,

$$I = \{0,1,\ldots,N-1\}.$$

The system ($\mathcal{S}$) under consideration is the finite dimensional linear discrete-time system. Mathematically it is described by a linear difference equation,

$$\begin{aligned} x(i+1)-x(i) &= A(i)x(i)+B(i)u(i)+C(i)w(i) \\ y(i) &= H(i)x(i) \ , \ \forall\, i \in I \\ x(0) &= x_o \ ; \end{aligned} \tag{5.1}$$

for each i in I,

$x(i) \in R^n$ represents the state of the system at time i,

$y(i) \in R^{\ell}(\ell \leq n)$, the output of the system at time i,

$u(i) \in R^m(m \leq n)$, the input (control) to the system at time i

and

$$w(i) \in R^k (k \leq n), \quad \text{the disturbance in the system at time } i;$$

for each time i in I,

	$A(i)$	is an $n \times n$ matrix
	$B(i)$	is an $n \times m$ matrix
	$C(i)$	is an $n \times k$ matrix
and	$H(i)$	is an $\ell \times n$ matrix;

all the above listed matrices are defined on I; finally $x_o \in R^n$ is a vector describing the state of ($\mathcal{S}$) at time 0.

In order to complete the mathematical description of the system ($\mathcal{S}$) a few definitions are necessary:

Definition 5.1: A control u is a map on the time set I into the m-dimensional Euclidean space R^m. The set of all such maps will be denoted by $1^N(R^m)$.

Definition 5.2: A disturbance w is a map on the time set I into the k-dimensional Euclidean space R^k. The set of all such maps will be denoted by $1^N(R^k)$.

Definition 5.3: The control set U is defined as

$$U = \{u \in 1_q^N(R_{q'}^m): ||u||_{1_q^N(R_{q'}^m)} \leq \rho\}$$

where the space $1_q^N(R_{q'}^m)$ is the space $1^N(R^m)$ with norm

$$\left\{ \sum_{i=o}^{N-1} \left[\sum_{j=1}^{m} |u_j(i)|^{q'} \right]^{q/q'} \right\}^{1/q} , \quad (1 \leq q,q' \leq \infty).$$

Definition 5.4: The disturbance set W is defined as

$$W = \{w \in l_r^N(R_{r'}^k): \ ||w||_{l_r^N(R_{r'}^k)} \leq \beta\}$$

where the space $l_r^N(R_{r'}^k)$ is the space $l^N(R^k)$ with norm

$$\left\{ \sum_{i=o}^{N-1} \left[\sum_{j=1}^{m} |w_j(i)|^{r'} \right]^{r/r'} \right\}^{1/r} , \quad (1 \leq r,r' \leq \infty).$$

It is clear that the above defined finite dimensional normed linear spaces are reflexive Banach spaces satisfying the requirements on the spaces X_2 and X_3 defined in Chapter II (see [9], page 244, Corollary 2, page 248, Corollary 8).

It is still necessary to study the solutions of the linear difference equation before defining the state space X_1 and the operators s, S and T. The solution of the linear difference equation is

$$\varphi(x_o;u,v,i+1)=\Phi(i+1,0)x_o+ \sum_{j=o}^{i} \Phi(i+1,j+1)B(j)u(j) \quad (5.2)$$

$$+ \sum_{j=o}^{i} \Phi(i+1,j+1)C(j)w(j) , \quad \text{for all i in I}$$

where

$$\Phi(k,k_o) = \begin{cases} \prod_{n=k_o}^{k} (I+A(n)) & , \quad k > k_o \\ I & , \quad k = k_o \end{cases} \qquad (5.3)$$

It should be noted that in the case of linear difference equations in finite dimensional linear spaces, all the elements of the subsets U of $l_q^N(R_{q'}^m)$ and W of $l_r^N(R_{r'}^k)$ are admissible. Finally, for simplicity, the matrices H(i) describing the output of the system ($\mathcal{S}$) will be taken to be identity matrices; nevertheless all the results which appear in this chapter are valid for any H(i), so that no loss of generality results from this simplification.

5.1 Point-Controllability

The problem of reachability of a desired state x_d from an initial state x_o at time zero is to determine the conditions for the existence of an admissible control $\overline{u}$ in U which steers the initial state x_o of the system at time zero to or within an appropriately defined ε-neighborhood of x_d at the final time N. This idea can now easily be extended to the perturbed case:

Definition 5.5: Point Controllability in the Presence of Disturbances

The system ($\mathcal{S}$) is point-controllable in the presence of disturbances with respect to

$(x_o, U, W, \|\ \|_{R^n}, x_d, \varepsilon)$ if there exists a control $\overline{u}$

in U such that

$$|| \varphi(x_o : \bar{u}, w, N) - x_d ||_{R^n} \leq \varepsilon$$

for all disturbances w in W.

In the case of point-controllability the state space X_1 is taken as R^n_p, where $1 \leq p \leq \infty$, and

$$||x||_{R^n_p} = [\sum_{j=1}^{n} |x_j|^p]^{1/p}$$

is the norm on R^n. It then immediately follows that s is an element of R^n defined as

$$s = \Phi(N,0)\, x_o,$$

and that the operators S and T are maps defined as

$$S : l^N_q(R^m_q,) \rightarrow R^n_p$$

$$u \rightarrow Su$$

$$Su = \sum_{j=o}^{N-1} \Phi(N,j+1)B(j)u(j)$$

and

$$T : l_r^N(R_{r'}^k) \to R_p^n$$

$$w \to Tw$$

$$Tw = \sum_{j=o}^{N-1} \Phi(N,j+1)C(j)w(j).$$

As a consequence, for any u in U and any w in W, the state of ($\mathcal{S}$), $\varphi(x_o;u,w,N)$, at time N is given by

$$\varphi(x_o;u,w,N) = s + Su + Tw.$$

It is clear that the finite dimensional character of the problem makes R_p^n a reflexive Banach space and S and T continuous linear operators on the reflexive Banach spaces $l_q^N(R_{q'}^m)$ and $l_r^N(R_{r'}^k)$, respectively. Thus after the next propositions concerning the representation of the conjugate spaces, the main Theorem 4.2 can be directly applied here.

Proposition 5.1:

Given the finite dimensional normed linear space $R_p^n (1 \le p \le \infty)$, its conjugate space can be chosen as $R_{\overline{p}}^n$ where

$$(p)^{-1} + (\overline{p})^{-1} = 1 \qquad (1 \le p \le \infty) ;$$

moreover, given a functional x* in $R_{\overline{p}}^n$ and a point x in R_p^n,

$$x^*(x) = \sum_{i=1}^{n} \alpha_i \beta_i = (x^*, x)$$

where $x^* = [\alpha_1, \ldots, \alpha_n]$,

$$x = [\beta_1, \ldots, \beta_n]$$

and (x^*, x) denotes the scalar product in R^n.

Proof:

The proof of this proposition can be found in [9], (page 247, Theorem 9).

Proposition 5.2:

Given the finite dimensional normed linear space $l_p^N(R_q^n)$ $(1 \leq p \leq \infty,\ 1 \leq q \leq \infty)$, its conjugate space can be chosen as $l_{\bar{p}}^N(R_{\bar{q}}^n)$ where

$$(p)^{-1} + (\bar{p})^{-1} = 1 \quad , \quad (q)^{-1} + (\bar{q})^{-1} = 1 \ ;$$

moreover given a functional x^* in $l_{\bar{p}}^N(R_{\bar{q}}^n)$ and a point x in $l_p^N(R_q^n)$,

$$x^*(x) = \sum_{i=o}^{N-1} \sum_{j=1}^{n} \alpha_j(i)\beta_j(i) = \sum_{i=o}^{N-1} (\alpha(i), \beta(i))$$

where

$$x^* = [\alpha(0), \alpha(1), \ldots, \alpha(N-1)] \ , \ \alpha(i) \in R_{\bar{q}}^n$$

$$x = [\beta(0), \beta(1), \ldots, \beta(N-1)] \ , \ \beta(i) \in R_q^n$$

and $(\alpha(i),\beta(i))$ is the scalar product in R^n.

Proof:

This proposition requires a proof which will be based on the proof of the previous proposition given in [9] (page 247, Theorem 9).

It will now be shown that for $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $p^{-1}+\overline{p}^{-1}=1$, $q^{-1} + \overline{q}^{-1} = 1$, that the mapping χ,

$$\chi : [l_p^N(R_q^n)]^* \to l_{\overline{p}}^N(R_{\overline{q}}^n)$$

$$x^* \to [\alpha(0),\ldots,\alpha(N-1)],\ \alpha(i) \in R_{\overline{q}}^n$$

is an isometric isomorphism of $[l_p^N(R_q^n)]^*$ onto $l_{\overline{p}}^N(R_{\overline{q}}^n)$.

It is easily seen that χ is an isomorphism of $[l_p^N(R_q^n)]^*$ onto $l_{\overline{p}}^N(R_{\overline{q}}^n)$; in order to prove that is is also an isometric map, it is now necessary to show that

$$|x^*| = [\sum_{i=o}^{N-1} |\alpha(i)|_{R_{\overline{q}}^n}^{\overline{p}}]^{1/\overline{p}} .$$

Let us first assume that $1 < p < \infty$, $1 < q < \infty$, then by successive applications of Holder's inequality

$$|x^*(x)| \leq [\sum_{i=o}^{N-1} |\alpha(i)|_{R_{\overline{q}}^n}^{\overline{p}}]^{1/\overline{p}} [\sum_{i=o}^{N-1} |\beta(i)|_{R_q^n}^{p}]^{1/p} .$$

In particular

$$|x^*| = \sup_{|x| \leq 1} |x^*(x)| \leq [\sum_{i=o}^{N-1} |\alpha(i)|^{\overline{p}}_{R^n_q}]^{1/\overline{p}} \tag{5.4}$$

Now let

$$\overline{\beta}_j(i) = \begin{cases} |\alpha(i)|^{\overline{p}-\overline{q}}_{R^n_q} \dfrac{|\alpha_j(i)|^{\overline{q}}}{\alpha_j(i)} & , \alpha_j(i) \neq 0 \\ 0 & , \alpha_j(i) = 0 \end{cases}$$

then for $\overline{x}$ defined by the above $\overline{\beta}_j(i)$,

$$x^*(x) = \sum_{i=o}^{N-1} |x^*(i)|^{\overline{p}}_{R^n_q} = \sum_{i=o}^{N-1} |\alpha(i)|^{\overline{p}}_{R^n_q}$$

Moreover

$$|\overline{x}| = \{\sum_{i=o}^{N-1} [\sum_{j=1}^{n} |\beta_j(i)|^q]^{p/q}\}^{1/p} = [\sum_{i=o}^{N-1} |x^*(i)|^{\overline{p}}_{R^n_q}]^{1/p} .$$

However,

$$x^*(x) \leq |x^*| \quad |x| \quad , \quad \forall x,$$

and then

$$|x^*| \geq \frac{x^*(\overline{x})}{\overline{x}} = [\sum_{i=o}^{N-1} |\alpha(i)|^{\overline{p}}_{R^n_q}]^{1-\frac{1}{p}} = \frac{1}{\overline{p}} \quad ; \tag{5.5}$$

consequently inequalities (5.4) and (5.5) imply the equality and show that the mapping is isometric.

Similarly with some obvious changes in notation, the isometry can be established for $p = 1$ or ∞ and $q = 1$ or ∞.

This completes the proof of Proposition 5.2.

<u>Theorem 5.1</u> <u>Point-Controllability</u>

Given the system ($\mathcal{S}$) with

$$U = \{u \in l_q^N(R_{q'}^m) \; : \; ||u||_{l_q^N(R_{q'}^m)} \leq \rho\}, \; \rho \geq 0,$$

and

$$W = \{w \in l_r^N(R_{r'}^k) \; : \; ||w||_{l_r^N(R_{r'}^k)} \leq \beta\} \; , \; \beta \geq 0,$$

$$S = [\Phi(N,1)B(0) \;\vdots\; \Phi(N,2)B(1) \;\vdots \ldots \vdots\; \Phi(N,N)B(N-1)]$$

$$T = [\Phi(N,1)C(0) \;\vdots\; \Phi(N,2)C(1) \;\vdots \ldots \vdots\; \Phi(N,N)C(N-1)]$$

$$s = \Phi(N,0)\; x_o,$$

the system ($\mathcal{S}$) is point-controllable in the presence of disturbances with respect to $(x_o, U, W, || \;||_{R_p^n}, x_d, \varepsilon)$ if and only if

$$\sup\{\beta[\sum_{i=o}^{N-1} ||[\Phi(N,i+1)C(i)]^T g||_{R_{\bar{r}'}^k}^{\bar{r}}\;]^{1/\bar{r}} \; : ||g||_{R_{\bar{p}}^n} = 1\} \leq \varepsilon \tag{5.6}$$

and

$$[(h,\Phi(N,0)x_o-x_d)-\rho\{\sum_{i=o}^{N-1}||[\Phi(N,i+1)B(i)]^T h||_{R^m_{\bar{q}'}}^{\bar{q}}\}^{1/\bar{q}}$$

$$-\max\{(h,x):x\in\partial T_D^{x_d}\}]\leq 0\ ,\ \forall\ h\in R^n\sim\{0\} \tag{5.7}$$

where $(q,\bar{q})$, $(q',\bar{q}')$ and $(p,\bar{p})$ are conjugate exponents, respectively, and

$$\partial T_D^{x_d}=\{x\in R^n:\max[(g,x)+\beta\{\sum_{i=o}^{N-1}||[\Phi(N,i+1)C(i)]^T g||_{R^k_{\bar{r}'}}^{\bar{r}}\}^{1/\bar{r}}:$$

$$||g||_{R^n_{\bar{p}}}=1]=\varepsilon\}$$

where $(r,\bar{r})$ and $(r',\bar{r}')$ are conjugate exponents, respectively; furthermore, condition (5.7) which must be true for all non-identically zero h in R^n is equivalent to verifying that the supremum of the left-hand side of Equation 5.7 on the unit sphere in $R^n_{\bar{p}}$ is negative or zero. In all cases "T" denotes the transposed matrix.

<u>Corollary 5.1</u>

Under the premises of Theorem 5.1, if β is zero (unperturbed case), then a necessary and sufficient condition for point controllability is

$$\sup\{(h,\Phi(N,0)x_o-x_d)-\rho\{\sum_{i=o}^{N-1}||[\Phi(N,i+1)B(i)]^T h||_{R^m_{\bar{q}}}^{\bar{q}}\}^{1/\bar{q}}:||h||_{R^n_{\bar{p}}}=1\}-\varepsilon\leq 0.$$

Proof:

The above results directly follow from Theorem 4.2 and Propositions 5.1 and 5.2; the only important detail which is worth mentioning is that the adjoint operators S* and T* are combinations of the transposed matrices defining S and T.

This completes the proof of Theorem 5.1.

5.2 Functional Controllability

In this section the interest will be shifted from the state of the system at time N to the set of its states over the time set $I \cup \{N\}$; the latter set now defines the state of the system. However, to avoid confusion with state of the system at any particular time, the set of its states for all time in $I \cup \{N\}$ will be called a trajectory.

Definition 5.6: The trajectory space of ($\mathcal{S}$) is the set of all maps from the set $I \cup \{N\}$ into the n-dimensional Euclidean space R^n; it will be denoted by $l^{N+1}(R^n)$ and its elements called trajectories. The norms on $l^{N+1}(R^n)$ which will be considered are characterized by two integers, p and p',

$$||x|| = \{ \sum_{i=o}^{N} [\sum_{j=1}^{n} |x_j(i)|^{p'}]^{p/p'} \}^{1/p} \qquad (1 \leq p,p' \leq \infty)$$

and the corresponding relfexive Banach space generated will be denoted by $l_p^{N+1}(R_{p'}^n)$.

Definition 5.7: Given x_o, u and w the corresponding trajectory of the system ($\mathcal{S}$) in the trajectory space $1^{N+1}(R^n)$ will be denoted by $\varphi(x_o;u,w)$.

The problem of reproducibility of a desired trajectory x_d is to determine the conditions for the existence of an admissible control $\bar{u}$ in U such that the corresponding trajectory of the system $\varphi(x_o;u,w)$ is ε-close to the desired one. This again can be extended to the perturbed case as follows:

Definition 5.8: Functional Controllability in the Presence of Disturbances

The system ($\mathcal{S}$) is functionally controllable in the presence of disturbances with respect to $(x_o,U,W,\|\ \|_{1^{N+1}(R^n)},x_d,\varepsilon)$ if there exists a control $\bar{u}$ in U such that

$$\|\varphi(x_o;\bar{u},w) - x_d\|_{1^{N+1}(R^n)} \leq \varepsilon$$

for all disturbances w in W.

Before going into the details of the solution of the problem of functional controllability, it must be mentioned that this problem has already been studied by R.W. Brockett and M.D. Mesarović [6]. In that particular case the term used is "functional reproducibility" and some extremely interesting necessary and sufficient conditions were obtained for linear differential equation systems. However the definition of reproducibility adopted by Brockett and Mesarović does not coincide with the definition of controllability adopted here and no direct comparison can be arrived at. In fact a recent letter by

M.K. Sain [14] shows that the concept of reproducibility used by Brockett and Mesarović appears to be related to the idea of sensitivity. The relation between the concepts of reproducibility and controllability has not been undertaken here, but it would be worth while to have a comparative study of reproducibility, sensitivity and controllability for linear time invariant differential equations.

In the case of functional controllability it is easily verified that s is an element of $1^{N+1}(R^n)$ defined as

$$s = \begin{bmatrix} I \\ \Phi(1,0) \\ \Phi(2,0) \\ \cdots\cdots \\ \Phi(N,0) \end{bmatrix} x_o,$$

and that the operators S and T are maps defined as

$$S : 1_q^N(R_q^m,) \to 1_p^{N+1}(R_p^n,)$$

$$u \to Su$$

$$Su = \begin{bmatrix} 0 & \vdots\ 0 & \vdots & \cdots & \vdots\ 0 \\ \Phi(1,1)B(0) & \vdots\ 0 & \vdots & \cdots & \vdots\ 0 \\ \Phi(2,1)B(0) & \vdots\ \Phi(2,2)B(1) & \vdots & \cdots & \vdots\ 0 \\ \cdots\cdots & \vdots\ \cdots\cdots & \vdots & \cdots & \vdots\ \cdots\cdots \\ \Phi(N,1)B(0) & \vdots\ \Phi(N,2)B(1) & \vdots & \cdots & \vdots\ \Phi(N,N)B(N-1) \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \cdots\cdots \\ u(N-1) \end{bmatrix}.$$

and

$$T : l_r^N(R_{r'}^k) \to l_p^{N+1}(R_{p'}^n)$$

$$w \to Tw$$

$$Tw = \begin{bmatrix} 0 & \vdots\; 0 & \vdots \cdots \vdots & 0 \\ \Phi(1,1)C(0) & \vdots\; 0 & \vdots \cdots \vdots & 0 \\ \Phi(2,1)C(0) & \vdots\; \Phi(2,2)C(1) & \vdots \cdots \vdots & 0 \\ \cdots & \vdots\; \cdots & \vdots \cdots \vdots & \cdots \\ \Phi(N,1)C(0) & \vdots\; \Phi(N,2)C(1) & \vdots \cdots \vdots & \Phi(N,N)C(N-1) \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \\ w(2) \\ \cdots \\ w(N-1) \end{bmatrix}$$

As a consequence of the above remarks, for any u in U and any w in W, the trajectory of ($\mathcal{S}$), $\varphi(x_1;u,w)$, in the trajectory space $l^{N+1}(R^n)$ is given by

$$\varphi\,(x_o;u,w) \;=\; s + Su + Tw.$$

It is clear that the finite dimensional character of the problem makes $l_p^{N+1}(R_{p'}^n)$ a reflexive Banach space and S and T continuous linear operators on the reflexive Banach spaces $l_q^N(R_{q'}^m)$ and $l_r^N(R_{r'}^k)$. Thus after the next proposition concerning the representation of the conjugate space of $l_p^{N+1}(R_{p'}^r)$, Theorem 4.2 can again be applied.

<u>Proposition 5.3</u>:

Given the finite dimensional normed linear space $l_p^{N+1}(R_q^n)$ $(1 \leq p \leq \infty,\ 1 \leq q \leq \infty)$, its conjugate space can be chosen as $l_{\bar{p}}^{N+1}(R_{\bar{q}}^n)$ where

$$\frac{1}{p} + \frac{1}{\bar{p}} = 1 \quad , \quad \frac{1}{q} + \frac{1}{\bar{q}} = 1 \quad ;$$

moreover, given a functional x^* in $l_{\bar{p}}^{N+1}(R_{\bar{q}}^{n})$ and a point x in $l_{p}^{N+1}(R_{q}^{n})$, then

$$x^*(x) = \sum_{i=o}^{N+1} \sum_{j=1}^{n} \alpha_j(i)\beta_j(i) = \sum_{i=o}^{N+1} (\alpha(i),\beta(i))$$

where $x^* = [\alpha(0),\alpha(1),\ldots,\alpha(N+1)]$, $\alpha(i) \in R_{\bar{q}}^{n}$

$x = [\beta(0),\beta(1),\ldots,\beta(N+1)]$, $\beta(i) \in R_{q}^{n}$

and $(\alpha(i),\beta(i))$ is the scalar product in R^n.

Proof:

The proof is identical to the proof of Proposition 5.2.

Theorem 5.2 Functional Controllability

Given the system ($\mathcal{S}$) with

$$U = \{u \in l_q^N(R_{q'}^m) : ||u||_{l_q^N(R_{q'}^m)} \leq \rho\} \; , \; \rho \geq 0$$

and

$$W = \{w \in l_r^N(R_{r'}^k) : ||w||_{l_r^N(R_{r'}^k)} \leq \beta\} \; , \; \beta \geq 0$$

$$S = \begin{bmatrix} 0 & \vdots & 0 & \vdots & \cdots & \vdots & 0 \\ \Phi(1,1)B(0) & \vdots & 0 & \vdots & \cdots & \vdots & 0 \\ \Phi(2,1)B(0) & \vdots & \Phi(2,2)B(1) & \vdots & \cdots & \vdots & 0 \\ \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ (N,1)B(0) & \vdots & \Phi(N,2)B(1) & \vdots & \cdots & \vdots & \Phi(N,N)B(N-1) \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & \vdots & 0 & \vdots & \cdots & \vdots & 0 \\ \Phi(1,1)C(0) & \vdots & 0 & \vdots & \cdots & \vdots & 0 \\ \Phi(2,1)C(0) & \vdots & \Phi(2,2)C(1) & \vdots & \cdots & \vdots & 0 \\ \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ \Phi(N,1)C(0) & \vdots & \Phi(N,2)C(1) & \vdots & \cdots & \vdots & \Phi(N,N)C(N-1) \end{bmatrix}$$

$$s = \begin{bmatrix} I \\ \Phi(1,0) \\ \Phi(2,0) \\ \cdots \\ \Phi(N,0) \end{bmatrix} \quad x_o,$$

The system ($\mathcal{S}$) is functionally controllable in the presence of disturbances with respect to $(x_o, U, W, \|\ \|_{l_p^{N+1}(R^n_{p'})}, x_o, \varepsilon)$ if and only if

$$\sup\{\beta[\sum_{\ell=1}^{N} \|\sum_{i=\ell}^{N} [\Phi(i,\ell)C(\ell-1)]^T g(i)\|^{\bar{r}}_{R^k_{\bar{r}'}}]^{1/\bar{r}} : \|g\|_{l_{\bar{p}}^{N+1}(R^n_{\bar{p}'})} = 1\} \leq \varepsilon \tag{5.8}$$

and

$$[\sum_{i=o}^{N} (h(i), \Phi(i,0)x_o - x_d(i)) - \rho\{\sum_{\ell=1}^{N} \|\sum_{i=\ell}^{N} [\Phi(i,\ell)B(\ell-1)]^T h(i)\|^{\bar{q}}_{R^m_{\bar{q}'}}\}^{1/\bar{q}}$$

$$-\max\{\sum_{i=o}^{N-1} (h(i), x(i)) : x \in \partial T_D^{x_d}\}], \forall\, h \in l^{N+1}(R^n) \sim \{0\} \tag{5.9}$$

where $(q,\bar{q}),(q',\bar{q}'),(p,\bar{p}),(p',\bar{p}'),(r,\bar{r})$ and $(r',\bar{r}')$ are conjugate exponents, respectively, and the elements of $\mathcal{F}_D^{x_d}$ are elements of $l^{N+1}(R^n)$ satisfying

$$\sup\{\sum_{i=o}^{N}(g(i),x(i))+\beta[\sum_{\ell=1}^{N}||\sum_{i=\ell}^{N}[\Phi(i,\ell)C(\ell-1)]^T g(i)||^{\bar{r}}_{R^{k}_{\bar{r}'}}]^{1/\bar{r}}: ||g||_{l^{N+1}_{\bar{p}}(R^{n}_{\bar{p}'})}=1\}=\varepsilon \; ; \qquad (5.10)$$

furthermore, condition (5.9) which is on all non-identically zero h in $l^{N+1}(R^n)$ is equivalent to verifying that the "sup" on the unit sphere in $l^{N+1}(R^n)$ is negative or zero. In all cases "T" denotes the transposed matrix.

Corollary 5.2

Under the premises of Theorem 5.2, if β is zero (unperturbed case), then a necessary and sufficient condition for functional controllability is

$$\sup\{\sum_{i=o}^{N}(h(i),\Phi(i,0)\,x_o-x_d(i))\} \qquad (5.11)$$

$$-\rho\{\sum_{\ell=1}^{N}||\sum_{i=\ell}^{N}[\Phi(i,\ell)B(\ell-1)^T h(i)||^{\bar{q}}_{R^{m}_{\bar{q}'}}\}^{1/\bar{q}}: ||h||_{l^{N+1}_{\bar{p}}(R^{n}_{\bar{p}'})}=1\}-\varepsilon\leq 0.$$

Proof of Theorem 5.2:

The above results again directly follow from Theorem 4.2 and Propositions 5.1, 5.2, and 5.3; however, a discussion of the representation of the adjoint operators S* and T* is deemed necessary to elucidate the somewhat complicated form of the results.

The adjoint operators S* and T* of S and T are their respective transposed matrices:

$$S^* = \begin{bmatrix} 0 & \vdots & [\Phi(1,1)B(0)]^T & \vdots & [\Phi(2,1)B(0)]^T & \vdots & \cdots & \vdots & [\Phi(N,1)B(0)]^T \\ 0 & \vdots & 0 & \vdots & [\Phi(2,2)B(1)]^T & \vdots & \cdots & \vdots & [\Phi(N,2)B(1)]^T \\ \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ 0 & \vdots & 0 & \vdots & 0 & \vdots & \cdots & \vdots & [\Phi(N,N)B(N-1)]^T \end{bmatrix}$$

$$T^* = \begin{bmatrix} 0 & \vdots & [\Phi(1,1)C(0)]^T & \vdots & [\Phi(2,1)C(0)]^T & \vdots & \cdots & \vdots & [\Phi(N,1)C(0)]^T \\ 0 & \vdots & 0 & \vdots & [\Phi(2,2)C(1)]^T & \vdots & \cdots & \vdots & [\Phi(N,2)C(1)]^T \\ \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ 0 & \vdots & 0 & \vdots & 0 & \vdots & \cdots & \vdots & [\Phi(N,N)C(N-1)]^T \end{bmatrix}$$

Consequently

$$(h,s) = \sum_{i=o}^{N} ((h(i), \Phi(i,0)\, x_o)$$

$$S^*h = \begin{bmatrix} 0 + [\Phi(1,1)B(0)]^T h(1) + [\Phi(2,1)B(0)]^T h(2) + [\Phi(N,1)B(0)]^T h(N) \\ 0 + 0 + [\Phi(2,2)B(1)]^T h(2) + [\Phi(N,2)B(1)]^T h(N) \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ 0 + 0 + 0 + [\Phi(N,N)B(N-1)]^T h(N) \end{bmatrix}$$

and

$$||S^*h||_{l^N_{\bar q}(R^m_{\bar q'})} = \{ \sum_{\ell=1}^{N} || \sum_{i=\ell}^{N} [\Phi(i,\ell)B(\ell-1)]^T h(i)||^{\bar q}_{R^m_{\bar q'}} \}^{1/\bar q} \quad ;$$

similarly

$$||T^*h||_{l^N_{\bar q}(R^k_{\bar r'})} = \{ \sum_{\ell=1}^{N} || \sum_{i=\ell}^{N} [\Phi(i,\ell)C(\ell-1)]^T g(i)||^{\bar r}_{R^k_{\bar r'}} \}^{1/\bar r} \quad .$$

The substitution of the above results in the conditions (4.6), (4.7), (4.8) and (4.9) of Theorem 4.2 yields the conditions (5.9), (5.10), and (5.11) of this Proposition.

5.3 Particular Cases

In this section, Theorems 5.1 and 5.2 will be further specialized to targets in the R^n_∞ and $l^{N+1}_\infty(R^n_\infty)$ spaces; that is

$$||x-x_d||_{R^n_\infty} = \max\{ |x_j - x_{d_j}| \ : \ j = 1,\dots,n\}$$

and

$$||x-x_d||_{l^{N+1}_\infty(R^n_\infty)} = \max\{\max[|x_j(i)-x_{d_j}(i)| : j=1,..,n] : i=0,..,N\}$$

Theorem 5.3

Given the system ($\mathcal{S}$) with

$$U = \{u \in l^N_q(R^m_{q'}) : ||u||_{l^N_q(R^m_{q'})} \le \rho \ \} \ , \ \rho \ge 0$$

and

$$W=\{w\in l_\infty^N(R_2^k):||w||_{l_\infty^N(R_2^k)} \leq \beta\} \ , \ \beta \geq 0$$

S, T and s as defined in Theorem 5.1, then there exists a control $\bar{u}$ in U such that

$$||\varphi(x_o;\bar{u},w,N) - x_d||_{R_\infty^n} \leq \varepsilon \ , \ \forall \ w\in W,$$

if and only if

$$(h,\Phi(N,0)x_o-x_d)-\rho\{\sum_{i=o}^{N-1}||[\Phi(N,i+1)B(i)]^T h||_{R_{\bar{q}'}^n}^{\bar{q}}\}^{1/\bar{q}} \tag{5.12}$$

$$-\sum_{j=1}^{n}[\varepsilon-\beta\sum_{i=o}^{N-1}||[\Phi(N,i+1)C(i)]_j||_{R_2^k}]|h_j| \leq 0, \ \forall \ h\in R_1^n\sim\{0\};$$

where $(q,\bar{q})$ and $(q',\bar{q}')$ are conjugate exponents, respectively; furthermore, condition (5.12) is equivalent to

$$\sup\{(h,\Phi(N,0)x_o-x_d)-\rho[\sum_{i=o}^{N-1}||[\Phi(N,i+1)B(i)]^T h||_{R_{\bar{q}'}^n}^{\bar{q}}]^{1/\bar{q}} \tag{5.13}$$

$$+\beta\sum_{j=1}^{n}\sum_{i=o}^{N-1}||[\Phi(N,i+1)C(i)]_j||_{R_2^k}|h_j|: \sum_{j=1}^{n}|h_j|=1\}-\varepsilon \leq 0.$$

Corollary 5.3

Under the premises of Theorem 5.3, if β is zero (unperturbed case), then a necessary and sufficient condition for controllability with respect to $(x_o, U, W, \| \|_{R^n_\infty}, x_d, \varepsilon)$ is

$$\sup\{(h,\Phi(N,0)x_o - x_d) - \rho\Big[\sum_{i=o}^{N-1} \|[\Phi(N,i+1)B(i)]^T h\|^{\bar{q}}_{R^n_{\bar{q}}},\Big]^{1/\bar{q}} : \sum_{j=1}^{n} |h_j| = 1\} - \varepsilon \leq 0. \tag{5.14}$$

Proof:

This theorem is a direct application of Theorem 5.1 with the additional feature that the elements of the set $\partial T_D^{x_d}$ can be explicitly calculated. In order to show this, note that the elements of the set $\partial T_D^{x_d}$ can be alternatively defined as all x in R^n such that

$$\max_W \|x+Tw\|_{R^n_\infty} = \max_W \max_{j=1,..,n} |x_j + (Tw)_j| = \varepsilon; \tag{5.15}$$

but

$$Tw = \sum_{i=o}^{N-1} \Phi(N,i+1)C(i)w(i)$$

and

$$(Tw)_j = \sum_{i=o}^{N-1} (T_{j\,i}, w(i))$$

where T_{ji} is the j-th row of the matrix $\Phi(N,i+1)C(i)$ also denoted as $[\Phi(N,i+1)C(i)]_j$. Equation 5.13 now becomes

$$\max_{W} \max_{j=1,..,n} |x_j + \sum_{i=o}^{N-1} (T_{ji}, w(i))|$$

and by changing the order of the maximization

$$\max_{j=1,...,n} \max_{W} |x_j + \sum_{i=o}^{N-1} (T_{ji}, w(i))| ;$$

then clearly there exists an upper bound on each component

$$|x_j + \sum_{i=o}^{N-1} (T_{ji}, w(i))| \leq |x_j| + \beta \sum_{i=o}^{N-1} ||T_{ji}||_{R_2^k} ; \qquad (5.16)$$

in addition the existence of a $\overline{w}$ in W,

$$\overline{w}(i) = \begin{cases} \beta \dfrac{x_j}{|x_j|} \dfrac{T_{ji}}{||T_{ji}||_{R_2^k}} , & \text{if } x_j \neq 0 \text{ and } T_{ji} \neq 0 \\ 0 , & \text{if } x_j \text{ or } T_{ji} \text{ is zero} \end{cases} ,i=0,..,N-1 \qquad (5.17)$$

such that

$$|x_j + \sum_{i=o}^{N-1} (T_{ji}, w(i))| = |x_j| + \beta \sum_{i=o}^{N-1} ||T_{ji}||_{R_2^k}$$

makes the left-hand side of Equation 5.16 equal to the upper bound for each component and the elements of $\partial T_D^{x_d}$ can now be defined as all x in R^n such that

$$\max_{j=1,\ldots,n} [|x_j| + \beta \sum_{i=o}^{N-1} ||T_{ji}||_{R_2^k}] = \varepsilon \quad . \tag{5.18}$$

To see the major simplification Equation 5.18 will introduce in our results, notice that if h is in R^n, then the x which maximizes (h,x),

$$(h,x) = \sum_{j=1}^{n} h_j x_j \ ,$$

and simultaneously satisfies Equation 5.18 is defined by

$$x_j = \begin{cases} \dfrac{h_j}{|h_j|} [\ \varepsilon - \beta \sum_{i=o}^{N-1} ||T_{ji}||_{R_2^k}] & , \text{ if } h_j \neq 0; \\ 0 & , \text{ if } h_j = 0 \end{cases}$$

for all j = 1,...,n; that is,

$$\max\{(h,x) : x\ \partial T_D^{x_d}\} = \sum_{j=1}^{n} |h_j| [\varepsilon - \beta \sum_{i=o}^{N-1} ||T_{ji}||_{R_2^k}] , \forall h \ \varepsilon \ R^n \tag{5.19}$$

or

$$\max\{(h,x):x\in \partial T_D^{x_d}\} = \varepsilon-\beta \sum_{j=1}^{n} \sum_{i=o}^{N-1} ||T_{ji}||_{R_2^k}|h_j| \qquad (5.20)$$

if h is taken as an element of R_1^n, the conjugate of R_∞^n, with unit norm:

$$\sum_{j=1}^{n} |h_j| = 1.$$

One interesting feature of the alternate form of the controllability condition given by Equation 5.13 is that ε appears outside of the maximization bracket; hence a lower bound is obtained for ε:

$$\varepsilon \geq \max_{|h|_{R_1^n}=1} [(h,\Phi(N,0)x_o-x_d)-\rho\{\sum_{i=o}^{N-1} ||[\Phi(N,i+1)B(i)]^T h||^q_{R_{q'}^n}\}^{1/q}$$

$$+\beta \sum_{j=1}^{n} \sum_{i=o}^{N-1} ||[\Phi(N,i+1)C(i)]_j||_{R_2^k}|h_j|]. \qquad (5.21)$$

This completes the comments on Theorem 5.3.

Due to the success of the method in the previous case, we shall now turn to the $l_\infty^{N+1}(R_\infty^n)$ space and try to obtain similar results.

Theorem 5.4

Given the system ($\mathcal{S}$) with

$$U=\{u\in l_q^N(R_{q'}^n):||u||_{l_q^N(R_{q'}^m)}\leq \rho\},\ \rho\geq 0$$

and

$$W=\{w\ \ l_\infty^N(R_2^k):||w||_{l_\infty^N(R_2^k)}\leq \beta\},\ \beta\geq 0\ ,$$

S, T, s as defined in Theorem 5.2, then there exists a control function $\bar{u}$ in U such that

$$||\varphi(x_o;\bar{u},w)-x_d||_{l_\infty^{N+1}(R_\infty^n)}\leq \varepsilon\ ,\ \forall\ w\in W,$$

if and only if

$$[\sum_{i=o}^{N}(h(i),\Phi(i,0)x_o-x_d(i))-\rho\{\sum_{\ell=1}^{N}|\sum_{i=\ell}^{N}[\Phi(i,\ell)B(\ell-1)]^T h(i)||_{R_{\bar{q}'}^m}^{\bar{q}}\}^{1/\bar{q}}$$

$$-\sum_{i=o}^{N}\sum_{j=1}^{n}|h(i)_j|[\varepsilon-\beta\sum_{k=1}^{i}||T_{ijk}||_{R_2^k}]],\ \forall h\varepsilon l^{N-1}(R^n)\sim\{0\}$$

(5.22)

where $(q,\bar{q})$ and $(q',\bar{q}')$ are conjugate exponents, respectively, and T_{ijk} is the j-th row of the matrix $\Phi(i,k)C(k-1)$ also denoted as $[\Phi(i,k)C(k-1)]_j$; furthermore, condition (5.22) is equivalent to

$$\sup\{\sum_{i=o}^{N}(h(i),\Phi(i,0)x_o-x_d(i))-\rho\{\sum_{\ell=1}^{N}\|\sum_{i=\ell}^{N}[\Phi(i,\ell)B(\ell-1)]^T h(i)\|_{R^m_{\bar{q}'}}^{\bar{q}}\}^{\frac{1}{\bar{q}}}$$

$$+\beta\sum_{i=o}^{N}\sum_{j=1}^{n}\sum_{k=1}^{i}\|T_{ijk}\|_{R_2^k}|h(i)_j| \;:\; \|h\|_{1_1^{N+1}(R_1^n)}=1\}-\varepsilon\leq 0 \tag{5.23}$$

Corollary 5.4

Under the premises of Theorem 5.4, if β is zero (unperturbed case), then a necessary and sufficient condition for controllability with respect to $(x_o,U,W,\|\ \ \|_{1_p^{N+1}(R^n_{p'})},x_d,\varepsilon)$:

$$\sup\{\sum_{i=o}^{N}(h(i),\Phi(i,0)x_o-x_d(i)) \tag{5.24}$$

$$-\rho[\sum_{\ell=1}^{N}\|\sum_{i=\ell}^{N}[\Phi(i,\ell)B(\ell-1)]^T h(i)\|_{R^m_{\bar{q}'}}^{\bar{q}}]^{1/\bar{q}}:\|h\|_{1_1^{N+1}(R_1^n)}\}-\varepsilon\ \leq\ 0$$

Proof of Theorem 5.4:

This theorem is a direct application of Theorem 5.2 with the additional feature that the elements of the set $\partial T_D^{x_d}$ can be explicitly calculated. In order to show this, note that the elements of $\partial T_D^{x_d}$ can be alternately defined as all x in R^n such that

$$\max\{||x+Tw||_{1_\infty^{N+1}(R_\infty^n)} : w \varepsilon W\} = \max\{\max\{\max[|x_j(i)+(Tw)_j(i)| : j=1,..,n] : i=o,..,N\} : w\varepsilon W\}; \qquad (5.25)$$

note that Tw is defined in Theorem 5.2:

$$[Tw](i) = \begin{cases} 0 & , \quad i = 0 \\ \sum_{k=1}^{i} \Phi(i,k)C(k-1)w(k-1), & i \neq 0 \end{cases}$$

and

$$[Tw]_j(i) = \begin{cases} 0 & , \quad i = 0 \\ \sum_{k=1}^{i} ([\Phi(i,k)C(k-1)]_j, w(k-1)) & , \quad i \neq 0 \end{cases}$$

for all j = 1,...,n.

So let

$$T_{ijk} = [\Phi(i,k)C(k-1)]_j$$

where T_{ijk} is an element of R_2^k.

Now the order of the maximization in Equation 5.25 can be changed:

$$\max_{w\varepsilon W} \max_{i=0,..,N} \max_{j=1,..,n} \left|x_j(i)+ \sum_{k=1}^{i} (T_{ijk},w(k-1))\right|$$

$$= \max_{i=0,..,N} \max_{j=1,..,n} \max_{w\varepsilon W} \left|x_j(i)+ \sum_{k=1}^{i} (T_{ijk},w(k-1))\right| \quad .$$

But each component has an upper bound

$$\left|x_j(i)+ \sum_{k=1}^{i} (T_{ijk},w(k-1))\right| \leq |x_j(i)|+ \beta \sum_{k=1}^{i} ||T_{ijk}||_{R_2^k} \quad ;$$

moreover, there exists $\overline{w}$ in W,

$$\overline{w}(k) = \begin{cases} \beta \dfrac{x_j}{|x_j|} \dfrac{T_{ijk}}{|T_{ijk}|_{R_2^k}} & \text{, if } x_j \neq 0 \text{ and } T_{ijk} \neq 0, \\ 0 & \text{, if } x_j = 0 \text{ or } T_{ijk} = 0, \end{cases}$$

for all $k = 0,\ldots,i-1$ and $i = 1,\ldots,N$, such that

$$|x_j(i) + \sum_{k=1}^{i} (T_{ijk}, \overline{w}(k-1)| = |x_j(i)| + \beta \sum_{k=1}^{i} ||T_{ijk}||_{R_2^k}$$

as can be easily verified by substitution.

Equation 5.25 now becomes

$$\max\{\max[|x_j(i)| + \beta \sum_{k=1}^{i} ||T_{ijk}||_{R_2^k} : j=1,..,n] : i=0,..,N\} \quad (5.26)$$

so that the elements of $\partial T_D^{x_d}$ are defined as all x in $1^{N+1}(R^n)$ such that Equation (5.26) is equal to ε.

To see the major simplification Equation (5.26) will introduce in the results, notice that if h is an element of $1^{N+1}(R^n)$, then the x which maximizes

$$(h,x) = \sum_{i=o}^{N} \sum_{j=1}^{n} h_j(i)\, x_j(i) ,$$

and simultaneously satisfies Equation (5.26) is defined by

$$x(i)_j = \begin{cases} \dfrac{h(i)_j}{|h(i)_j|} [\varepsilon - \beta \sum_{k=1}^{i} ||T_{ijk}||_{R_2^k}] & , \text{ if } h(i)_j \neq 0 \\ 0 & , \text{ if } h(i)_j = 0 \end{cases}$$

for all j = 1,..,n and i = 0,..,N; that is

$$\max_{x \in \partial T_D^{x_d}} (h,x) = \sum_{i=o}^{N} \sum_{j=1}^{n} |h(i)_j| [\varepsilon - \beta \sum_{k=1}^{i} |T_{ijk}|_{R_2^k}], \forall\, h \varepsilon 1^{N-1}(R^n), \quad (5.27)$$

or

$$\max_{x \in \partial T_D^{x_d}} (h,x) = \varepsilon - \beta \sum_{i=o}^{N} \sum_{j=1}^{n} \sum_{k=1}^{i} ||T_{ijk}||_{R_2^k} \quad (5.28)$$

if h is an element of $1_1^{N+1}(R_1^n)$, the conjugate of $1_\infty^{N+1}(1_\infty^n)$, with unit norm:

$$\sum_{i=o}^{N} \sum_{j=1}^{n} |h(i)_j| = 1.$$

One interesting feature of the alternate form of the controllability condition given by Equation (5.23), is that ε can be taken out of the maximization bracket:

$$\sup\{ \sum_{i=o}^{N} (h(i),\Phi(i,0)x_o - x_d(i)) - \rho[\sum_{\ell=1}^{N} || \sum_{i=\ell}^{N} [\Phi(i,\ell)B(\ell-1)]^T h(i)||^{\bar{q}}_{R^m_{\bar{q}}},]^{1/\bar{q}}$$

$$+ \beta \sum_{i=o}^{N} \sum_{j=1}^{n} \sum_{k=1}^{i} |h(i)_j| \quad ||T_{ijk}||_{R^k_2} : ||h||_{1^{N+1}_1(R^n_1)} = 1\} \leq \varepsilon \qquad (5.29)$$

The right-hand side of Equation (5.29) is now a lower bound on the "size", ε, of the target tube.

This completes the comments on Theorem 5.4.

CHAPTER VI

THE CHARACTERIZATION PROBLEM

The aspects of the characterization problem treated here are:

1) impossibility of strict controllability for the perturbed system ($\mathcal{L}$),

2) existence of a maximum norm on the disturbance set to insure controllability,

3) existence of a minimum norm on the control set to insure controllability,

4) existence of a minimal target to insure controllability,

5) analytical implicit equations to find the above parameters.

The aspects 1) and 2) of the characterization problem will be solved for the abstract system ($\mathcal{L}$) with U and W as defined in Theorem 4.2; as for 3) and 4), it was decided, for simplicity, to illustrate only the basic ideas behind the solutions of 3) and 4) by considering the simpler case of Theorem 5.3.

Proposition 6.1:

Given a non-trivial form of the abstract linear system ($\mathcal{L}$) as defined in Theorem 4.2, strict controllability ($\varepsilon=0$) cannot be achieved in the perturbed case ($\beta>0$).

Proof:

Let β be different from zero and ε be zero, then the reduced target set T_D is non-void if and only if

$$\sup\{\beta|T^*g|_{3^*} : |h|_{1^*}=1\} \leq 0;$$

that is, since β is not zero, if

$$|T^*g|_{3*} = 0 \quad , \forall\, g \in X_1^*$$

where $|g|_{1*} = 1$, or, more precisely, if the operator T is identically zero, which is a trivial form of the system ($\mathcal{L}$). Thus, if T is not identically zero

$$\sup\{\beta|T^*g|_{3*} : |g|_{1*} = 1\} > 0$$

and the reduced target set is empty making strict controllability impossible.

This completes the proof of Proposition 6.1.

Proposition 6.2:

Given a non-trivial form of the abstract linear system ($\mathcal{L}$) as described in Theorem 4.2, the bounds ε on the target set and β on the disturbance set, and the fact that for some non-negative ρ^* the system is controllable, then there exists a unique minimum required bound $\tilde{\rho}$ such that controllability is achieved only for ρ greater or equal to $\tilde{\rho}$.

Furthermore, the minimal $\tilde{\rho}$ is given by

(i) $\tilde{\rho} = 0$ if

$$\sup\{h(s-x_d)-\max[h(x):x\in \partial T_D^{x_d}]:|h|_{1*} = 1\} \leq 0 \qquad (6.1)$$

(ii) $\tilde{\rho} = \bar{\rho}$, the unique solution of Equation 6.2

$$\sup\{h(s-x_d)-\bar{\rho}|S^*h|_{2*}-\max[h(x):x\in \partial T_D^{x_d}]:|h|_{1*}=1\}=0 \quad (6.2)$$

if Equation 6.1 is strictly positive.

Proof:

Let the map f

$$f : R^+ \cup \{0\} \times X_1^* \rightarrow R$$

$$(\rho,h) \rightarrow f(\rho,h)$$

be defined as

$$f(\rho,h) = h(s-x_d)-\rho|S^*h|_{2*}-\max[h(x):x\in \partial T_D^{x_d}]$$

and let the map g

$$g : R^+ \cup \{0\} \rightarrow R$$

$$\rho \rightarrow g(\rho)$$

be defined as

$$g(\rho) = \sup\{f(\rho,h) \; : \; |h|_{1*} = 1\} \; .$$

It will be proven that g is a monotonically decreasing, convex and continuous function of $R^+ \cup \{0\}$. Hence if $g(0) \leq 0$, then the minimal ρ, $\tilde{\rho}$, is zero; and if $g(0) > 0$, then the properties of the function g and the fact that $g(\rho^*)$ is non-positive for some non-negative ρ^* guarantee the existence of a unique $\bar{\rho} > 0$ such that $g(\bar{\rho}) = 0$, proving the Proposition.

1) g is monotonically decreasing on $R^+ \cup \{0\}$

Given

$$\rho_2 \geq \rho_1 \geq 0 \; ,$$

$$f(h,\rho_2) \leq f(h,\rho_1)$$

for all h in X_1^* and hence

$$g(\rho_2) \leq g(\rho_1) \; .$$

2) g is convex on $R^+ \cup \{0\}$

Given

$$\rho_2 \neq \rho_1 \quad \text{and} \quad \lambda \quad \text{in } [0,1],$$

$$f(h,\lambda\rho_1+(1-\lambda)\rho_2) = \lambda f(h,\rho_1)+(1-\lambda)f(h,\rho_2), \forall\, h \in X_1^*$$

and

3) <u>g is continuous on $R^+ \cup \{0\}$</u>

Given

$$\rho_2 \geq \rho_1 \geq 0 \ ,$$

$$|g(\rho_2)-g(\rho_1)|=g(\rho_1)-g(\rho_2) \leq \sup\{(\rho_2-\rho_1)|S^*h|_{2*}:|h|_{1*} = 1\}$$

$$\leq |\rho_2-\rho_1|B$$

where $B = \sup\{|S^*h|_{2*}:|h|_{1*}=1\}$ is bounded since S^* is a continuous linear operator. The above inequality shows that g satisfies the ε-δ definition of continuity.

This completes the proof of Proposition 6.2.

The next two propositions will be proven only for the system ($\mathcal{S}$) as described in Theorem 5.3 where the necessary and sufficient conditions for controllability are

$$\sup\{\beta m_j : j = 1,\ldots,n \} \leq \varepsilon \tag{6.3}$$

and

$$\sup\{(h,c)-\rho||S^*h||_Q+\beta \sum_{j=1}^{n} m_j|h_j|:|h|_{R_1^n}=1\}- \varepsilon \leq 0 \tag{6.4}$$

where

$$c = \Phi(N,0)x_o - x_d$$

$$||S^*h||_Q = \{ \sum_{i=o}^{N-1} ||[\Phi(N,i+1)B(i)]^T h||^{\bar{q}}_{R^n_{\bar{q}'}} \}^{1/\bar{q}}$$

$$m_j = \sum_{i=o}^{N-1} ||[\Phi(N,i+1)C(i)]_j||_{R^k_2} \geq 0 .$$

Proposition 6.3:

Given the system ($\mathcal{S}$) defined in Theorem 5.3 and the bounds ρ and β on the control set U and the disturbance set W, respectively, then there exists a unique minimal bound $\tilde{\varepsilon}$ on the target set such that only for ε greater or equal to $\tilde{\varepsilon}$, is controllability achieved.

Furthermore $\tilde{\varepsilon}$ is given by

$$\tilde{\varepsilon} = \min(\hat{\varepsilon},\bar{\varepsilon}) \tag{6.5}$$

where

$$\hat{\varepsilon} = \beta \max\{m_j : j=1,\dots,n\} \tag{6.6}$$

and

$$\bar{\varepsilon} = \max(0,\varepsilon') \tag{6.7}$$

where

$$\varepsilon' = \sup\{(h,c)-\rho||S^*h||_Q + \beta \sum_{j=1}^{n} m_j|h_j| : \sum_{j=1}^{n} |h_j| = 1\}. \qquad (6.8)$$

Proof:

First of all note that the requirement on the non-emptiness of the reduced target set gives a lower bound ,ε', on the "size" of the target. Secondly, the necessary and sufficient condition also give a lower bound ,ε', which could be negative; if it is negative $\bar{\varepsilon}$ is taken as being zero; if it is positive $\bar{\varepsilon}$ is set equal to ε'.

This completes the proof of Proposition 6.3.

Proposition 6.4:

Given a non-trivial form of system ($\mathcal{S}$) as described in Theorem 5.3, the bounds ε on the target set, ρ on the control set U, and the fact that ($\mathcal{S}$) is controllable in the unperturbed case (β=0), then there exists a maximum allowable bound $\tilde{\beta}$ on the disturbance set W for which and below which the system is controllable, and above which controllability cannot be achieved.

Furthermore, this maximum bound $\tilde{\beta}$ is unique and given by

$$\tilde{\beta} = \min\ (\hat{\beta},\bar{\beta})$$

where

$$\hat{\beta} = \frac{\varepsilon}{\max\{m_j : j=1,\ldots,n\}}$$

and $\overline{\beta}$ the unique solution of the equation

$$\sup\{(h,c)-\rho||S^*h||_Q+\beta\sum_{j=1}^{n}m_j|h_j|\}-\varepsilon=0. \qquad (6.9)$$

Proof:

a) The non-emptiness of the reduced target set gives the first upper bound $\hat{\beta}$.

b) To show that β' is also a lower bound, it will be proven that for a non-trivial form of the system ($\mathcal{S}$) (that is, $\max\{m_j:j=1,\ldots,n\}>0$), the map g

$$g \ : \ R^+\cup\{0\}\to R$$

$$\beta \to g(\beta)$$

defined by

$$g(\beta)=\sup\{f(\beta,h) \ : \ |h|_{R_1^n}=1\} \ ,$$

where f is a map

$$f \ : \ R^+\cup\{0\}\times X_1^*\to R$$

$$(\beta,h)\to f(\beta,h)$$

defined by

$$f(\beta,h) = (h,c)-\rho||S^*h||_Q + \beta \sum_{j=1}^{n} m_j|h_j| - \varepsilon ,$$

is a continuous, monotonically increasing convex functional on $R^+ \cup \{0\}$.

That g is monotonically increasing and convex on $R^+ \cup \{0\}$ is obvious; to show it is continuous simply note that given $\beta_1 \leq \beta_2$

$$|g(\beta_2)-g(\beta_1)|=g(\beta_2)-g(\beta_1)\leq \sup\{(\beta_2-\beta_1)\sum_{j=1}^{n} m_j|h_j| : \sum_{j=1}^{n} |h_j|=1\}$$

and

$$|g(\beta_2)-g(\beta_1)| \leq \max\{m_j:j=1,\ldots,n\}|\beta_2-\beta_1|$$

where $\max\{m_j:j=1,\ldots,n\}$ is bounded above. As a consequence of the above inequality the ε-δ definition of continuity is satisfied.

Now g(0) is less than or equal to zero by assumption; then, since $\max\{m_j:j=1,\ldots,n\}$ is strictly greater than zero, there exists a large enough $\overline{\beta}$ such that $g(\overline{\beta}) = 0$. Moreover, the convexity and the monotonically increasing character of the function g guarantee that $\overline{\beta}$ is unique.

This completes the proof of Proposition 6.4.

CHAPTER VII

THE DECOMPOSITION SCHEME AND THE NUMERICAL PROBLEM

The relatively simple structure of the analytical expressions derived in the previous chapters is a sufficient motivation to look for a general iterative scheme involving successive approximations. In the most general case treated here (Theorem 4.2), the necessary and sufficient conditions for controllability are:

$$\sup\{\beta|T^*g|_{3*}:|g|_{1*} = 1\} \leq \varepsilon \tag{7.1}$$

and

$$h(s-x_d) - \rho|S^*h|_{2*} - \max\{h(x):x\varepsilon\partial T_D^{x_d}\} \leq 0, \forall h \in X_1^* \sim \{0\} \tag{7.2}$$

where the elements of $\partial T_D^{x_d}$ are points of X_1 satisfying

$$\sup\{g(x) + \beta|T^*g|_{3*} :|g|_{1*} = 1\} = \varepsilon. \tag{7.3}$$

It is clear that the controllability condition represented by Equation 7.2 is only a function of the non-identically zero elements of X_1^* and does not depend in any way on the natural norm $|\ |_{1*}$

induced on X_1^*. This makes it possible to define another norm on X_1^* different from $| \ |_{1*}$ and judiciously constructed in order to simplify the problems to be solved; in fact what is true for X_1^* is also true for X_1. The ability to define new norms on the space X_1 and its conjugate X_1^* is the key to the decomposition of the equations arising from the necessary and sufficient conditions for controllability. The decomposition will generate several sub-problems, almost identical in structure, involving the maximization of a continuous function on a unit sphere in X_1 or X_1^*.

Theorem 7.1

a) Given the abstract system ($\mathcal{L}$) and

$$\sup[\beta|T^*g|_{3^*} : |g|_{1*} = 1] < \varepsilon \tag{7.4}$$

the controllability condition expressed by Equation 7.2 is satisfied if and only if

$$\sup[\ h(s-x_d)-\rho|S^*h|_{2^*} : ||h||^* = 1] - \varepsilon \leq 0 \tag{7.5}$$

where

$$||h||^* = \frac{1}{\varepsilon} \max\{ h(x) \ : \ ||x|| \ = 1\} \tag{7.6}$$

is a norm on the elements of X_1^* and

$$||x|| = \frac{1}{c} \text{ if } x \neq 0, \ ||x|| \ = 0 \text{ if } x = 0, \tag{7.7}$$

c being the unique non-negative solution of the equation

$$\sup\{c\, g(x) \ + \beta|T^*g|_{3*} \ : \ |g|_{1*} = 1\} \ = \ \varepsilon \ , \tag{7.8}$$

is a norm on the elements of X_1; furthermore, the left-hand side of Equation (7.8) as a function of c is non-negative, monotonically increasing, convex and continuous for all non-negative real c.

b) If, in addition to the conditions of part a), $|T^*g|_{3*}$ and $|S^*h|_{2*}$ have continuous gradients and X_1 is a finite dimensional Euclidean space, then Goldstein's iterative method (see Proposition 7.5) can be used to find the extrema of the left hand sides of Equations 7.4, 7.5, 7.6, and 7.8.

Corollary 7.1

If, in addition to the conditions in part a) of the above theorem, β is zero (unperturbed system), then the controllability condition is satisfied if and only if

$$\sup\{(h,s-x_d)-\rho|S^*h|_{2*} : |h|_{1*} = 1\} - \varepsilon \leq 0 \quad ;$$

hence if $|S^*h|_{2*}$ has a continuous gradient and X_1 is a finite dimensional Euclidean space the extrema of the above equation can be found by Goldstein's iterative method.

The above Theorem and its Corollary do not require any special proof since the Corollary is a consequence of the Theorem, which itself is a direct consequence of Propositions 7.1 to 7.5, which will be proven later in this chapter.

Only part b) needs further comment. If X_2 and X_3 are $l_q^N(R_{q'}^k)$-type spaces, it would appear from the structure of the natural norms on X_2^* and X_3^* that the continuous functions $|T^*g|_{3*}$ and $|S^*h|_{2*}$ have a well-defined gradient. Hence if an iterative scheme in which the continuity requirement on the gradients of the above functions can be relaxed were available; that is, an iterative method to find the extrema on a unit sphere of a continuous functional with a well-defined gradient. One of the reasons for orienting the interest in that direction is the "successive approximation method" given by Gabasov and Kirillova ([13]) for a particular case (R_2^n).

Proposition 7.1:

If

$$\sup\{\beta|T^*g|_{3*} : |g|_{1*} = 1\} < \varepsilon , \tag{7.9}$$

and if the functional $|| \; ||^*$ on X_1^* is defined as

$$||h||^* = \frac{1}{\varepsilon} \max\{h(x) : x \varepsilon \partial T_D^{x_d}\}$$

where $\partial T_D^{x_d}$ is defined in Equation 7.3, then $|| \; ||^*$ is a well defined norm on the elements of X_1^* and the norm-topology generated by $|| \; ||^*$ is the same as the norm-topology generated by the natural norm.

Proof:

The functional $|| \; ||^*$ is clearly well-defined and non-negative. It is continuous because $\partial T_D^{x_d}$ is a bounded set:

$$\sup\{|x|_1 : x \varepsilon \partial T_D^{x_d}\} \leq \varepsilon.$$

Given any h_1, h_2 in X_1^*, it is clear that

$$\left|\ ||h_2||^* - ||h_1||^*\ \right| \leq \varepsilon\ |h_2 - h_1|_{1*}$$

and that by the previous inequality the ε-δ definition of continuity is satisfied.

It then suffices to show that $||\ ||^*$ has the three properties characterizing a norm:

(i) $||h^*|| = 0 \Leftrightarrow h = 0$

(ii) $||\alpha h||^* = |\alpha|\ ||h||^*, \forall\, h \in X_1^*$ and $\alpha \varepsilon R$

(iii) $||h_1 + h_2||^* \leq ||h_1||^* + ||h_2||^*$.

(i) It is readily seen that h=0 implies $||h||^* = 0$; conversely, let $||h||^*$ be equal to zero and assume that h is not the origin. Let α be defined as

$$\alpha = \sup\{\beta|T^*g|_{3*} : |g|_{1*} = 1\} \tag{7.10}$$

and consider the neighborhood $N_\delta(0)$ of the origin in X_1

$$N_\delta(0) = \{x \varepsilon X_1 : |x_1|_1 \leq \delta = \varepsilon - \alpha\}$$

(note that by hypothesis $\alpha < \varepsilon$ and $\varepsilon - \alpha > 0$).

It is clear that the inequality

$$\sup\{g(x)+\beta|T^*g|_{3*} : |g|_{1*} = 1\} \leq \varepsilon - \alpha + \alpha = \varepsilon, \forall\, x\varepsilon N_\delta(0)$$

implies that

$$N_\delta(0) \subset T_D^{x_d} ;$$

now for all non-identically zero h in X_1^*,

$$||h||^* = \frac{1}{\varepsilon}\max\{h(x):x\in T_D^{x_d}\} \geq \frac{1}{\varepsilon}\max\{h(x):x\epsilon N_\delta(0) = \frac{\delta}{\varepsilon}|h|_{1*} > 0.$$

The last inequality contradicts the hypothesis that $||h||^*$ is null. Hence if $||h||^*$ is zero, h must be 0.

(ii) $||\alpha h||^* = |\alpha| \quad ||h||^*$

In the proof of Theorem 4.2, it was shown that for any h in X_1^*

$$\max\{h(x):x\varepsilon\partial T_D^{x_d}\} = \max\{|h(x)|:x\varepsilon\partial T_D^{x_d}\} ;$$

this last remark shows that (ii) is true.

(iii) $||h_1+h_2||^* \leq ||h_1||^* + ||h_2||^*, \forall\ h_1, h_2$ in X_1^*

This is again obvious from the linearity of h:

$$|(h_1+h_2)(x)| \leq |h_1(x)| + |h_2(x)|, \forall\ x \varepsilon \partial T_D^{x_d}$$

and

$$||h_1+h_2||^* \leq ||h_1||^* + ||h_2||^*$$

The second part of the proof is to show the equivalence of the two norm-topologies on X_1^*. Since both topologies are metric it suffices to verify that given $\eta > 0$ and an element a in X_1^*, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$||h-a||^* < \delta_1 \implies |h-a|_{1*} < \eta \qquad (7.11)$$

and

$$|h-a|_{1*} < \delta_2 \implies ||h-a||^* < \eta \qquad (7.12)$$

let

$$\delta_2 = \frac{\eta}{2\varepsilon} \quad \text{and} \quad \delta_1 = \frac{\eta}{2(\varepsilon-\alpha)}$$

where

$$\alpha = \sup\{\beta|T^*g|_{3*} : |g|_{1*} = 1\}$$

(note that $\alpha < \varepsilon$ by hypothesis), then a straight substitution of δ_1 and δ_2 in Equation 7.11 and 7.12 concludes the proof of the equivalence of the two norm topologies of X_1^*.

This completes the proof of Proposition 7.1.

<u>Proposition 7.2:</u>

If $|| \; ||^*$ is the norm defined in Proposition 7.1 and if the functional v on X_1^* is defined as

$$v(h)=h(s-x_d)-\rho|S^*h|_{2*}-\max\{h(x):x\varepsilon\partial T_D^{x_d}\} \tag{7.13}$$

where $\partial T_D^{x_d}$ is defined in Equation 7.3, then

$$(v(h)\leq 0, \forall \; h\varepsilon X_1^*\sim\{0\}) \Leftrightarrow (\sup\{v(h):||h||^*=1\} \leq 0) .$$

Proof:

1) ($\Rightarrow$)

This is obvious.

2) ($\Leftarrow$)

Note that for all non-identically zero h in X_1^*,

$$v(h) = v(h/\ ||h||^*)\ ||h||^* ,$$

and

$$v(h) \leq ||h||^* \sup\{v(h): ||h||^*=1\} \leq 0 ;$$

the result is again obvious.

This completes the proof of Proposition 7.2.

Proposition 7.3:

The map v,

$$v : R^+ \cup \{0\} \to R$$

$$c \to v(c)$$

defined by

$$v(c)=\sup\{c\ g(x)+\beta|T^*g|_{3*} : |g|_{1*}=1\} \tag{7.14}$$

is well-defined, non-negative, monotonically increasing, continuous and convex on $R^+\cup\{0\}$ (non-negative reals).

<u>Proof</u>:

Let the map f,

$$f : R^+\cup\{0\} \times \{g \in X_1^* : |g|_{1*}=1\} \to R$$

$$(c,g) \to f(c,g)$$

be defined as

$$f(c,g) = c\ g(x) + \beta|T^*g|_{3*} ; \tag{7.15}$$

then the map v can be equivalently redefined as

$$v(c) = \sup\{f(c,g) : |g|_{1*} = 1\} . \tag{7.16}$$

As noted in the proof of Proposition 4.5, $v(c)$ is non-negative; in addition, given c_1, c_2 such that $0 \leq c_1 \leq c_2$, then

$$c_1 |\, g(x)\, | + \beta |T^*g|_{X_3^*} \leq c_2 |\, g(x)\, | + \beta |T^*g|_{X_3^*}, \forall\, g \in X_1^*$$

and

$$v(c_1) \leq v(c_2) ,$$

showing the monotonically increasing property of the map v.

That v is continuous follows from the next series of inequalities:

$$\begin{aligned} |v(c_2)-v(c_1)| &\leq \sup\{ |(c_2-c_1)(g,x)| : |g|_{1*} = 1\} \\ &\leq |c_2-c_1| \; |x|_1 \; ; \end{aligned}$$

from the above inequality, it is clear that $v(c)$ satisfies the ε-δ definition of continuity on $R^+ \cup \{0\}$.

Finally, the convexity of the functional v is obvious: for all $c_2 \geq c_1 \geq 0$ and $\lambda \varepsilon [0,1]$,

$$f(\lambda c_1+(1-\lambda)c_2,g) = \lambda f(c_1,g)+(1-\lambda)f(c_2,g), \forall g \in X_1^* ;$$

and

$$v(\lambda c_1+(1-\lambda)c_2) \leq \lambda v(c_1)+(1-\lambda)v(c_2), \forall \lambda \in [0,1].$$

This completes the proof of Proposition 7.3.

Proposition 7.4:

If

$$\sup\{\beta|T^*g|_{3*} : |g|_{1*} = 1\} < \varepsilon \qquad (7.17)$$

and the function $|| \; ||$ on X_1 is defined as

$$||x|| = \begin{cases} 1/c & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases} \qquad (7.18)$$

where c is the unique non-negative real satisfying the equation

$$\sup\{cg(x)+\beta|T^*g|_{3*} : |g|_{1*} = 1\} = \varepsilon \tag{7.19}$$

then $||\ ||$ is a well defined norm on the X_1 space; furthermore, the topology induced by $||\ ||$ is identical to the topology induced by the initial norm $|\ |_1$.

Proof:

It can be shown that given a real linear space X_1 and a convex, symmetric and absorbing subset A of X_1, a norm $||\ ||$ for the elements of X_1 can be defined as

$$||x|| = 1/\alpha \text{ if } x \neq 0, \text{ and, } ||x|| = 0 \text{ if } x = 0$$

where

$$\alpha = \sup_A \{\lambda \mid \lambda x \in A\} .$$

The proof of the above result, which is not particularly difficult, will not be given here.

It is clear that $T_D^{x_d}$ is convex and symmetric. In order to show that it is absorbing, it suffices to note that for any x in $X_1 (x \neq 0)$: if $c = \varepsilon\text{-}\sup\{\beta|T^*g|_{3*} : |g|_{1*} = 1\}$, cx is an interior point of $X_D^{x_d}$, and, if $c > 2[\varepsilon + \beta \sup\{|T^*g|_{3*} : |g|_{1*} = 1\}] / |x|_1$, cx is a point exterior to $T_D^{x_d}$.

As was shown in Proposition 4.1 the set $T_D^{x_d}$ is convex and weakly compact; hence the set Λ

$$\Lambda = \{\lambda \mid \lambda x \in T_D^{x_d}\} \tag{7.20}$$

is also convex and compact. Consequently there exists a unique non-negative $\bar{\lambda}$ such that

$$\bar{\lambda} = \sup\{\lambda \mid \lambda x \in T_D^{x_d}\}.$$

Moreover, since Λ is compact

$$\bar{\lambda} = \max \{\lambda \mid \lambda \in \partial\Lambda\}$$

and hence, since it can be proved by contradiction that

$$\partial\Lambda = \{\lambda \mid \lambda x \in \partial T_D^{x_d}\} \quad ,$$

$$\overline{\lambda} = \max \{\lambda \mid \lambda x \in \partial T_D^{x_d}\}. \tag{7.21}$$

But since any element x in $\partial T_D^{x_d}$ satisfies the equation

$$\sup\{g(x)+\beta|T^*g|_{3*} : |g|_{1*}\} = \varepsilon \tag{7.22}$$

and since it is known the $\overline{\lambda}$ is non-negative, Equation 7.21 can be rewritten as

$$\overline{\lambda} = \max \{\lambda \mid \max_{|g|_{1*}=1}[g(\lambda x)+\beta|T^*g|_{3*}] = \varepsilon \ , \ \lambda \geq 0 \} \ ; \tag{7.23}$$

by Proposition 7.3 we know that the function $v(\lambda)$,

$$v(\lambda) = \sup\{\lambda g(x)+\beta|T^*g|_{3*} : |g|_{1*} = 1\} \ , \tag{7.24}$$

is a well-defined continuous, convex, monotonically increasing and non-negative map on the non-negative reals. Consequently, since for λ equal to zero, the function is strictly less that ε, there exists a unique $\lambda > 0$ such that the function be equal to ε; the uniqueness follows from the convexity of the function.

The last part of the proof concerns the equivalence of the topologies. Since they are both metric it must be verified that given $\eta > 0$ and an element a in X_1, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$||x-a|| < \delta_1 \implies |x-a|_1 < \eta \tag{7.25}$$

and such that

$$|x-a|_1 < \delta_2 \implies ||x-a|| < \eta \; ; \tag{7.26}$$

the previous statement is easily verified if

$$\delta_1 = \frac{\eta}{2(\varepsilon-\alpha)} \qquad \text{and} \qquad \delta_2 = \frac{\eta}{2\varepsilon}$$

where

$$\alpha = \sup\{\beta|T^*g|_{3*} : |g|_{1*} = 1\}.$$

This completes the proof of Proposition 7.4.

The next proposition is a quotation of Goldstein's iterative method ([9]) for finding the extrema of a real continuous function with continuous gradient on a unit sphere. As quoted, Proposition 7.5 will find the minima of a real continuous function with a continuous gradient; however, the maximum of a function f can easily be obtained by the following well-known relation:

$$\sup[f(h)] = -\inf[-f(h)]. \tag{7.27}$$

Proposition 7.5: (see [12], pages 60-64)

Assume that f is in C^1 on ∂S (unit sphere in R^n). Take x^o arbitrarily in ∂S. Choose μ such that

$$3\mu \leq \max\{||\nabla f(x)||^{-1} : x \in \partial S\}.$$

For $x \in \partial S$, set

$$x'(\gamma) = \frac{x-\gamma\nabla f(x)}{||x-\gamma\nabla f(x)||} ,$$

$$\Delta(x,\gamma) = f(x) - f(x'(\gamma))$$

and

$$g(x,\gamma) = \frac{-\Delta(x,\gamma)}{(\nabla f(x), x'(\gamma)-x)} ;$$

the iterative scheme now proceed as follows:

Choose δ, $0 < \delta \leq \min(\frac{1}{2},\mu)$. Set

$$\theta_k = \min\{1, \tfrac{1}{3} \, |(\nabla f(x^k), x^k)|^{-1}\} .$$

Take $\gamma_k \leq \theta_k$ such that $\delta \leq g(x^k,\gamma_k) \leq 1 - \delta$ if $g(x^k,\theta_k) < \delta$ or $\gamma_k = \theta_k$ otherwise. Set

$$x^{k+1} = \frac{x^k - \gamma_k\nabla f(x^k)}{||x^k-\gamma_k\nabla f(x^k)||} .$$

Then:

(i) $\{||\nabla f(x^k)||^2 - (\nabla f(x^k),x^k)^2\}$ converges to 0, every cluster point z of the sequence $\{x^k\}$ satisfies $\nabla f(z)= \pm||\nabla f(z)||z$, $\{f(x^k)\}$ converges downward to a limit, and $\{x^{k+1}-x^k\}$ converges to 0.

(ii) Assume that for x in S, the roots of the equation

$$\nabla f(x) = \pm \ ||\nabla f(x)||x$$

are finite in number. Then $\{x^k\}$ converges. If this equation has only one root in S, then $\lim\{x^k\}$ is a minimizer of f on S.

Proof:

The above proposition is quoted from a Theorem proven by Goldstein ([12], page 60-64).

At this point the iterative scheme is completely general except for the case where

$$\sup\{\beta|T^*g|_{3*} : |g|_{1*} = 1\} = \varepsilon \ ;$$

for that case it is evident that in the R_p^n and $l_q^{N+1}(R_{q'}^k)$ spaces with p, q and q' neither equal to one or infinity, the only element of the set $\partial T_D^{x_d}$ is the origin 0 in the X_1 space. This means, that the last term in Equation 7.2 is zero:

$$\max\{(h,x): x \in \partial T_D^{x_d}\} \ , \ \forall\, h \in X_1^* \ .$$

Hence by Proposition 7.1, the controllability condition is satisfied if and only if

$$\sup\{(h,s-x_d)-\rho|S^*h|_{X_3^*} \ : \ |h|_{1^*} = 1\} \leq 0 \ .$$

In the other cases the set $T_D^{x_d}$ has no interior, but in addition to the origin 0 it may have other points; in fact, although this has not yet been proven for the general case, $T_D^{x_d}$ would seem to be a subset of a hyperplane through the origin.

CHAPTER VIII

CONCLUSIONS

The results obtained constitute a complete answer to the four parts of the problem as outlined in section 2.3. More specifically, very general geometrical and analytical necessary and sufficient conditions for the controllability of an abstract linear system ($\mathcal{L}$) in the presence of disturbances are obtained by the introduction of the unperturbed attainable set and the reduced target set.

As an application of these general results the problem of controllability for linear difference equation systems has been completely solved.

The aspects of the characterization problem discussed in section 2.3 have been almost completely covered, confirming the intuitive ideas of impossibility of strict controllability in the presence of disturbances, upper bound on the size of the disturbances and lower bounds on the sizes of the target and the controls.

Finally a decomposition scheme has been developed to facilitate the numerical solution of the equations arising from the controllability problem; with this decomposition shceme and some numerical iterative schemes the problem of controllability of linear systems can be completely programmed on a digital computer.

As is quite obvious, there are numerous possible extensions of this work. For instance, the results can be extended to an unbounded closed convex target; the geometrical necessary and sufficient condition for controllability (Theorem 4.1) and its equivalent analytical statement (Theorem 3.7) remain true. Among the applications covered by the theory one can include the linear differential equation and linear partial differential equation systems with the appropriate continuity conditions on the matrices describing them.

Another interesting problem would be to derive a similar theory for free-end time difference equation systems and generalize the results. Finally, as a logical continuation to the work on the decomposition scheme, a better suited numerical iterative scheme could be obtained by making use of the special structure of the equations involved.

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