MULTILEVEL CONTROL OF LINEAR SYSTEMS

by

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ABSTRACT

The "satisfaction approach" of Takahara [5] reduces the complexity of the computation of the control for a large scale system. Its aim is not to reach the optimum, but, given a reference control, to compute a new control such as to improve the performance. The fact that the optimum point is not reached is compensated by the fact that the computation is simplified. The numerical application of this technique is studied here (Chapter 2). This viewpoint of the problem leads (in Chapter 3) to an algorithm which improves the performance and deals with each subsystem separately. The iteration of this algorithm is shown to give the optimum, if some assumptions are satisfied.

Another approach for this kind of problem is the "decomposition technique": it reaches the optimal control of a large system by dealing with each subsystem separately and then coordinating the results. This technique was applied by S. Reich [3] in the case of linear systems with quadratic performances and is extended (Chapter 4) to the use of linear systems with disturbances. Moreover, with regard to the solution of the global system, the results do not show any reduction in the computing time of the optimal solution.

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LIST OF SYMBOLS

A	Positive self adjoint linear operator from X into X.
B	Positive self adjoint linear operator from M into M.
C(T)	Set of continuous real-valued functions defined on [T].
D(T)	Set of continuous real-valued functions defined on [T].
E	Expectation operator.
H	Hamiltonian.
H_m	$\frac{\partial H}{\partial m}$.
M	Set of manipulated variables.
M_i	$i^{\underline{th}}$ projection of M.
m	Element of M $\equiv$ manipulated variable.
m^r	Reference control for the first level.
m_j^r	$P_j \ m^r$.
n	Number of the subsystems of the first level.
N	Vector Wiener Process: Noise in the system.
N_i	P_i N.
P_i	$i^{\underline{th}}$ projection operator.
q	Performance index on M x U.
q_j	Performance index of the $j^{\underline{th}}$ subsystem on $M_j \times U_j$.
r	Desired value of a state variable.
r_j	$j^{\underline{th}}$ projection of r.

S_i	First kind of sensitivity functional for the $i^{\underline{th}}$ subsystem.
$Sm(t)$	Segment of $m(t)$.
s_j	Sensitivity (first kind) of the $j^{\underline{th}}$ subsystem - Uncertainty of the $j^{\underline{th}}$ subsystem.
T	Time index set : $[o, t_e]$.
t	Time index.
$t_{S(i)}$	Starting time of the $i^{\underline{th}}$ adaptation.
t_e	End time of control.
V	Uncertainty set - Posterior variance $\equiv E\,[(x - \bar{x})(x - \bar{x})^T]$ = matrix (n,n)
V_j	Set of v_j.
v_{ij}	Element of the matrix V.
v_j	Uncertainty for the $j^{\underline{th}}$ subsystem.
X	Set of state vector.
X_i	$i^{\underline{th}}$ projection of X.
x_i	Element of X_i.
x^r	State vector corresponding to m^r.
$\bar{x}_i$	$E(x_i)$.
$\bar{x}$	$E(x)$ = posterior mean.
Y	Observation of the system.
Z	Vector Wiener Process: Noise in the measurements.
z_i	$P_i\ Z$.
ζ_j	Coordination variable for the $j^{\underline{th}}$ subsystem.
τ	Time index.
ϕ	Performance functional on X x M.

ϕ_i Performance functional of the $i^{\underline{th}}$ subsystem on $X_i \times M_i$.

$\leq_{(q)}$ Ordering relation on M.

$\leq_{(qj)}$ Ordering relation on M_j.

NOTATIONS

A^T = Transpose of A.

$\dot{Y}_{(t)}$ = Time derivative of $y_{(t)}$.

$\mathcal{D}, \mathcal{E}, \mathcal{L}, \mathcal{C}$: Time varying matrices defining the successive sweep method.

$h(t), w_1, w_2$: Time varying vectors defining the successive sweep method.

g_1, g_2 : Real valued functions defined on T.

W : $\frac{1}{dt}[d\ N(t)\ .\ d\ N^T(t)]$.

Q : $\frac{1}{dt}[d\ Z(t)\quad d\ Z^T(t)]$.

$v'_{11}, v'_{22}, \bar{x}'_1, \bar{x}'_2, v'_{12}, v''_{12}$: Dummy variables.

CHAPTER I

INTRODUCTION

This thesis is concerned with the decomposition of the computation of optimal controls for large scale systems.

There are two parts:

The first part (Chapters 2 and 3) deals with a particular technique, the "satisfaction approach" studied by Takahara [5]. The satisfaction approach proceeds as follows: it allows performance improvement of a large system by dealing with separate subsystems, and the interconnections between the subsystems are viewed as internal disturbances. The second chapter is an application of the satisfaction approach to linear quadratic systems. The third chapter is another way of improving the performance by dealing with separate subsystems. With some assumptions this heuristic is shown to lead by iterations to the optimal control.

The second part (Chapter 4) deals with the use of the decomposition technique in large systems with disturbances. This decomposition technique is derived from a technique developed by Dantzig and Wolfe [1], and by Arrow [2]. The technique was studied and used by S. Reich [3] for deterministic linear systems, and deterministic linear systems with non-linear coupling. In this chapter, noises are introduced in an additive way, both in the processes and the measurements. The noises are supposed to be Gaussian, uncorrelated, and their characteristics (mean and variance) are supposed to be known. The noises in the measurements introduce a new kind of interconnection between the subsystems so the decomposition technique cannot be applied directly. The Kalman technique [4] allows, however, in some specific cases the transformation of a stochastic problem into a deterministic one. Once the deterministic equations are found, the application of the decomposition technique presents no computational difficulty. However, it was not shown that this computation, based on a saddle-value argument should, in the general case, converge to the optimal point.

In all the numerical examples, the computation technique is the successive sweep method for the global method and the first level of the decomposition technique. The gradient method is used for the second level. The notations for the computation are those explained in Appendix 1.

CHAPTER II

THE SATISFACTION TECHNIQUE
-
APPLICATION TO LINEAR SYSTEMS

2.1 Goals and Methods of the Satisfaction Approach:

The decomposition technique is a way to break a large scale system into subsystems and then by a multi-level procedure find the overall optimum.

This is not the only way to approach the problem. The satisfaction approach, with the notion of "internal disturbances" leads to another multilevel procedure, dealing with separate subsystems. All the details on the method are taken from Takahara [5].

A multilevel system is a control system where a given controlled system is controlled by a group of goal seeking systems in a hierarchic arrangement.

By referring to Table 1 we find that: G_{11}, G_{12}, G_2 are goal seeking systems or controllers. G_{11}, G_{12} belong to the first level. G_2 belong to the second level.

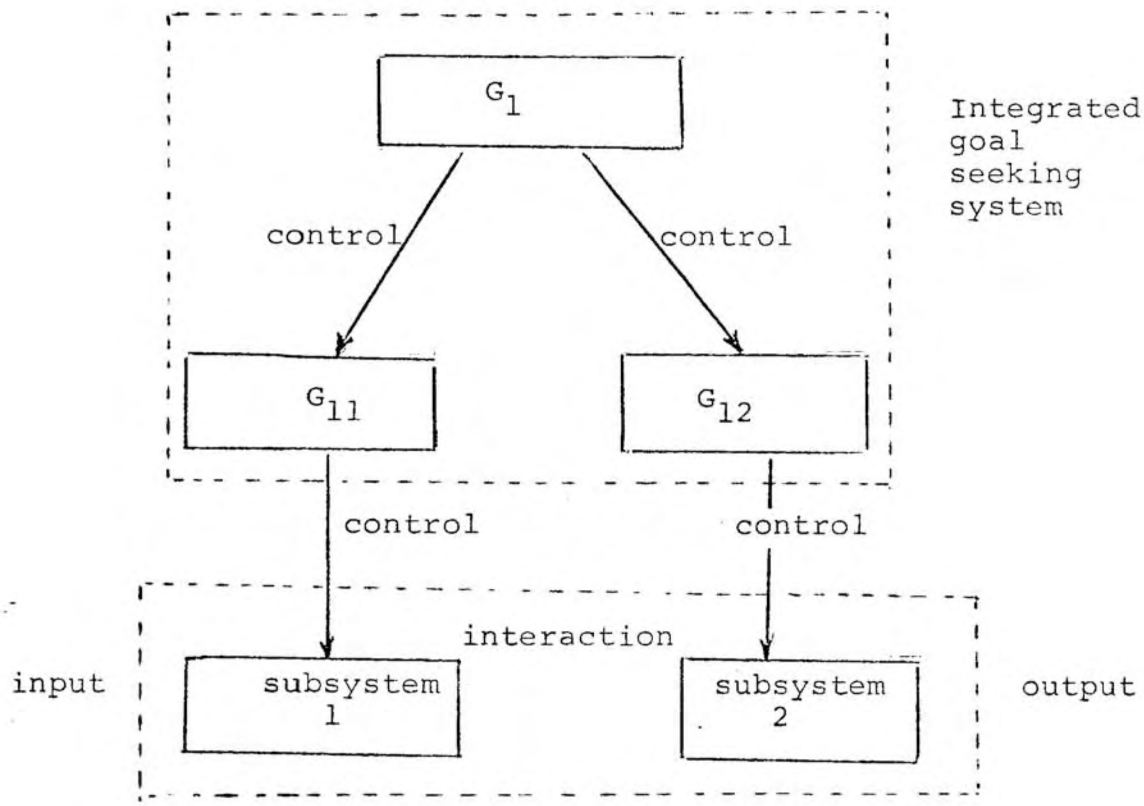


Table 1

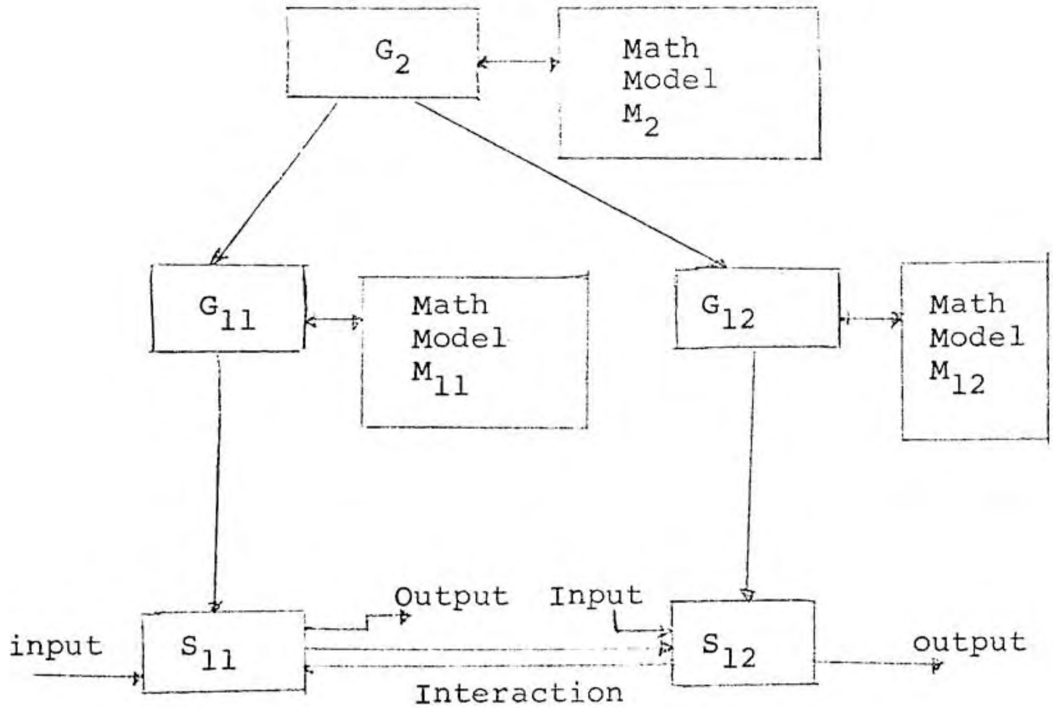
Integrated Controlled System

G_{11}, G_{12} are assumed to control S_{11} and S_{12} separately. But in general S_{11} and S_{12} are interacting. So we must introduce a goal seeking G_2 in order to coordinate the G_{11}, G_{12}. G_2 improves the integrated performance by compensating the negligences of the interactions through the controllers G_{11}, G_{12}. Since these interactions are unknown for G_{11} and G_{12}, they are called internal disturbances (the term "external disturbances" being used for the noises).

The satisfaction approach uses a so-called on-line coordination, i.e., a coordination which does not use an iteration technique. The reason is that the use of an iteration technique, in a so-called off-line coordination technique supposes an harmoniously coordinable system, i.e., the system is such that there exists a best value for the coordination variable and this value can be reached by iteration technique.

Therefore, the goal of the satisfaction technique is not to get the optimum solution of the problem but to improve the overall performance, with respect to a reference control.

In a given control system, each controller has its own mathematical model for the control purpose.



The whole problem is to determine a mathematical model for each goal seeking system in relation to the integrated mathematical model.

We can repeat the problem of the satisfaction approach in the following way:

Let S be a real normed linear space on which an inner product is defined. The inner product is assumed to be continuous with respect to the topology derived from the norm. Let U, M and X be subsets of S. The relative topologies are defined on them. Let Ψ and ϕ be continuous mappings such that:

$$\Psi : M \times U \to X$$

$$\phi : X \times M \to R$$

Let $q(m,u) = \phi(\Psi(m,u), m)$.

The satisfaction approach consists in finding:

1) If there exists $m^* \varepsilon M$ such that:

$$\forall u \varepsilon U \qquad q(m^*,u) = V(u)$$

where $V(u)$ is the satisfaction threshold.

2) If m^* exists, then it is desired to find it explicitly.

2.2 Mathematical Formulation.

Hypothesis and Assumptions:

We consider a multilevel deterministic system in a real Banach space B. Let X and M be subsets of B; we call them the state set and the manipulated variable set. The linear manifolds spanned by X and M will be written as $\underline{X}$ and $\underline{M}$. Let the system have as its state equation

$$x = \Psi (m)$$

and performance functional:

$$q(m) = \phi (\Psi (m), m).$$

Assumptions:

1. Ψ and ϕ have Frechet derivatives.
2. M and X are convex sets in B and $X = \underline{X}$ $Int(m) \neq \phi$. with respect to the relative topology.
3. $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$

   $\oplus$ represents the direct sum operation.

   $$M_i = P_i M$$
4. Let us call F_1 and F_2 the Frechet derivatives of ϕ with respect to x and m_1. We assume that F_1 and F_2 have also continuous Frechet derivatives.

With these assumptions we can give some definitions very helpful for the classification of multilevel systems:

-The operator K_i: $M \rightarrow X$ such that

$$K_i: P_i \Psi(m) - \dot{\Psi}(m) P_i$$

where P_i is the projection operator, is called the first kind of interaction operator.

If $K_i = 0$, we can write $x_i = \Psi(m_i) + x_0^i$ where x_0^i is a constant, i.e., the subsystems of the first level are isolated.

-The bounded linear functional $S_i : M \to R$ such that: $S_i = K_i^* F_1 (x,m)$ where K_i^* is the adjoint of K_i and $F_1 (x,m)$ is the Frechet derivative of ϕ with respect to x is called the first kind of sensitivity functional.

It can be shown that the variations of q with respect to the variations of m_i consist of two parts: one is the direct consequence of δm_i on x_i. The second, $\{-(S_i, \delta m_i)\}$ (inner product), is the consequence of the interactions among the subsystems.

Lastly we can deal with the interactions in the integrated performance functional with the help of the second kind of interaction operators, i.e.,

$$R_{i1} : X \to \underline{X}' \qquad R_{i2} : X \to \underline{M}'$$

$$T_{i1} : M \to \underline{X}' \qquad T_{12} : M \to \underline{M}'$$

with $\underline{X}'$, $\underline{M}'$ conjugate spaces of the linear manifold spanned by X and M, and such that:

$$R_{ij} = P_i^* D_x F_j (x,m) - D_x F_j (x,m) P_i$$

$$T_{ij} = P_i^* D_m F_j (x,m) - D_m F_j (x,m) P_i$$

where:

P_i^* is the adjoint of P_i.

F_1 is the Frechet derivative of ϕ with respect to x.

F_2 is the Frechet derivative of ϕ with respect to m.

$D_x F_j$ is the Frechet derivative of F_j with respect to x.

We say that ϕ (x,m) is <u>additive</u> if ϕ (x,m) is represented as $\phi(x,m) = \sum_{i=1}^{n} \phi_i(x_i, m_i)$.

It was proved [5] that ϕ (x,m) is additive if and only if $R_{ij}(I - P_i) = 0$ and $T_{ij}(I - P_i) = 0$.

So, if the two kinds of interactions, K_i on one hand, R_{ij} and T_{ij} on the other hand, are equal to zero, then, the interactions are zero, the system can be reduced to n independent control subsystems. If any one of these is not zero, then we have interactions called internal disturbances.

In order to give a mathematical formulation of the subsystems we have to make some more assumptions on the system:

The integrated system is given as follows:

$x = \Psi m + x^F$ state equation

$\phi(x,m) = [x - r, A(x - r)] + (m, B m)$ performance functional.

<u>Supplementary assumptions:</u>

-M is compact.

-$q(m) = \phi(\Psi(m), m)$ is convex and it takes its unique unitical point in the interior of M.

-Ψ is a linear operator such that: $\Psi : M \rightarrow X$.

-A and B are linear, bounded, self adjoint positive operators.

-$r \in X$ is a constant.

-$\phi(x,m)$ is additive, i.e., ϕ can be rewritten in the following form:

$$\phi(x,m) = (x_1-r_1, A_1(x_1-r_1)) + \ldots + (x_n-r_n, A_n(x_n-r_n)) + (m_1, B_1 m_1) + \ldots\ldots\ldots + (m_n, B_n m_n)$$

With these assumptions, we can write the following formulation:

The $j^{\underline{th}}$ subsystem control problem of the first level is defined as follows:

$$x_j = \Psi_j m_j + v_j + x_j^F$$

$$\min_{m_j \in M_j} \phi_j(x_j, m_j, s_j) = (x_j - r_j, A_j(x_j - r_j)) + (m_j, B' m_j) - (s_j, m_j)$$

$$q_j(m_j, v_j, s_j) = \phi_j(\Psi_j m_j + v_j + x_j^F, m_j, s_j)$$

where

$$x_j \in X_j,\ v_j \in V_j \subset X_j,\ m_j \in M_j,\ s_j \in S_j$$

and

$$\Psi_j = \Psi + P_j \Psi - \Psi P_j.$$

$$x_j^F = P_j x^F$$

$$B_j = B_j + K_j^* A K_j$$

$$N_j = K_j \bar{m}_j$$

It was proved [5] that:

If

m'' & m' ϵ admissible set of controls

and

$$q_i(m_i'', v_i, s_i) \le q_i(m_i', v_i, s_i) \; \forall_i \; \forall v_i \in V_i, \; \forall s_i \in S_i$$

then

$$q(m'') \le q(m').$$

But the following theorem is also true and the demonstration is exactly the same:

Theorem: For m'' and m' ϵ admissible set of controls, such that:

$$q_i(m_i'', v_i'', s_i'') \le q_i(m_i', v_i'', s_i'') \; \forall_i$$

we have:

$$q(m'') \le q(m').$$

with

$$v_i'' = K_i \bar{m}_i''$$

$$s_i'' = 2(K_i^* A(x'' - r) + K_i^* A K_i m'')$$

This formulation of the problem, i.e., the mathematical models of the subsystems, is a second order approximation of the integrated system. Since the integrated system we shall deal with is assumed to be linear-quadratic a second order approximation can represent the global property precisely and if a general system can be approximated by a linear quadratic system, then this formulation will be applicable.

2.3 Theoretical Way of Applying the Satisfaction Technique

We shall study a linear example:

$$\frac{dx}{dt} = C\ x + m.$$

or

$$\begin{cases} \dfrac{dx_1}{dt} = c_{11}\ x_1 + c_{12}\ x_2 + m_1 \\ \dfrac{dx_2}{dt} = c_{21}\ x_1 + c_{22}\ x_2 + m_2 \end{cases}$$

with the performance:

$$\phi = \int_0^{t_e} \{ (x - r)^T A\ (x - r) + m^T B\ m\}\ dt$$

with

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \qquad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

Notations:

For a linear example like the one we are studying we can write:

$$x_1(t) = x_1^f + (\Psi_1\ m_1)(t) + (K_1\ m_2)(t).$$

with

x_1^f = free movement of the subsystem 1.

$\Psi_1\ m_1$ = control action within subsystem 1.

$K_1\ m_2$ = interaction between the two subsystems.

This can be shown in the following way: $x(t)$, solution of a linear differential equation can be written:

$$x(t) = \varphi(t)\ x(0) + \int_0^t \varphi(t - \quad)\ m(\quad)\ d \quad .$$

with $\varphi(t)$ = transition matrix.

If we call

$$\varphi = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}$$

we get

$$\begin{cases} x_1^f(t) = \varphi_{11}(t)\ x_1(0) + \varphi_{12}(t)\ x_i(t) \\ \Psi_1\ m_1(t) = \int_0^t \varphi_{11}(t - \tau)\ m_1\ (\tau)\ d\tau\ . \\ K_1\ m_2(t) = \int_0^t \varphi_{12}(t - \tau)\ m_2\ (\tau)\ d\tau\ . \end{cases}$$

Fundamental theorem:

Given a reference control $[m_1^r,\ m_2^r]$, the internal disturbances $v_1^r = K_1\ m_2^r$

$$v_2^r = K_2\ m_1^r$$

$$s_1^r = 2[K_1^*\ A\ (x^r - r) + K_1^*\ A\ K_1\ m^r]$$

$$s_2^r = 2[K_2^*\ A\ (x^r - r) + K_2^*\ A\ K_2\ m^r]$$

and the following systems, equations and performances:

-sub system 1:

$$\begin{cases} x_1 = \Psi_1\ m_1 + x_1^f + v_1^r \\ \phi_1(x_1, m_1) = \int_0^{t_e} \{A_1(x_1 - r_1)^2 + B_1'\ m_1^2 - s_1^r\ m_1\}\ dt \end{cases}$$

$$= q(m_1,\ v_1^r,\ s_1^r)$$

-sub system 2:

$$x_2 = \Psi_2 \; m_2 + x_2^f + v_2^r$$

$$\phi(x_2,m_2) = \int_0^{t_e} (A_2(x_2-r_2)^2 + B_2' \; m_2^2 - s_2^r \; m_2) \; dt$$

$$= q_2 \; (m_2, \; v_2^r, \; s_2^r).$$

If we can find controls $m_1'(t)$ and $m_2'(t)$ such that

$$\begin{cases} q_1(m_1', \; v_1^r, \; s_1^r) \le q_1(m_1^r, \; v_1^r, \; s_1^r) \\ q_2(m_2', \; v_2^r, \; s_2^r) \le q_2(m_2^r, \; v_2^r, \; s_2^r) \end{cases}$$

and

Then we have, for the overall problem:

$$q(m') \le q(m).$$

Proof: This is exactly the fundamental theorem we stated in the preceeding paragraph.

The heuristic is now obvious:

1st level - 1st subsystem - ith stage:

equation $\quad x_1 = \Psi_1 \; m_1 - x_1^f + v_1^r$

performance:

$$\phi(i,x_1,m_1) = q_1(i,m_1,v_1^r,s_1^r) = \int_{t_S(i)}^{t_e} \{A_1(x_1-r_1)^2 + B_1' m_1^2 - s_1^r m_1\} dt$$

Given m_1^r, v_1^r, s_1^r by the second level, find:

$$m_1'(t) \quad \text{such that:}$$

$$q_1(i,m_1',v_1^r,s_1^r) \le q_1(i,m_i^r,v_1^r,s_1^r).$$

2nd level - ith stage:

Let us call $m(i,t)$ the ith adaptation stage control of the first level, $q(i,m)$ the first level performance index for the $i^{\underline{th}}$ adaptation stage, $t_S(i)$ the starting time of the $i^{\underline{th}}$ adaptation stage and

$Sm^r(i-1,t)$ the restriction of $m^r(i-1,t)$ to $[t_S(i),t_e]$

Let us call $Sm'(i-1,t)$ the restriction of $m'(i-1,t)$ to $[t_S(i),t_e]$

Given the control in the following way:

$$m^r(i,t) = \lambda\ S\ m^r(i-1,t) + (1-\lambda)\ S\ m'(i-1,t)$$

and the system:

$$x^r(i,t) = \Psi(i) m^r\ (i,t) + x^f(i,t)$$

the performance:

$$\phi = \int_{t_S(i)}^{t_e} \{(x^r-r)^T A(x^r-r) + m^{r^T} B\ m^r\}\ dt$$

Find:

$-\lambda_0$ such that: λ_0 minimizes ϕ

-Compute: $v_1^r = K_1\ m_2^r \qquad v_2^r = K_2\ m_1^r$

$$s_1^r = 2[K_1^* A(x^r-r) + K_1^* A\ K_1\ m^r]$$

$$s_2^r = 2[K_2^* A(x^r-r) + K_2^* A\ K_2\ m^r]$$

With this scheme we have:

$$\begin{matrix} q[i,\ S\ m^r(i-1)] \\ q[i,\ S\ m'(i-1)] \end{matrix} \geq q[i,\ m^r(i) \geq q[i,\ m'\ (i)]$$

This scheme is perfect from a theoretical point of view, but not easy to implement with a computer.

2.4 A Heuristic for the Satisfaction Technique

In this paragraph, no new concept is introduced. The preceeding theoretical scheme is adapted in order to be computable.

This computation scheme is an adaptation of the scheme outlined in [5].

The system we study is always the same, i.e., a linear quadratic system. But the trick is to exchange the roles of x and m, i.e.,

$$\begin{cases} m_1(t) = \frac{dx_1}{dt} - c_{11}\, x_1 - c_{12}\, x_2. \\ m_2(t) = \frac{dx_2}{dr} - c_{22}\, x_2 - c_{21}\, x_1. \end{cases}$$

with the performance:

$$\phi = \int_0^{t_e} \{(x - r)^T A\, (x - r) + m^T B\, m\}\, dt$$

Fundamental theorem:

Given a reference control (m_1^r, m_2^r) that is to say a reference trajectory (x_1^r, x_2^r) and given the sets of disturbances V_1, V_2, S_1, S_2 which contain respectively v_1^r, v_2^r, s_1^r, s_2^r and the following systems, equations and performances:

*subsystem 1:

$$m_1(t) = \frac{dx_1}{dt} - c_{11}\, x_1 - v_1$$

$$q_1(i,x_1,v_1,s_1)=\phi_1(i,x_1,m_1)=\int_{t_S(i)}^{t_e} [A_1+c_{21}^2 B_2)(x_1-r_1)^2 + B_1 m_1^2 - 2x_1 s_1] dt$$

*subsystem 2:

$$m_2(t) = \frac{dx_2}{dt} - c_{22}\, x_2 - v_2$$

$$q_2(i,x_2,v_2,s_2)=\phi_2(i,x_2,m_2)=\int_{t_S(i)}^{t_e} [A_2+c_{12}^2 B_1)(x_2-r_2)^2 + B_2 m_2^2 - 2x_2 s_2] dt$$

If we can find controls $m_1'(t)$ and $m_2'(t)$ such that:

$$q_1(i,x_1',v_1,s_1) \le q_1(i,x_1^r,v_1,s_1) \quad \forall\, v_1 \in V_1,\ \forall\, s_1 \in S_1$$

and $\quad q_2(i,x_2',v_2,s_2) \le q_2(i,x_2^r,v_2,s_2) \quad \forall\, v_2 \in V_2,\ \forall\, s_2 \in S_2$

Then we have, for the overall problem:

$$q(i,m') \le q(i,\, m^r)$$

Proof:

$$q_1(i,x_1',v_1^r,s_1^r) = \int_{t_S(i)}^{t_e} \{A_1+B_2 c_{21}^2)(x_1'-r_1)^2 + B_1 m_1^2 - 2s_1^r x_1'] \, dt$$

But

$$v_1^r = c_{12}\, x_2^r$$

$$s_1^r = B_2\, c_{21}\, m_2^r + c_{21}^2\, B_2\, (x_1^r - r_1)$$

Then replacing v_1^r, s_1^r, m_1' by their value, we get:

$$q_1(i,x_1',v_1^r,s_1^r)=\int_{t_S(i)}^{t_e}\{(A_1+B_2c_{21}^2)(x_1'-r_1)+B_1[\frac{dx_1'}{dt}-c_{11}\,x_1'$$

$$-c_{12}\,x_2^r]$$

$$-2[B_2c_{21}^2m_2^r+c_{21}^2(x_1^r-r_1)]\,x_1'\}\,dt$$

Now we write:

$$q_1(i,x_1^r,v_1^r,s_1^r)=\int_{t_S(i)}^{t_e}\{[A_1+B_2c_{21}^2][x_1^r-r_1]^2+B_1[\frac{dx_1'}{dt}-c_{11}\,x_1^r$$

$$-c_{12}\,x_2^r]$$

$$-2[B_2c_{21}m_2^r+c_{21}^2(x_1^r-r_1)]\,x_1^r\}\,dt$$

We call

$$\delta=\begin{pmatrix}\delta_1\\ \\ \delta_2\end{pmatrix}=\begin{pmatrix}x_1^r-x_1'\\ \\ x_2^r-x_2'\end{pmatrix}=x^r-x'.$$

By hypothesis we have:

$$q_1(i,x_1',v_1^r,s_1^r)-q(i,x_1^r,v_1^r,s_1^r)\le 0$$

Replacing

$$x_1' \text{ by } (x_1^r-\delta_1)$$

and

$$q_1(i,x_1',v_1^r,s_1^r)\ ,\ q_1(i,x_1^r,v_1^r,s_1^r)$$

by their values, we get the following inequality = (2) :

$$\int_{t_S(i)}^{t_e} \{(A_1+B_2c_{21}^2)(x_1^r-\delta_1-r_1)^2+B_1[\frac{dx_1^r}{dt} - c_{11}\ x_1^r - c_{12}\ x_2^r - \frac{d\delta_1}{dt} + c_{11}\ \delta_1]^2$$

$$-2[B_2c_{21}[\frac{dx_2^r}{dt} - c_{22}\ x_2^r - c_{21}\ x_1^r] + c_{21}^2\ (x_1^r-r_1)[x_1^r-\delta_1]$$

$$-[A_1+B_2c_{21}^2)[x_1^r-r_1]^2 - B_1[\frac{dx_1^r}{dt} - c_{11}\ x_1^r - c_{12}\ x_2^r]^2$$

$$+2[B_2c_{21}\{\frac{dx_2^r}{dt} - c_{22}\ x_2^r - c_{21}\ x_1^r] + c_{21}^2(x_1^r-r_1)]x_1^r\}\ dt \leq 0$$

Now we compute:

$$q(x^r - \delta_1) - q(x^r) =$$

$$\int_{t_S(i)}^{t_e} \{A_1(x_1^r-\delta_1-r_1)^2+B_1[\frac{dx_1^r}{dt} - c_{11}\ x_1^r - c_{12}\ x_2^r - \frac{d\delta_1}{dt} + c_{11}\ \delta_1]^2$$

$$+B_2c_{21}^2[x_1^r-r_1-\delta_1]^2-2(x_1^r-r_1)(x_1^r-\delta_1)-2B_2c_{21}(\frac{dx_1^r}{dt} - c_{22}\ x_2^r - c_{21}\ x_1^r)(x_1^r - \delta_1)$$

$$-A_1(x_1^r-r_1)^2-B_2c_{21}^2(x_1^r-r_1)^2-B_1[\frac{dx_1^r}{dt} - c_{11}\ x_1^r - c_{12}\ x_2^r]^2$$

$$+2x_1^r[B_2c_{21}(\frac{dx_2^r}{dt} - c_{22}\ x_2^r - c_{21}\ x_1^r)+c_{21}^2B_2(x_1^r-r_1)]\ dt$$

Developing this expression algebraically we can find that $q(x^r-\delta_1) - q(x^r)$ is equal to the left hand side of the inequality ②.

So

$$q(x^r - \delta_1) - q(x^r) \le 0.$$

In the same way we could have shown that:

$$q(x^r - \delta_1 - \delta_2) \le q(x^r - \delta_1)$$

so

$$q(x') \le q(x^r)$$

End of the proof.

Corollary:

Given a reference control (m_1^r, m_2^r), i.e., a reference trajectory (x_1^r, x_2^r) and given

$$v_1^r = c_{12}\, x_2^r \qquad s_1^r = B_2\, c_{21}\, m_2^r + c_{21}^2\, B_2 (x_1^r - r_1)$$

$$v_2^r = c_{21}\, x_1' \qquad s_2^r = B_1\, c_{12}\, m_1' + c_{21}^2\, B_1\, (x_2^r - r_2).$$

and the following systems, equations and performances:

-subsystem 1:

$$m_1(t) = \frac{dx_1}{dt} - c_{11}\, x_1 - v_1^r$$

$$q_1(i, x_1, v_1^r, s_1^r) = \phi_1(i, x_1, m_1) = \int_{t_S(i)}^{t_e} [(A_1 + c_{21}^2 B_2)(x_1 - r_1)^2 + B_1 m_1^2 - 2x_1 s_1^r]\, dt$$

-subsystem 2:

$$m_2(t) = \frac{dx_2}{dt} - c_{22}\, x_2 - v_2^r$$

$$q_2(i,x_2,v_2^r,s_2^r)=\phi_2(i,x_2,m_2)=\int_{t_S(i)}^{t_e} [(A_2+c_{12}^2B_1)(x_2-r_2)^2$$

$$+ B_2m_2^2 - 2x_2s_2^r]\ dt$$

If we can find controls $m_1'(t)$ and $m_2'(t)$ such that:

$$q_1(i,x_1',v_1^r,s_1^r) \le q_1(i,x_1^r,v_1^r,s_1^r)$$

and $q_2(i,x_2',v_2^r,s_2^r) \le q_2(i,x_2^r,v_2^r,s_2^r)$.

Then we have for the overall problem:

$$q(i,m') \le q(i,m^r).$$

The proof of the corollary is straightforward.

A procedure was derived by Takahara [5] based on the preceeding theories.

<u>Procedure (1^{st} stage only):</u>

1. Given the system without any correlation, (we make $c_{12} = c_{21} = 0$), we compute the optimal solution, which we call: $x_1^r(0,t)$, $x_2^r(0,t)$, $m_1^r(0,t)$, $m_2^r(0,t)$ with

reference control → r

$x_1^r\ (0,t)$ ← time

subsystem 1 → 1; stage 0 → 0

In this part 1, the problem is divided in two independent sub problems.

2. We compute

$$v_1^r\ (0,t) = c_{12}\ x_2^r\ (0,t)$$

$$s_1^r\ (0,t) = B_2\ c_{21}\ m_2^r\ (0,t) + c_{21}^2\ B_2\ [x_1^r\ (0,t) - r_1].$$

3. <u>subsystem 1:</u>

$$\begin{cases} m_1(t) = \frac{dx_1}{dt} - c_{11}\, x_1 - v_1^r \\ q_1(0, x_1, v_1^r, s_1^r) = \phi_1(0, x_1, m_1) = \int_{t_s(0)}^{t_e} [(A_1 + c_{21}^2 B_2)(x_1 - r_1)^2 + B_1 m_1^2 - 2x_1 s_1^r]\, dt \end{cases}$$

$$\min_{m_1} \phi_1(0, x_1, m_1) = \phi_1(0, x_1', m_1^0)$$

4. Compute:

$$v_2^r(0,t) = c_{21}\, x_1'$$

$$s_2^r(0,t) = B_1\, c_{12}\, [\frac{dx_1'}{dt} - c_{11}\, x_1' - c_{12}\, x_2^r] + c_{12}^2\, B_1\, (x_2^r - r_2).$$

5. <u>subsystem 2:</u>

$$m_2(t) = \frac{dx_2}{dt} - c_{22}\, x_2 - v_2^r$$

$$q_2(0, x_2, v_2^r, s_2^r) = \phi_2(0, x_2, m_2) = \int_{t_s(0)}^{t_e} [(A_2 + c_{12}^2 B_1)(x_2 - r_2)^2 + B_2 m_2^2 - 2x_2 s_2^r]\, dt$$

$$\min_{m_2} \phi_2(0, x_2, m_2) = \phi_2(0, x_2', m_2^0)$$

6. Computation of $m_1'(0,t)$, $m_2'(0,t)$.

$$m_1'(0,t) = \frac{dx_1'}{dt} - c_{11}\, x_1' - c_{12}\, x_2'$$

$$m_2'(0,t) = \frac{dx_2'}{dt} - c_{21}\, x_1' - c_{22}\, x_2'$$

7. Computation of the new performance:

$$q[m_1'(0,t), m_2'(0,t)] = A_1(x_1'-r_1)^2 + A_2(x_2'-r_2)^2 + B_1\, m_1'^2 + B_2\, m_2'^2$$

And at this stage we have:

$$q[m_1'(0,t), m_2'(0,t)] \leq q[m_1^r(0,t), m_2^r(0,t)]$$

And the procedure can be applied again with a new reference control; in a new stage:

8. Computation of $m_1^r(1,t)$, $m_2^r(1,t)$:

Given the feedback information

$$Sm_1(i-1,t)$$
$$Sm_2(i-1,t)$$

consider the control:

$$m^r(1,t) = \lambda\ Sm^r(0,t) + (1-\lambda)\ Sm(0,t)$$

the system

$$\frac{dx^r}{dt}(1,t) = C\ x^r\ (1,t) + m^r\ (1,t)$$

the performance

$$\phi = \int_{t_s(1)}^{t_e} \{[x^r(1,t)-r]^T A[x^r(1,t)-r] + m^{r^T}(1,t)\ B\ m^r(1,t)\}\ dt$$

Find λ such that $\min\limits_{\lambda} \phi$

which gives the new reference control for the next stage:

$$m^r(1,t) = \begin{pmatrix} m_1^r \ (1,t) \\ \\ m_2^r \ (1,t) \end{pmatrix}$$

Now the cycle is complete: we can start the computation again.

Survey of the procedure:

The procedure is a two level, multistage, updating procedure:

1st level:

steps 1, 3, 5: computation of a better control for the subsystems.

2nd level:

steps 2, 4: computation of v_1^r, v_2^r, s_1^r, s_2^r.

steps 6, 7: computation of the new control, and the new performance.

step 8: updating.

2.5 Application of the Procedure to 2 Numerical Examples

Example 1

Let us consider the following integrated system and performance functional:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$$

$$\phi(x,m) = \int_0^1 \{10\ (x_1^2 + x_2^2) + m_1^2 + m_2^2\}\ dt$$

$$x_1(0) = 5 \qquad \text{and} \qquad x_2(0) = 2.$$

Then

$$c = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \quad A = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The procedure is the following:

Step 1

part a

$$\begin{cases} \text{system} & \dot{x}_1 = 2\ x_1 + m_1 \\ \text{performance} & \phi_1\ (x_1, m_1) = 10\ x_1^2 + m_1^2 \\ \underset{m_1}{\text{Min}} & \phi_1\ (x_1, m_1) = \phi_1[x_{1F}(0,t),\ m_1^r(0,t)] \end{cases}$$

part b

$$\begin{cases} \text{system} & \dot{x}_2 = 3\ x_2 + m_2 \\ \text{performance} & \phi_2\ (x_2, m_2) = 10\ x_2^2 + m_2^2 \\ \underset{m_2}{\text{min}} & \phi_2\ (x_2, m_2) = \phi_2[x_{2F}(0,t),\ m_2^r(0,t)]. \end{cases}$$

part c $$\frac{dx_1^r}{dt} = 2\ x_1^r + 2\ x_2^r + m_1^r\ (0,t)$$

$$\frac{dx_2^r}{dt} = 2\ x_1^r + 3\ x_2^r + m_2^r\ (0,t)$$

Step 2

$$v_1^r\ (0,t) = 2\ x_2^r\ (0,t)$$

$$s_1^r\ (0,t) = 2\ m_2^r\ (0,t) + 4\ [x_1^r\ (0,t) - r_1]$$

Step 3

system $$\frac{dx_1}{dt} = 2\ x_1 + m_1 + v_1^r\ (0,t)$$

performance $$\phi_1[0,x_1,m_1] = \int_0^{t_e}[14(x_1-r_1)^2 + m_1^2 - 2\ x_1 s_1^r]\ dt$$

$$\min_{m_1}\ \phi_1[0,x_1,m_1] = \phi_1[0,x_1',m_1^0]$$

Step 4

$$v_2^r\ [0,t] = 2\ x_1'$$

$$s_2^r\ (0,t) = 2[\frac{dx_1'}{dt} - 2\ x_1' - 2\ x_2^r] + 4\ (x_2^r - r_2]$$

Step 5

system $$\frac{dx_2}{dt} = 3\ x_2 + m_2 + v_2^r\ (0,t)$$

performance $$\phi_2(0,x_2,m_2) = \int_0^{t_e}\{14(x_2-r_2)^2 + m_2^2 - 2\ x_2 s_2^r\}\ dt$$

$$\min_{m_2}\ \phi_2(0,x_2,m_2) = \phi_2(0,x_2',m_2^0).$$

Step 6

$$\frac{dx_1'}{dt} = 2\ x_1' + 2\ x_2' + m_1'\ (0,t)$$

$$\frac{dx_2'}{dt} = 2\ x_1' + 3\ x_2' + m_2'\ (0,t)$$

Step 7

$$q[m_1'(0,t), m_2'(0,t)] = \int_0^{t_S} \{10(x_1'-r_1)^2 + 10(x_2'-r_2)^2 + m_1'^2 + m_2'^2\} dt$$

Step 8

Given the feedback information

$Sm_1\ (i-1,t)$

$Sm_2\ (i-1,t)$

system $\quad \frac{dx^r}{dt}(1,t) = C\ x^r(1,t) + m^r\ (1,t)$

control $\quad m^r\ (1,t) = \lambda\ Sm^r(0,t) + (1-\lambda)\ Sm(0,t)$

performance $\quad \phi = \int_{t_S(1)}^{t_e} \{A_1(x_1^r(1,t)-r_1)^2 + A_2(x_2^r(1,t) - r_2)^2 + B_1 m_1^r(1,t) + B_2 m_2^r(1,t)\}\ dt$

Find λ_0 such that

$$\min_{\lambda} \quad \phi\ (\lambda) = \phi\ [\lambda_0]$$

Computation:

Step 1:

-part a:

$$H = 10\ x_{1F}^2 + (m_1^r)^2 + p_1\ (2\ x_{1F} + m_1^r)$$

$\mathcal{A} = 2 \quad \mathcal{E} = -0.5 \qquad w_1 = \frac{1}{2} \qquad \delta Hm_1 \quad \mathcal{Y} = -20 \qquad w_2 = 0$

-part b:

$$H = 10\ x_{2F}^2 + (m_2^r)^2 + p_2\ (3\ x_{2F} + m_2^r)$$

$\mathcal{A} = 3 \quad \mathcal{E} = -0.5 \qquad w_1 = 0.5 \quad \delta Hm_{2r} \quad \mathcal{Y} = -20 \qquad w_2 = 0$

Step 3:

$$H = 14(x_1'-r_1)^2 + (m_1^0)^2 - 2\ x_1'\ s_1^r + q_1(2x_1'+m_1^0+v_1^r)$$

$\mathcal{A} = 2 \quad \mathcal{E} = -0.5 \qquad w_1 = 0.5 \quad \delta Hm_{11} \quad \mathcal{Y} = -28 \qquad w_2 = 0$

Step 5:

$$H = 14(x_2'-r_2)^2 + (m_2^0)^2 - 2\ x_2'\ s_2^r + q_2(3x_2'+m_2^0+v_2^r)$$

$\mathcal{A} = 3 \quad \mathcal{E} = -0.5 \qquad w_1 = 0.5 \quad \delta Hm_{22} \quad \mathcal{Y} = -28 \qquad w_2 = 0$

Example 2: A linear system with 5 variables:

The most general system is:

$$\frac{dx_i}{dt} = \sum_{j=1}^{5} c_{ij}\ x_j + m_i \qquad i = 1,2,3,4,5.$$

$$\phi = \int_0^{t_e} \sum_{j=1}^{5} [A_j(x_j - r_j)^2 + B_j\ m_j^2]\ dt.$$

We break the system in two parts:

$$\mathcal{X}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \mathcal{X}_2 = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

$$\mathcal{M}_1 = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \qquad \mathcal{M}_2 = \begin{vmatrix} m_1 \\ m_2 \\ m_3 \end{vmatrix}$$

$$\mathcal{C}_{11} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \quad \mathcal{C}_{12} = \begin{bmatrix} c_{12} & c_{14} & c_{15} \\ c_{23} & c_{24} & c_{25} \end{bmatrix}$$

$$\mathcal{C}_{21} = \begin{bmatrix} c_{31} & c_{32} \\ c_{41} & c_{42} \\ c_{51} & c_{52} \end{bmatrix} \quad \mathcal{C}_{22} = \begin{bmatrix} c_{33} & c_{34} & c_{35} \\ c_{43} & c_{44} & c_{45} \\ c_{53} & c_{54} & c_{55} \end{bmatrix}$$

$$\mathcal{A}_1 = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \qquad \mathcal{A}_2 = \begin{bmatrix} A_3 & 0 & 0 \\ 0 & A_4 & 0 \\ 0 & 0 & A_5 \end{bmatrix}$$

$$\mathcal{B}_1 = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \qquad \mathcal{B}_2 = \begin{bmatrix} B_3 & 0 & 0 \\ 0 & B_4 & 0 \\ 0 & 0 & B_5 \end{bmatrix}$$

The system can now be written:

$$\frac{d\mathcal{X}_1}{dt} = \mathcal{C}_{11}\ \mathcal{X}_1 + \mathcal{C}_{12}\ \mathcal{X}_2 + \mathcal{M}_1$$

$$\frac{d\mathcal{X}_2}{dt} = \mathcal{C}_{21}\ \mathcal{X}_1 + \mathcal{C}_{22}\ \mathcal{X}_2 + \mathcal{M}_2$$

We can compute now:

$$v_1^r = \mathcal{C}_{12}\ \mathcal{X}_2^r \qquad v_2^r = \mathcal{C}_{21}\ \mathcal{X}_1^r$$

$$s_1^r = \mathcal{C}_{21}^T\ \mathcal{B}_2\ \mathcal{M}_2^r + \mathcal{C}_{21}^T\ \mathcal{B}_2\ \mathcal{C}_{21}\ (\mathcal{X}_1^r - \mathcal{K}_1)$$

$$s_2^r = \mathcal{C}_{12}^T\ \mathcal{B}_1\ \mathcal{M}_1^r + \mathcal{C}_{12}^T\ \mathcal{B}_1\ \mathcal{C}_{12}\ (\mathcal{X}_2^r - \mathcal{K}_2)$$

Now we consider the two subsystems:

1st subsystem:

$$\frac{d\mathcal{X}_1}{dt} = \mathcal{C}_{11}\ \mathcal{X}_1 + v_1^r$$

$$\min_{\mathcal{X}_1} \quad \phi_1 = \int_0^{t_e} \{ (\mathcal{X}_1 - \mathcal{K}_1)^T [\mathcal{A}_1 + \mathcal{C}_{21}^T\ \mathcal{B}_2\ \mathcal{C}_{21}][\mathcal{X}_1 - \mathcal{K}_1] + \mathcal{M}_1^T\ \mathcal{B}_1\ \mathcal{M}_1 - 2\ s_1^{r^T}\ \mathcal{X}_1 \} dt$$

2nd subsystem:

$$\frac{d\mathcal{X}_2}{dt} = \mathcal{C}_{22}\ \mathcal{X}_2 + v_2^r$$

$$\min_{\mathcal{X}_2} \quad \phi_2 = \int_0^{t_e} \{ (\mathcal{X}_2 - \mathcal{K}_2)^T [\mathcal{A}_2 + \mathcal{C}_{12}^T\ \mathcal{B}_1\ \mathcal{C}_{12}][\mathcal{X}_2 - \mathcal{K}_2] + \mathcal{M}_2^T\ \mathcal{B}_2\ \mathcal{M}_2 - 2\ s_2^{r^T}\ \mathcal{X}_2 \} dt$$

For the computation, with the successive sweep method, we get:

1st subproblem:

$$\mathcal{A} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \quad \mathcal{E} = \begin{bmatrix} -\frac{1}{2}B_1 & 0 \\ 0 & -\frac{1}{2}B_2 \end{bmatrix} \quad w_1 = \begin{bmatrix} \frac{1}{2}B_1 & \delta Hm_1 \\ \frac{1}{2}B_2 & \delta Hm_2 \end{bmatrix}$$

$$\mathcal{L}= \begin{bmatrix} -2[A_1+B_3c_{31}^2+B_4c_{41}^2+B_5c_{51}^2 & -2[B_3c_{31}c_{32}+B_4c_{41}c_{42} \\ & +B_5c_{51}c_{52}] \\ -2[B_3c_{31}c_{32}+B_4c_{41}c_{42}+B_5c_{51}c_{52} & -2[A_2+B_3c_{32}^2+B_4c_{42}^2 \\ & +B_5c_{52}^2 \end{bmatrix}$$

$$w_2= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2<u>nd</u> subproblem:

$$\mathcal{D}= \begin{bmatrix} c_{33} & c_{34} & c_{35} \\ c_{43} & c_{44} & c_{45} \\ c_{53} & c_{54} & c_{55} \end{bmatrix} \quad \mathcal{E}= \begin{bmatrix} -\frac{1}{2}B_3 & 0 & 0 \\ 0 & -\frac{1}{2}B_4 & 0 \\ 0 & & -\frac{1}{2}B_5 \end{bmatrix}$$

$$\mathcal{L}= \begin{bmatrix} -2[A_3+B_1c_{13}^2+B_2c_{23}^2] & -2[B_1c_{13}c_{14}+B_2c_{23}c_{24}] \\ & -2[B_1c_{13}c_{15}+B_2c_{23}c_{25}] \\ -2[B_1c_{13}c_{14}+B_2c_{23}c_{24}] & -2[A_4+B_1c_{14}^2+B_2c_{24}^2] \\ & -2[B_1c_{14}c_{15}+B_2c_{24}c_{25}] \\ -2[B_1c_{13}c_{15}+B_2c_{23}c_{25}] & -2[B_1c_{14}c_{15}+B_2c_{24}c_{25}] \\ & -2[A_5+B_1c_{15}^2+B_2c_{25}^2] \end{bmatrix}$$

$$w_1 = \begin{bmatrix} \frac{1}{2}B_3 & \delta Hm_3 \\ \frac{1}{2}B_4 & \delta Hm_4 \\ \frac{1}{2}B_5 & \delta Hm_5 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The program of the computation is shown in Appendix 3, Program 1.

The results of this satisfaction approach are shown and compared to a new heuristic in the following chapter.

CHAPTER III

A HEURISTIC FOR A SATISFACTION APPROACH WITH SEPARATE SUBSYSTEMS

3.1 The Method:

Goal of the method:

In the last chapter we saw that the general technique of the satisfaction approach is rather complex when applied to linear systems. It is possible, at least when dealing with difference equations, to find a Heuristic much simpler and with the same advantages as the satisfaction approach.

What are the advantages of the application of the satisfaction approach with deterministic systems? Given a reference control, the technique allows us to find a control which improves the performance of the overall system, and this only by dealing with separate subsystems and considering the interconnections as uncertainties. The reduction of dimensionality and the fact that the computation is done in one iteration are the basic interests of this method. The latter is the one which makes the satisfaction approach different from the decomposition technique. The decomposition technique gets the overall optimum but this needs a second iterative level.

A single Heuristic can be implemented which has these advantages:

Consider the following discrete linear system:

$$x(t_{i+1}) = x(t_i) + h[A\ x(t_i) + Bm(t_i)]$$

with the t_i such that

$$t_0 = 0 < t_1 < t_2 < \dots < t_i < t_{i+1} < \dots < t_m = t_e$$

and

$$t_{i+1} - t_i = h$$

with h greater than 0

and the performance functional:

$$\phi = \phi\ [\ x(t_0),\ \dots,\ x(t_m),\ m(t_0)\ \dots\ m(t_m)\ ].$$

We call $x(t_0)$ the vector $(x_1(t_0) \dots x_n(t_0))$

x_j the array $(x_j(t_0),\ x_j(t_1) \dots x_j(t_m))$

$m(t_0)$ the vector $(m_1(t_0) \dots m_n(t_0))$

m_j the array $(m_j(t_0),\ m_j(t_1) \dots m_j(t_m))$

We make the following assumptions:

-Assumption 1:

B is non singular.

So we can express $m(t_i)$ as a function of $x(t_i)$ and $x(t_{i+1})$

$$m(t_i) = B^{-1}\ [\ \frac{x(t_{i+1}) - x(t_i)}{h} - A\ x(t_i)\ ]$$

If we replace $m(t_i)$ by its value in function of $x(t_i)$ and $x(t_{i+1})$ in the functional performance, we will get a functional:

$$\phi\,[x(t_0),\ \ldots,\ x(t_m),\ m(t_0),\ \ldots,\ m(t_m)]$$
$$= \phi'\,[x(t_0),\ \ldots,\ x(t_m)]$$
$$= \phi''\,[x_1,\ \ldots,\ x_n].$$

-Assumption 2:

ϕ'' is a continuous, non negative function with respect to the x_j, and it has Frechet derivatives.

-Assumption 3:

$\phi''\,[x_1, \ldots, x_n]$ is convex and it takes its unique critical point in the interior of M.

-Assumption 4:

$$X \text{ is convex and } X = X_1 \oplus X_2 \oplus \ldots \oplus X_n$$

with X = set of the arrays of vectors $(x(t_0), \ldots, x(t_m))$

X_1 = set of the arrays: $(x_1(t_0), \ldots, x_1(t_m))$

X_2 = set of the arrays: $(x_2(t_0), \ldots, x_2(t_m))$

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X_n = set of the arrays: $(x_n(t_0), \ldots, x_n(t_m))$.

-Assumption 5:

$$\forall_j,\ \forall\ x_1^0,\ \ldots,\ x_{j-1}^0,\ x_{j+1}^0,\ \ldots,\ x_n^0,$$

if

$$\min_{x_j \varepsilon X_j} \phi\,[x_1^0,\ \ldots,\ x_{j-1}^0,\ x_j,\ x_{j+1}^0,\ \ldots,\ x_n^0] =$$

$$\phi\,[x_1^0,\ \ldots,\ x_{j-1}^0,\ x_j^0,\ x_{j+1}^0,\ \ldots,\ x_n^0],$$

then

$$x_j^o \in \text{Int}\ (x_j).$$

With these assumptions, the procedure is the following:

Given a reference control $m_1^r, \ldots, m_n^r$ and the corresponding trajectories $x_1^r, \ldots, x_n^r$ and performance $\phi''\ (x_1^r, \ldots, x_n^r)$

Step 1:

We consider the function

$$f_1(x_1) = \phi''\ (x_1, x_2^r, \ldots, x_n^r)$$

By assumption 2, $f_1(x_1)$ is a continuous function of x_1. By assumption 4, if

$$x_1 \in X_1,\ (x_1, x_2^r, \ldots, x_n^r) \in X.$$

So we can compute x_1' such that:

$$\min_{x_1 \in X_1} \quad f_1(x_1) = f_1(x_1')$$

By assumption 4, we know that $(x_1', x_2^r, \ldots, x_n^r) \in X$. Further, by assumption 5, we can say that $(x_1', x_2^r, \ldots, x_n^r)$ is not a boundary point of X. So we can say:

$$\phi''\ (x_1', x_2^r, \ldots, x_n^r) < \phi''\ (x_1^r, x_2^r, \ldots, x_n^r) \qquad \text{if } x_1' \neq x_1^r$$

or

$$\frac{\partial\ \phi''\ (x_1', \ldots, x_n^r)}{\partial\ x_1} = 0 \quad \text{if } x_1' = x_1^r$$

We are using too assumption 2 (ϕ'' has derivatives).

Step 2:

We consider now the function:

$$f_2(x_2) = \phi''\ (x_1',\ x_2,\ x_3^r,\ \ldots,\ x_n^r)$$

In the same way $f_2(x_2)$ is a continuous function of x_2 and

$$x_2 \in X_2 \rightarrow (x_1',\ x_2,\ x_3^r,\ \ldots,\ x_n^r) \in X.$$

We cumpute x_2' such that:

$$\min_{x_2 \in X_2} \quad f_2(x_2) = f_2(x_2')$$

We can say again that: $(x_1',\ x_2',\ x_3^r,\ \ldots,\ x_n^r) \in X$

and that: $(x_1',\ x_2',\ x_3^r,\ \ldots,\ x_n^r) \in \text{Int } X$

$$\phi''(x_1',x_2',x_3^r,\ \ldots,\ x_n^r) < \phi''(x_1',x_2^r,x_3^r \ \ldots\ x_n^r)$$

$$\text{if } x_2' \neq x_2^r.$$

or

$$\frac{\partial\phi''}{\partial x_2}\ (x_1',x_2',x_3^r,\ \ldots,\ x_n^r) = 0 \qquad \text{if } x_2' = x_2^r.$$

We compute n steps in the same way. We have n problems of optimal control, each of which is only one dimensional.

At the $n^{\underline{th}}$ step we have:

either

$$\int \phi''\ [x_1',\ \ldots.,\ x_n'] < \phi''\ [x_1^r,\ \ldots.,\ x_n^r]$$

with $(x_1', \ldots, x_n') \in X$ and $(x_1', \ldots, x_n') \in \text{Int } X$.

or

$$\frac{\partial \phi''(x_1' \ldots x_n')}{\partial x_1} = \frac{\partial \phi''(x_1' \ldots x_n')}{\partial x_2} = \ldots = \frac{\partial \phi''(x_1' \ldots x_n')}{\partial x_n} = 0$$

if $\quad x_1' = x_1^r \;\&\; x_2' = x_2^r \;\&\; \ldots \;\&\; x_n' = x_n^r$

In the first case we improve the performance.

In the second, we got the same point but we know that the reference control is a local minimum, and hence, by assumption 3, the optimal control.

<u>Iteration of the technique:</u>

At the end of the procedure we improve the performance or we are at the optimum. If we are not at the optimum and if we iterate the procedure we will get

$$0 \leq \ldots < \phi''(x_1^k, x_2^k, \ldots, x_n^k) < \phi''(x_1^{k-1}, x_2^{k-1}, \ldots, x_n^{k-1}) < \ldots \ldots < \phi''(x_1^r, \ldots x_n^r).$$

So the iteration will give a sequence of performance functionals. This sequence is monotone decreasing, and has a lower bound (assumption 4). So this sequence is convergent. To compute the limit $\phi(x_1, \ldots, x_n)$ we have to write:

$$\phi''(x_1^k,\ldots,x_n^k) = \phi''(x_1^{k-1},\ldots,x_n^{k-1}) = \phi''(x_1^{\ell},\ldots,x_n^{\ell}).$$

But by assumption 5, this equality means that:

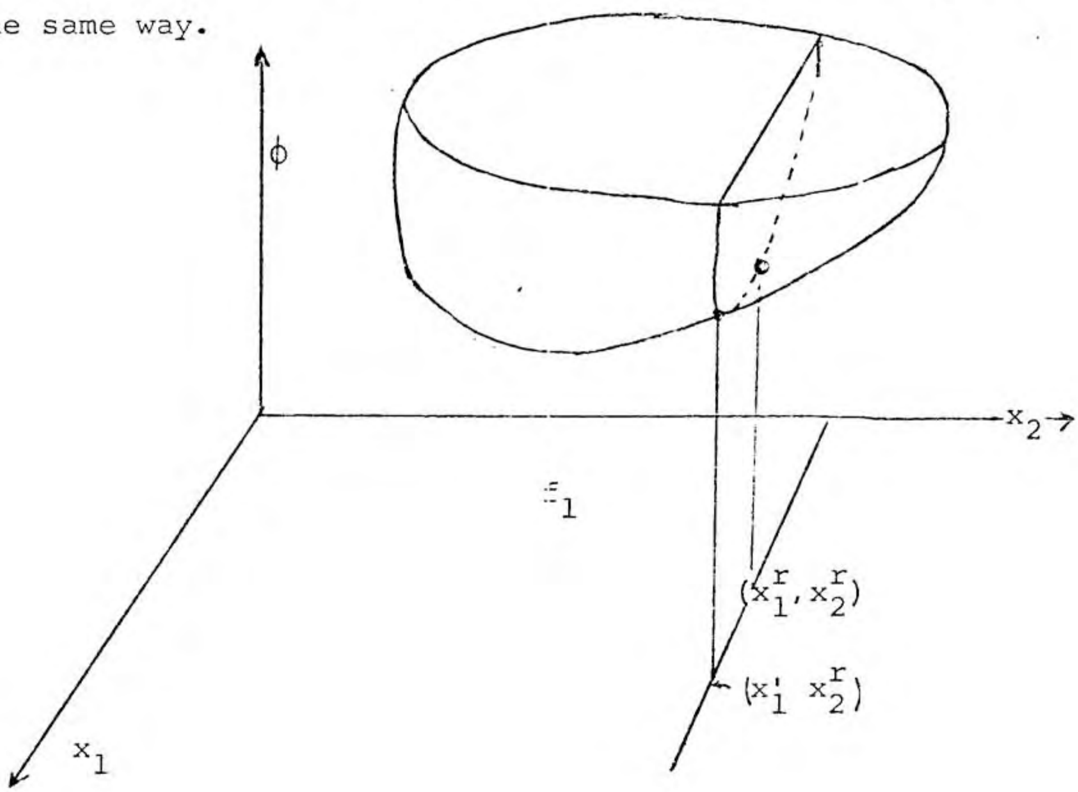
$$\rightarrow (x_1^k,\ldots,x_n^k) = (x_1^{k-1},\ldots,x_n^{k-1}) = (x_1^{\ell},\ldots,In^{\ell})$$

which means, by the preceeding demonstration that:

$$(x_1^{\ell},\ldots,x_n^{\ell})$$

is the global minimum of the performance functional.

The geometrical interpretation of this iterative heuristic is very simple. Consider the curve $f_1(x_1)$ section of the surface $z = \phi''(x_1 \ldots x_n)$ by the curves $x_2 = x_2^r, \ldots, x_n = x_n^r$. Then consider on this curve the minimum of ϕ'' call it x_1', and proceed again with x_2 in the same way.



-Comparison of this technique with iteration and the decomposition technique:

Suppose a linear discrete problem with n state variables. Suppose the performance index is additively separable, and that we want to deal only with one state variable sub-processes. Suppose B has an inverse. We have:

$$\phi = \phi\ [x_1, \ldots, x_n, m_1, \ldots, m_n]$$

$$\frac{x(t_{i+1}) - x(t_i)}{h} = C\ x(t_i) + Bm(t_i)$$

Since B has an inverse we can write the system in the following way:

$$\begin{cases} \phi = \phi''\ [x_1, \ldots, x_n] \\ \dfrac{x_k(t_{i+1}) - x_k(t_i)}{h} = \sum\limits_j c_{kj}\ x_k\ (t_i) + B_{kk}\ m_k(t_i) \end{cases}$$

or

$$\begin{cases} \dfrac{x_1(t_{i+1}) - x_1(t_i)}{h} = c_{11}\ x_1(t_i) + c_{12}\ x_2(t_i) + \ldots + c_{1n}\ x_n(t_i) \\ \qquad\qquad + B_{11}\ m_1(t_i) \\ \vdots \\ \dfrac{x_n(t_{i+1}) - x_n(t_i)}{h} = c_{n1}\ x_1(t_i) + \ldots + c_{nn}\ x_n(t_i) \\ \qquad\qquad + B_{nn}\ m_n(t_i) \end{cases}$$

The decomposition technique will proceed as follows:

$$x_1 = s_1, \;\ldots\ldots,\; x_n = s_n$$

<u>1st level:</u>

1st sub system

$$\begin{cases} \dfrac{x_1(t_{i+1}) - x_1(t_i)}{h} = c_{11}\, x_1(t_i) + c_{12}\, s_2(t_i) + \cdots + c_{1n}\, s_n(t_i) + B_{11}\, m_1(t_i) \\[2ex] \min\limits_{m_1, s_2, \ldots, s_n} \phi_1''[x_1, s_2, \ldots, s_n, K_1, \ldots, K_n] = \phi_1''^{0} \end{cases}$$

$$\vdots$$

nth sub system

$$\begin{cases} \dfrac{x_n(t_{i+1}) - x_n(t_i)}{h} = c_{n1}\, s_n(t_i) + \cdots + c_{nn}\, x_n(t_i) + B_{nn}\, m_n(t_i) \\[2ex] \min\limits_{m_n, s_1, \ldots, s_{n-1}} \phi_n''[s_1, \ldots, s_{n-1}, x_n, K_1, \ldots, K_n] = \phi_n''^{0} \end{cases}$$

<u>2nd level:</u>

$$\underset{K_1, \ldots, K_n}{\text{maximize}} \quad \varphi = \phi_1''^{0} + \phi_2''^{0} + \cdots + \phi_n''^{0} = \varphi^0$$

Once the min-max problem is solved (by a multilevel iteration procedure) we get:

$$\varphi^0 = \phi''^{0} \text{ and } x_1 = s_1,\; x_2 = s_2,\; \ldots\ldots,\; x_n = s_n.$$

The technique here is to deal with a min-max problem and with subsystems which actually do not represent anything for the real system itself. It adapts these subsystems (by the mean of the K's) in such a way that their optimal points correspond to the optimal real subsystems of the actual problem. But the coincidence takes place only at the optimum.

The heuristic proposed will proceed as follows:

$i^{\underline{th}}$ stage of the iteration:

$1^{\underline{st}}$ sub system

$$\begin{cases} \min_{m_1} \ \phi_1''^{i} = \phi''[x_1^i, x_2^{i-1}, x_3^{i-1}, \ldots, x_n^{i-1}] \\ \dfrac{x_1^i(t_{j+1}) - x_1^i(t_j)}{h} = c_{11}\, x_1^i(t_j) + c_{12}\, x_1^{i-1}(t_j) + \ldots + c_{1n}\, x_n^{i-1}(t_j) + B_{11}\, m_1^i(t_j) \end{cases}$$

$n^{\underline{th}}$ sub system

$$\begin{cases} \min_{m_n} \ \phi_n''^{i} = \phi''\, [x_1^i, x_2^i, \ldots, x_n^i] \\ \dfrac{x_n^i(t_{j+1}) - x_n^i(t_j)}{h} = c_{n1}\, x_1^i(t_j) + c_{n2}\, x_2^i(t_j) + \ldots + c_{nn}\, x_n^i(t_j) + B_{nn}\, m_n^i(t_j) \end{cases}$$

After the last $N^{\underline{th}}$ iteration we have to compute the controls $m_1^N \ldots m_n^N$.

The latter method is an iteration technique: with n subsystems. But it should be noted:

1) For each subsystem the minimization of the performance is to be computed with respect to one variable, instead of (n+1) in the decomposition technique.

2) We do not have a $2^{\underline{nd}}$ level but an iteration technique, which means an easier programming.

3) On the whole, the effectiveness of this heuristic is dependent upon the form of the performance. This heuristic can give the optimal point in one iteration, while with some other surfaces the convergence might be very slow.

3.2 Application to Linear Quadratic Systems:

system
$$\begin{cases} \dfrac{x_1(t_{i+1}) - x_1(t_i)}{h} = c_{11}\, x_1(t_i) + c_{12}\, x_2(t_i) + m_1(t_i) \\ \\ \dfrac{x_2(t_{i+1}) - x_2(t_i)}{h} = c_{21}\, x_1(t_i) + c_{22}\, x_2(t_i) + m_2(t_i) \end{cases}$$

performance

$$\phi = \sum_{i=0}^{m} \{A_1[x_1(t_i)-r_1]^2 + A_2[x_2(t_i)-r_2]^2 + B_1\, m_1^2(t_i) + B_2\, m_2^2(t_i)\}\, h$$

global method:

$$H = A_1[x_1(t_i)-r_1]^2 + A_2[x_2(t_i)-r_2]^2 + B_1\, m_1^2(t_i) + B_2\, m_2^2(t_i) + p_2[c_{21}(t_i)\, x_1(t_i) - c_{22}(t_i)\, x_2(t_i) + m_2(t_i)]$$

We get:

$$\mathcal{A} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \quad \mathcal{E} = \begin{bmatrix} -\frac{1}{2B_1} & 0 \\ 0 & -\frac{1}{2B_2} \end{bmatrix} \quad w_1 = \begin{bmatrix} \frac{1}{2B_1} & \delta Hm_1 \\ \frac{1}{2B_2} & \delta Hm_2 \end{bmatrix}$$

$$\mathcal{L} = \begin{bmatrix} -2A_1 & 0 \\ 0 & -2A_2 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The program of the computation is shown in Appendix 4, Program 2.

Use of the Heuristic:

We start with a reference control m_1^r, m_2^r. We compute the corresponding trajectory x_1^r, x_2^r by:

$$\frac{x_1^r(t_{i+1}) - x_1^r(t_i)}{h} = c_{11}\, x_1^r(t_i) + c_{12}\, x_2^r(t_i) + m_1^r(t_i)$$

$$\frac{x_2^r(t_{i+1}) - x_2^r(t_i)}{h} = c_{21}\, x_1^r(t_i) + c_{22}\, x_2^r(t_i) + m_2^r(t_i)$$

Then we have:

1$^{\underline{st}}$ subsystem:

$$\left\{ \begin{array}{l} \min\limits_{x_1=x_1'} \phi(x_1,x_2^r) = \sum\limits_{i=0}^{m} \{A_1[x_1(t_i)-r_1]^2 + A_2[x_2^r(t_i)-r_2]^2 \\ \qquad\qquad + B_1\, m_1^2(t_i) + B_2\, m_2^2(t_i)\}. \\ \text{with} \\ \left\{ \begin{array}{l} m_1(t_i) = \dfrac{x_1(t_{i+1}) - x_1(t_i)}{h} - c_{11}\, x_1(t_i) - c_{12}\, x_2^r(t_i) \\ m_2(t_i) = \dfrac{x_2^r(t_{i+1}) - x_2^r(t_i)}{h} - c_{21}\, x_1(t_i) - c_{22}\, x_2^r(t_i) \end{array} \right. \end{array} \right.$$

$2^{\underline{nd}}$ subsystem:

Given by x'_1 by the $1^{\underline{st}}$ system:

$$\begin{cases} \min\limits_{x_2=x'_2} \phi(x'_1, x_2) = \sum\limits_{i=0}^{m} \{A_1[x'_1(t_i)-r_1]^2 + A_2[x_2(t_i)-r_2]^2 \\ \qquad + B_1\, m_1^2(t_i) + B_2\, m_2^2(t_i)\}.\ h \\ \text{with} \\ \begin{cases} m_1(t_i) = \dfrac{x'_1(t_{i+1}) - x'_1(t_i)}{h} - c_{11}\, x'_1(t_i) - c_{12}\, x_2(t_i) \\ m_2(t_i) = \dfrac{x_2(t_{i+1}) - x_2(t_i)}{h} - c_{21}\, x'_1(t_i) - c_{22}\, x_2(t_i) \end{cases} \end{cases}$$

We then have either: $\phi\ (x'_1,\ x'_2) < \phi\ (x_1^r,\ x_2^r)$

or: $\left.\begin{array}{l} x'_1 = x_1^r \\ x'_2 = x_2^r \end{array}\right\}$ and the reference control is a local optimum.

We then compute the controls leading to these trajectories:

$$m'_1(t_i) = \frac{x'_1(t_{i+1}) - x'_1(t_i)}{h} - c_{11}\, x'_1(t_i) - c_{12}\, x'_2(t_i)$$

$$m'_2(t_i) = \frac{x'_2(t_{i+1}) - x'_2(t_i)}{h} - c_{21}\, x'_1(t_i) - c_{22}\, x'_2(t_i)$$

The program of the computation is shown in Appendix 4, Program 3.

3.3 Application of the Heuristic to Slightly Non-Linear Systems:

By slightly non linear system, we mean the following system:

$$\begin{cases} \dfrac{x_1(t_{i+1}) - x_1(t_i)}{h} = c_{11}\, x_1(t_i) + c_{12}\, x_2(t_i) \\ \qquad + D_{11}\, x_1^2(t_i) + D_{12}\, x_2^2(t_i) + m_1(t_i) \\ \dfrac{x_2(t_{i+1}) - x_2(t_i)}{h} = c_{21}\, x_1(t_i) + c_{22}\, x_2(t_i) \\ \qquad + D_{21}\, x_1^2(t_i) + D_{22}\, x_2^2(t_i) + m_2(t_i) \end{cases}$$

performance:

$$\phi = \sum_{i=0}^{m} \{A_1[x_1(t_i)-r_1]^2 + A_2[x_2(t_i)-r_2]^2 + B_1\, m_1^2(t_i) + B_2\, m_2^2(t_i)\}.\ h$$

The advantage of the heuristic is obvious from the following example. The computation of the coefficients in a successive sweep method for the overall problem is very complex, while, it is simple to do, with this heuristic:

subsystem 1:

$$\min_{x_1=x_1'} \phi\,(x_1, x_2^r) = \sum_{i=0}^{m} \{A_1[x_1(t_i)-r_1]^2 + A_2[x_2^r(t_i)-r_2]^2 + B_1\, m_1^2(t_i) + B_2\, m_2^2(t_i)\}.\ h$$

with

$$\begin{cases} \dfrac{x_1(t_{i+1}) - x_1(t_i)}{h} = c_{11}\, x_1(t_i) + c_{12}\, x_2^r(t_i) \\ \qquad + D_{11}\, x_1^2(t_i) + D_{12}\, (x_2^r(t_i))^2 \\ \qquad + m_1(t_i) \\ \dfrac{x_2^r(t_{i+1}) - x_2^r(t_i)}{h} = c_{21}\, x_1(t_i) + c_{22}\, x_2^r(t_i) \\ \qquad + D_{21}\, x_1^2(t_i) + D_{22}\, (x_2^r(t_i))^2 \\ \qquad + m_2(t_i) \end{cases}$$

<u>subsystem 2:</u>

Given x_1' from the subsystem 1:

$$\min_{x_2=x_2'} \phi\,(x_1', x_2) = \sum_{i=0}^{m} \{A_1[x_1'(t_i)-r_1]^2 + A_2[x_2(t_i)-r_2]^2 + B_1\, m_1^2(t_i) + B_2\, m_2^2(t_i)\}.\, h$$

with

$$\begin{cases} \dfrac{x_1'(t_{i+1}) - x_1'(t_i)}{h} = c_{11}\, x_1'(t_i) + c_{12}\, x_2(t_i) \\ \qquad + D_{11}\, (x_1'(t_i))^2 + D_{12}\, x_2^2(t_i) \\ \qquad + m_1(t_i) \\ \dfrac{x_2(t_{i+1}) - x_2(t_i)}{h} = c_{21}\, x_1'(t_i) + c_{22}\, x_2(t_i) \\ \qquad + D_{21}\, (x_1'(t_i))^2 + D_{22}\, x_2^2(t_i) \\ \qquad + m_2(t_i) \end{cases}$$

We then have $\phi\ (x_1', x_2') < \phi\ (x_1^r, x_2^r)$

or $\left.\begin{array}{l} x_1' = x_1^r \\ x_2' = x_2^r \end{array}\right\}$ and the reference control is a local minimum.

Application: Appendix 4, Program 4.

USE OF THE HEURISTIC OF CHAPTER III WITHOUT ITERATION

LINEAR SYSTEM

Initial performance	Performance after iteration
41.067 →	39.566
71.172 →	39.865
2 2 10 →	88.56
Optimal performance 39.560	

System

$$\begin{cases} \dfrac{x_1(t_{k+1}) - x_1(t_k)}{h} = [illegible]\, x_2(t_k) + m_1(t_k) \\ \dfrac{x_2(t_{k+1}) - x_2(t_k)}{h} = 2x_1(t_k) + 3x_2(t_k) + m_2(t_k) \end{cases}$$

Performance

$$\sum \{[illegible]\, x_1^2(t_k) + 10\, x_2^2(t_k) + m_1^2(t_k) + m_2^2(t_k)\} \cdot h$$

$h = 0.1$

NON LINEAR SYSTEM

Initial performance	Performance after one iteration
115.56 →	112.82
115.96 →	112.54
122.61 →	115.36
178.14 →	141.26
2165.5 →	154.60
2534 →	1552.2
optimal performance 112.82	

System

$$\begin{cases} \dfrac{x_1(t_{k+1}) - x_1(t_k)}{h} = 2x_1(t_k) + 10x_2(t_k) + [illegible] \\ \dfrac{x_2(t_{k+1}) - x_2(t_k)}{h} = 15x_1(t_k) + 10x_2(t_k) + [illegible] + m_2(t_k) \end{cases}$$

Performance

$$\sum \{10\, x_1^2(t_k) + 10\, x_2^2(t_k) + m_1^2(t_k) + m_2^2(t_k)\} h$$

3.4 Application of the Heuristic to a Linear System with 5 Variables. Computational Study of the Heuristic:

The system is the most general one:

$$\frac{x_i(t_{k+1}) - x_i(t_k)}{h} = \sum_{j=1}^{5} c_{ij}\, x_j(t_k) + m_i(t_k)$$

$$i = 1,2,3,4,5.$$

$$\phi = \sum_{k=0}^{m} \sum_{j=1}^{5} [A_j(x_j(t_k)-r_j)^2 + B_j\, m_j^2(t_k)].\ h$$

The computation at the $N^{\underline{th}}$ stage, $j^{\underline{th}}$ step is the following one:

$$\left\{\begin{array}{l} \dfrac{x_j^{(N)}(t_{k+1}) - x_j^{(N)}(t_k)}{h} = \displaystyle\sum_{1\le i\le j} c_{ji}\, x_i^{(N)}(t_k) \\ \qquad + \displaystyle\sum_{j<k\le 5} c_{jk}\, x_k^{(N-1)}(t_k) \\ \qquad + m_j^{(N)}(t_k) \\ \displaystyle\min_{x_j^{(N)}} \phi_j^{(N)} = \sum_{i=0}^{m} \{ \sum_{1\le i\le j} [A_i(x_i^{(N)}(t_k)-r_i)^2 \\ \qquad + B_i\, m_i^{(N)^2}(t_k)] \\ \qquad + \displaystyle\sum_{j<\ell\le 5} [A_\ell(x_\ell^{(N-1)}(t_k)-r_\ell)^2 \\ \qquad + B_\ell\, m_\ell^{(N-1)^2}(t_k)]\}.\ h \end{array}\right.$$

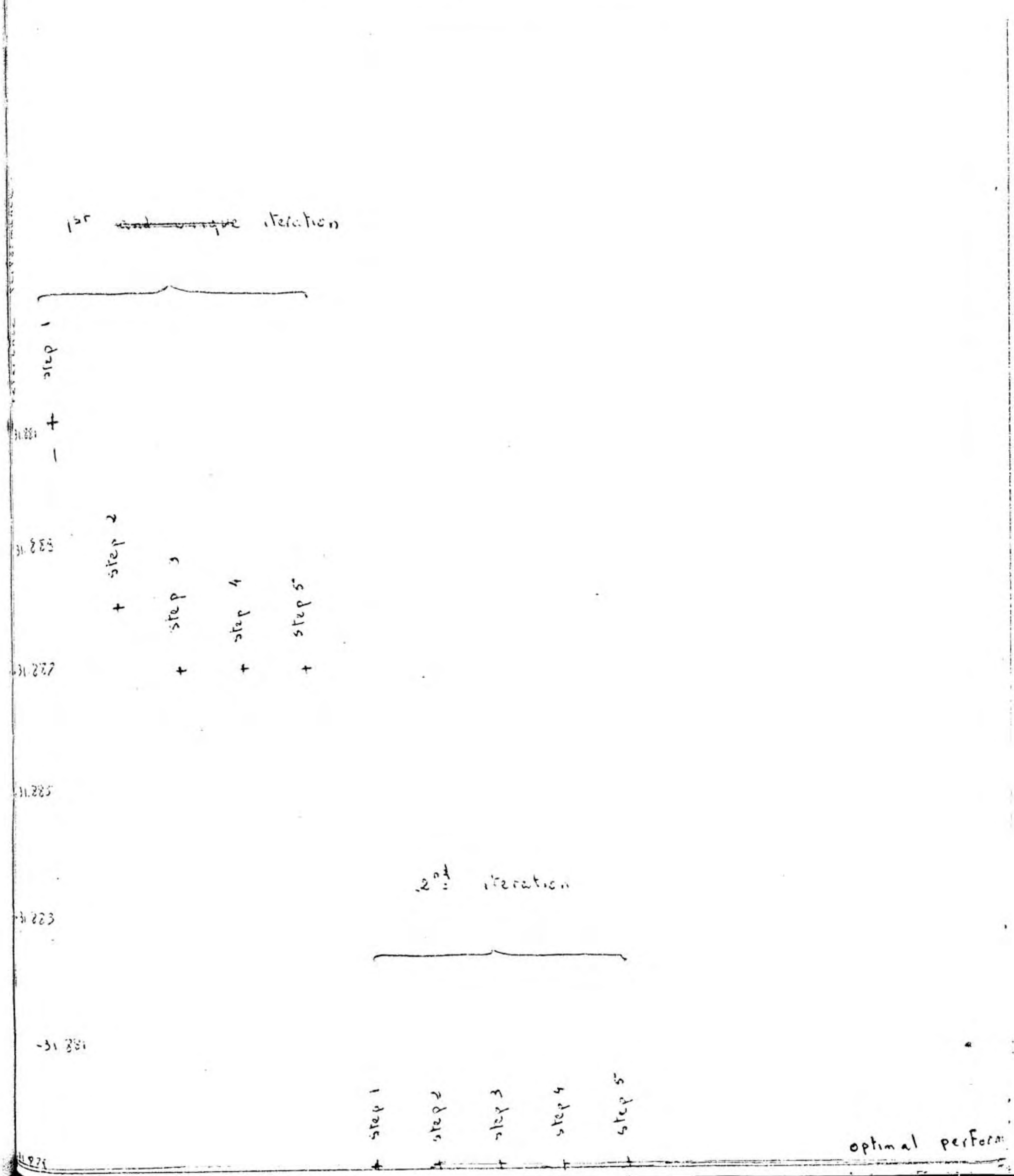
Program of computation: See Appendix 4, Program 5.

Performance with the reference control	Performance after application of the satisfaction approach	Number of iterations	Application of the Heuristic and iteration: step 1	step 2	step 3	step 4	step 5
31.891	31.879	1	31.891	31.888	31.887	31.887	31.887
		2	31.879	31.879	31.879	31.879	31.879
38.791	32.791	1	36.732	34.815	33.879	32.756	31.894
		2	31.892	31.881	31.880	31.879	31.879
		3	31.879	31.879	31.879	----	
731.29	33.67	1	531.36	351.79	255.68	164.42	32.865
		2	32.674	32.356	32.285	31.885	31.881
		3	31.879	31.879	31.879	31.879	31.879

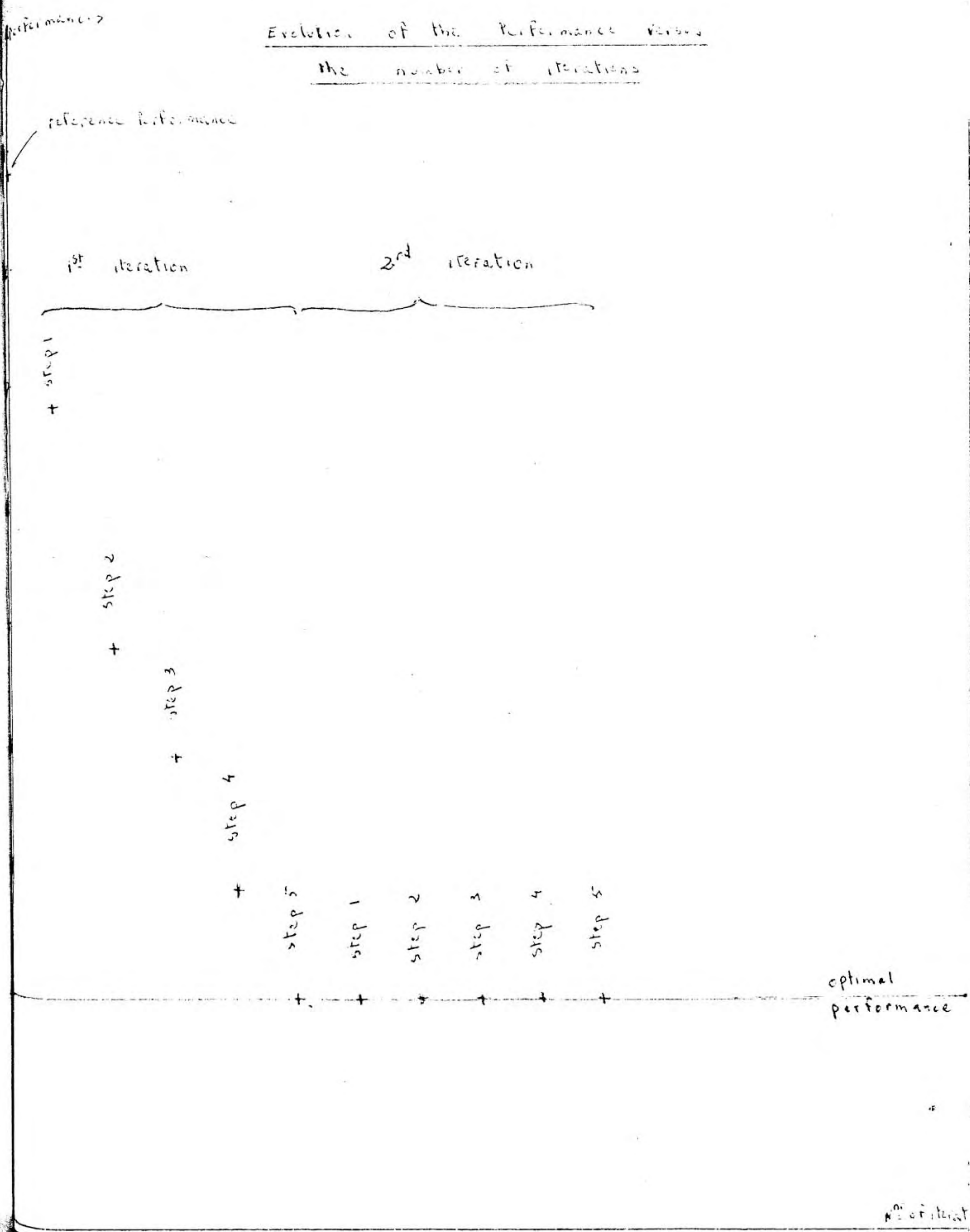
Iteration N°	Step N°	$\lvert H_{m1} \rvert$	$\lvert H_{m2} \rvert$	$\lvert H_{m3} \rvert$	$\lvert H_{m4} \rvert$	$\lvert H_{m5} \rvert$
1	1	8.175	.400	199.5	199.4	211.35
	2	-5.80	17.3	199.65	199.[illegible]	225.0
	3	5.80	12.8	5.06	192.7	229.7
	4	14.58	-17.66	4.77	12.93	232.60
	5	14.955	17.66	5.19	11.05	-0.004
2	1	$7.06 \cdot 10^{-3}$	11.24	3.64	12.91	0.127
	2	0.166	$6.2 \cdot 10^{-2}$	5.18	12.94	0.570
	3	0.166	0.119	0.00488	12.92	0.660
	4	1.168	1.35	0.0337	0.180	1.463
	5	1.086	1.35	0.0322	0.180	0.00080
3	1	0.00042	1.35	0.0322	0.180	0.0173
	2	0.0057	0.00209	0.0328	0.120	0.0375
	3	0.0057	0.00231	0.00145	0.120	0.0262
	4	0.00652	0.0217	0.00191	0.00164	0.0283
	5	0.00551	0.0217	0.00191	0.00164	0.0000012
4	1	0.000356	0.0217	0.00191	0.00164	0.000120
	2	0.000599	0.0015	0.00191	0.00170	0.000525
	3	0.00060	0.00159	0.000244	0.00171	0.000637
	4	0.000663	0.00161	0.000244	0.00235	0.000673
	5	0.000662	0.00161	0.000243	0.00237	0.0000030
5	1	0.000521	0.00161	0.000244	0.00237	0.00006.1
	2	0.000471	0.00155	0.000244	0.00237	0.000061
	3	0.000664	0.00161	0.000262	0.00237	0.00006
	4	0.000635	0.00161	0.000265	0.00240	0.0000057
	5	0.000665	0.00161	0.000265	0.00241	0.0000030

Median value of $\lvert H_{mi} \rvert$ on $[E, T_e]$ when the heuristic is applied and iterated.

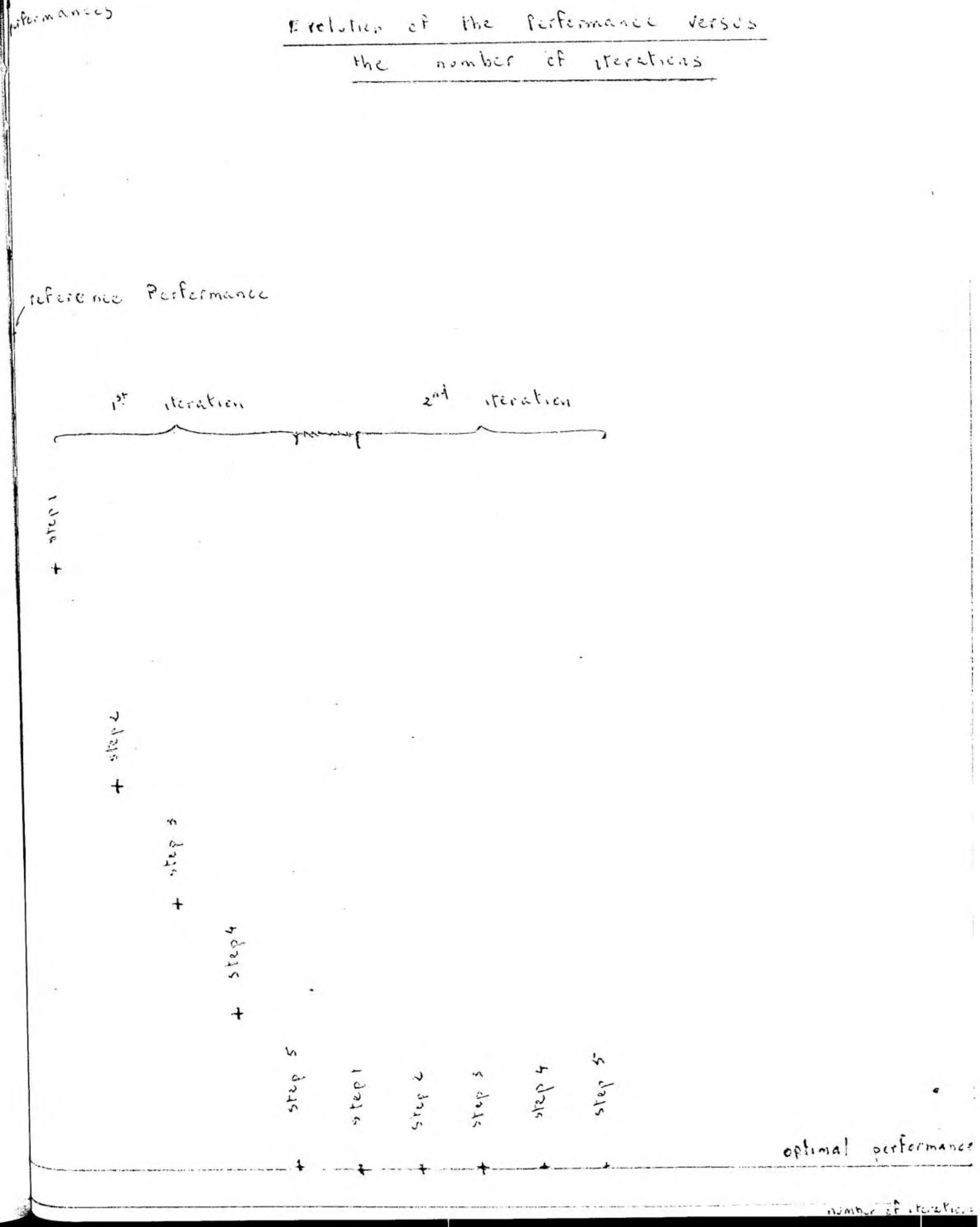
Evolution of the Performance versus the number of iterations



Evolution of the Performance versus the number of iterations



Evolution of the Performance versus the number of iterations



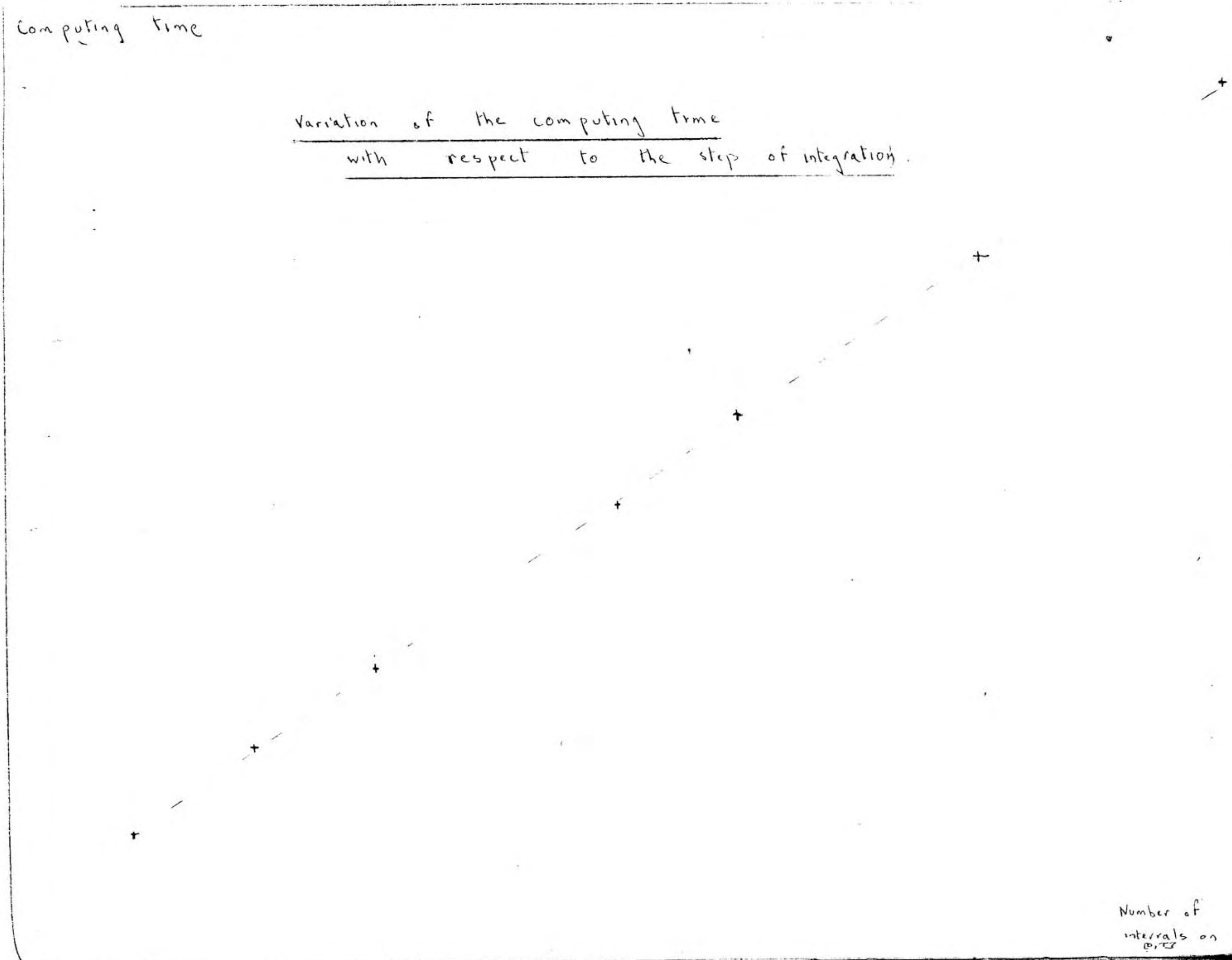
If, for the same system, we vary h/t_e from 0.1 to 0.01, we can see that:

-The computing time is proportional to the number of intervals in which $[t_0, t_e]$ is divided, or proportional to $1/(h/t_e)$.

-After 5 iterations, we get almost the same performance.

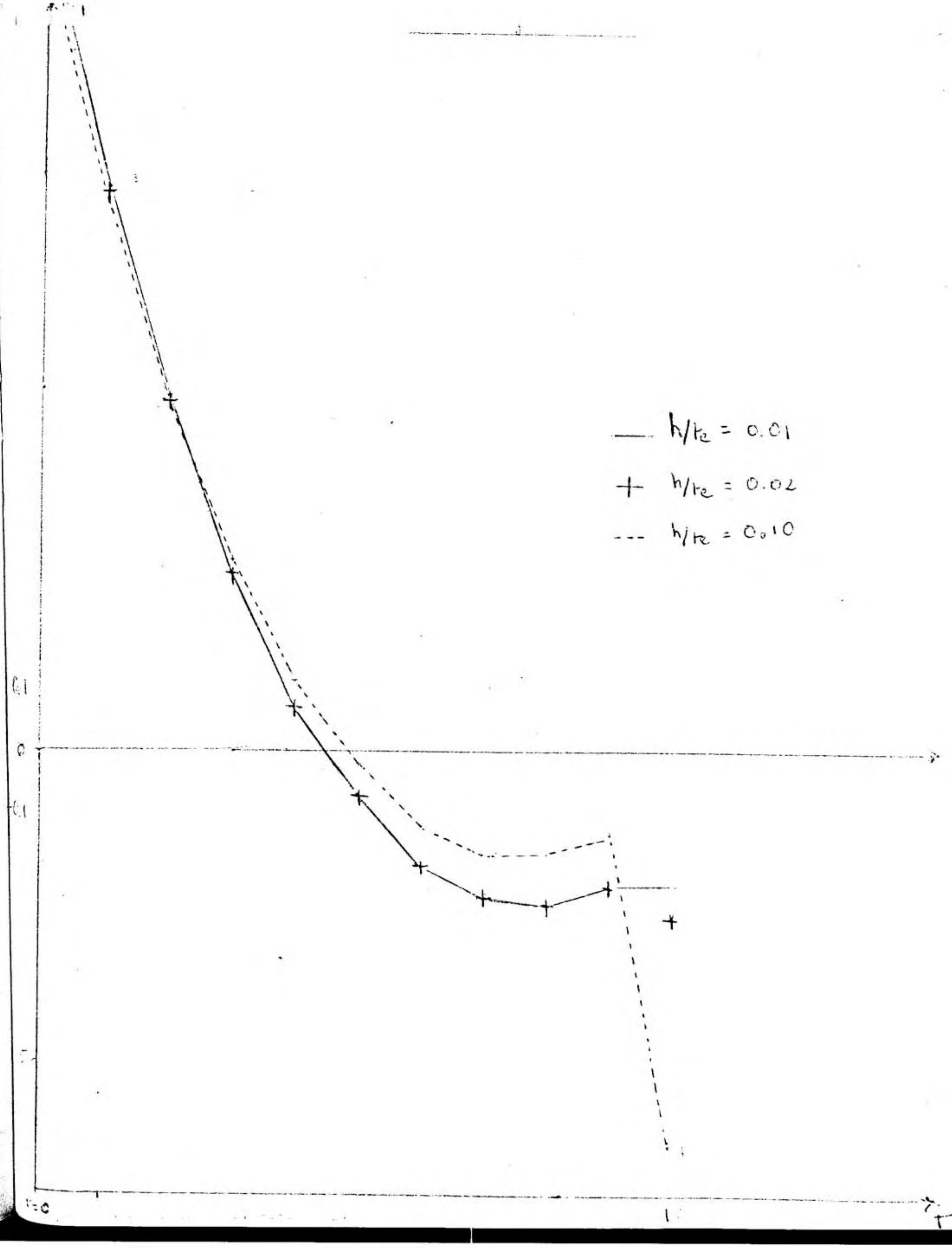
-When h decreases, we get a smoother curve of the control.

-The difference of the trajectories computed with different h/t_e can be neglected. The difference in the controls are more important; but this puts in evidence two facts: For this system, a small variation in the controls will not affect too much the trajectories, and the cost of the control is very little compared to the cost of the deviation from the desired trajectory.


Computing time
Variation of the computing time
with respect to the step of integration.
Number of
intervals on
[0,T]

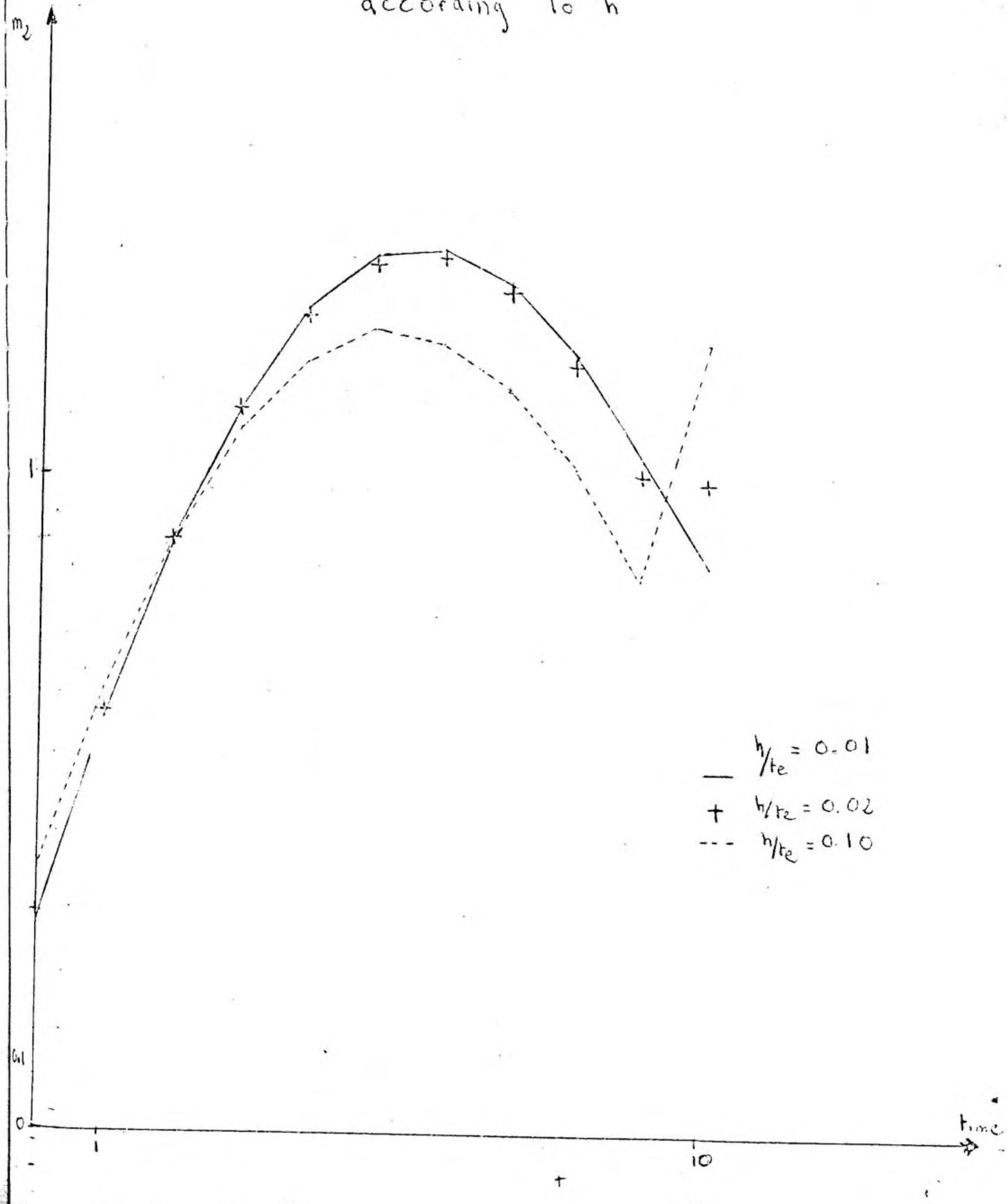
Variation of the evolution of the Performance index with respect to the step of integration h

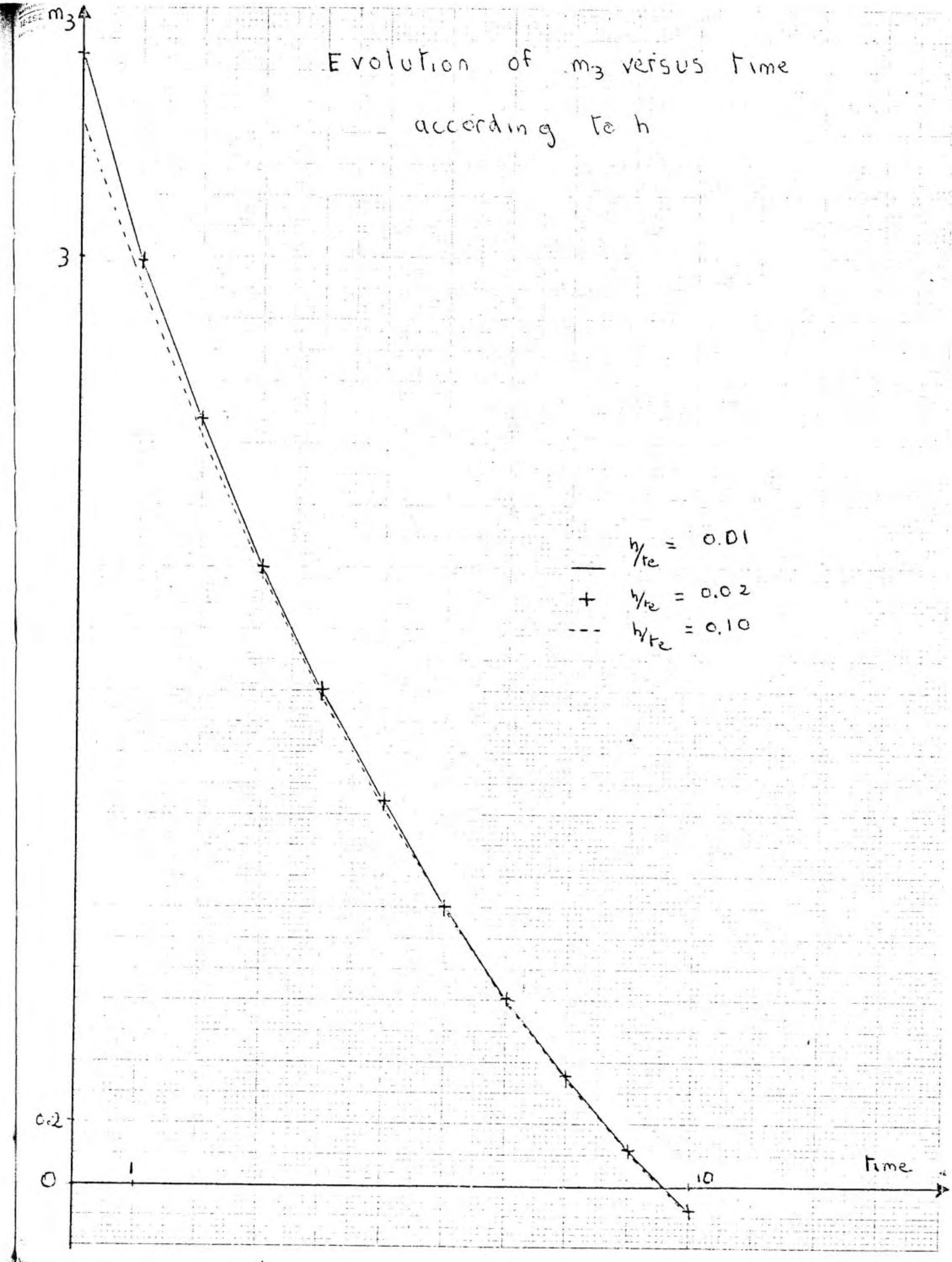
h/te	0.1	0.05	0.033	0.02	0.016	0.012	0.01
Number of intervals on [0,T]	10	20	30	50	60	80	100
Step N°							
1	8373.1	12087	17 263	27 886	33 241	43 986	54 753
2	6225.4	9409.2	12 877	19 967	23 541	30 709	37.889
3	5939.1	9266.6	12 778	19 908	23 490	30 669	37 856
4	5614.7	9098.2	12 660	19 830	23 420	30613	37 867
5	744.63	664.66	641.16	623.28	618.93	613.55	610.35
1	434.88	444.91	438.06	408.35	407.26	405.90	405.09
2	380.89	361.25	355.42	351.02	349.96	348.66	347.88
3	371.82	355.38	350.52	346.80	345.91	344.82	344.18
4	342.38	337.69	337.14	336.79	336.77	336.76	336.74
5	338.90	337.20	336.92	336.77	336.74	336.71	336.70
1	326.39	325.76	325.92	325.75	325.76	325.98	325.79
2	324.94	324.90	325.00	325.11	325.14	325.18	325.20
3	324.78	324.88	325.00	325.11	325.13	325.17	325.20
4	322.15	321.96	322.12	321.93	322.	321.93	321.94
5	321.55	321.63	321.68	321.71	321.72	321.73	321.74
1	320.33	320.26	320.26	320.27	320.27	320.27	320.28
2	318.84	318.93	318.97	319.01	319.02	319.03	319.04
3	318.77	318.89	318.94	318.98	318.99	319.00	319.01
4	318.00	317.96	318.03	318.02	318.02	317.95	318.02
5	317.86	317.79	317.77	317.75	317.74	317.73	317.73
1	317.73	317.62	317.58	317.55	317.55	317.54	317.53
2	317.16	317.04	317.00	316.97	316.96	316.95	316.95
3	317.15	317.02	316.98	316.95	316.94	316.93	316.92
4	316.97	316.80	316.77	316.73	316.72	316.70	316.70
5	316.96	316.75	316.68	316.63	316.61	316.59	316.58


h/re = 0.01
h/re = 0.02
h/re = 0.10
0.1
0
-0.1

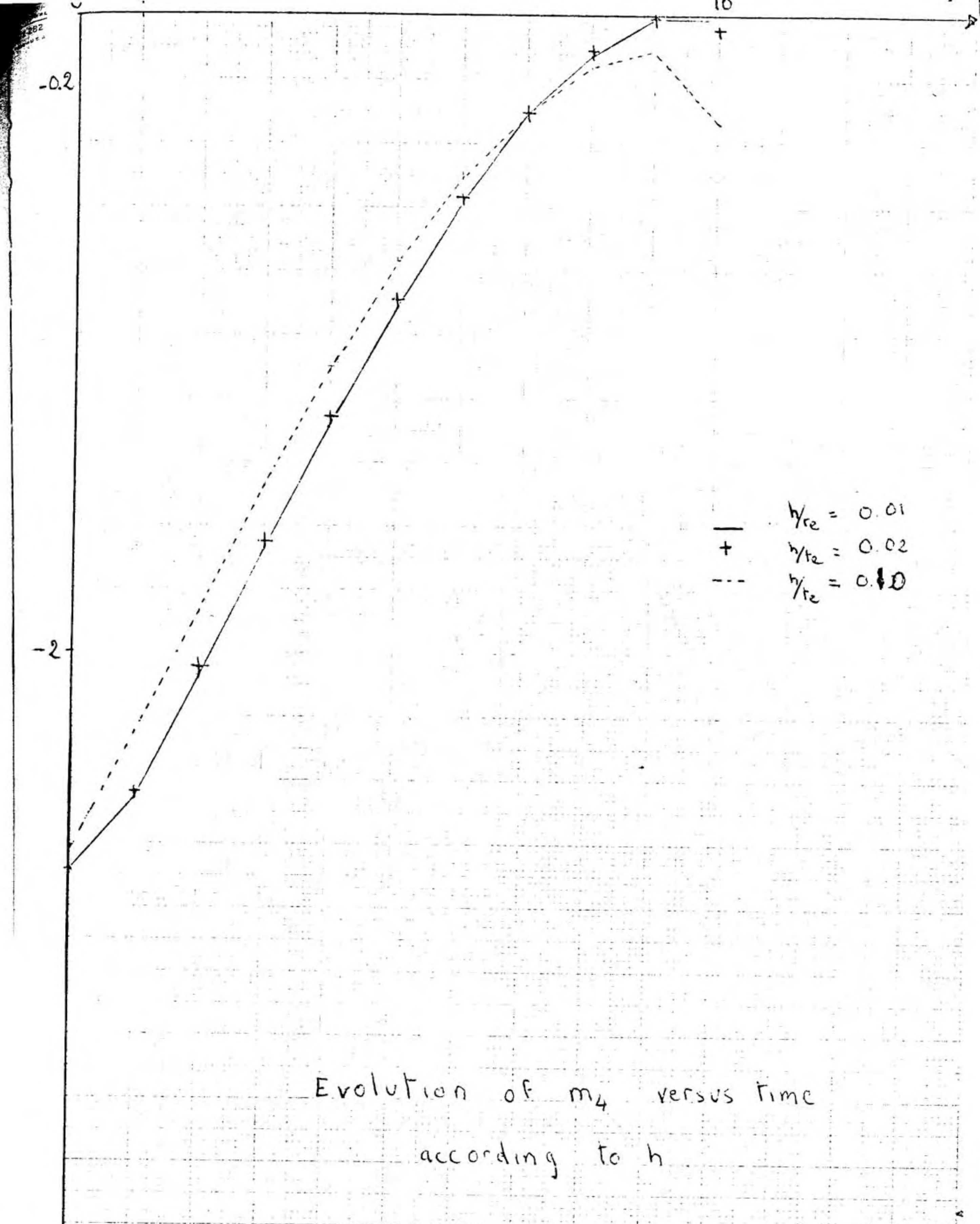
Evolution of m_2 versus time

according to h



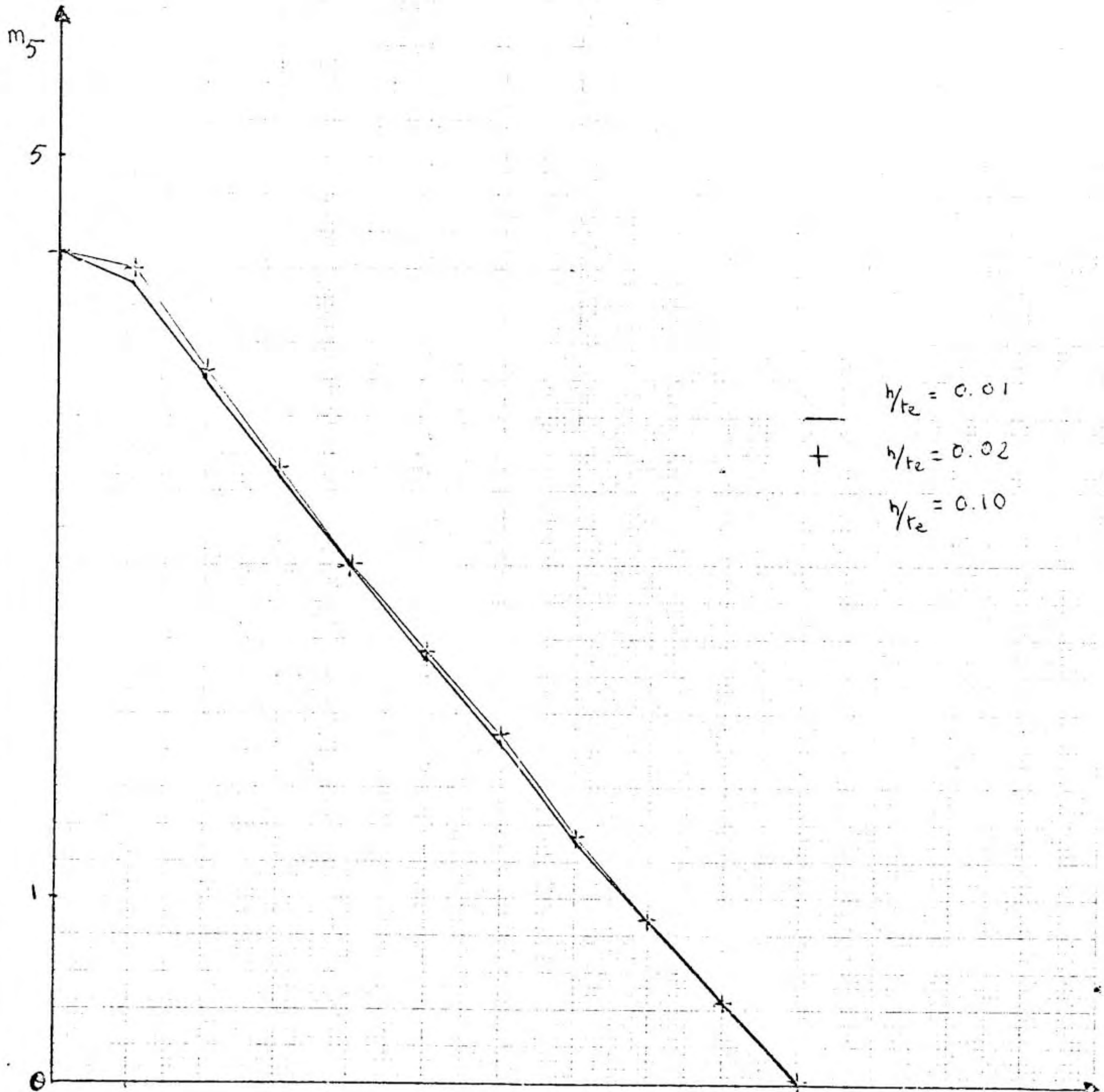


Squares to the Inch



Evolution of m_4 versus time according to h

Evolution of m_5 versus time according to h



Difference in % of the solutions with various h compared to the solution with h = 0.01

t_0

h/r_c	m_1	m_2	m_3	m_4	m_5	x_1	x_2	x_3	x_4	x_5
0.1	4	27	3.2	6.7	1.2	0	0	0	0	0
0.05	2.3	15	1.5	2.5	0.5	0	0	0	0	0
0.033	1.5	9.6	0.5	1.4	0.3	0	0	0	0	0
0.02	0.7	4.3	0.4	0.6	0.1	0	0	0	0	0
0.016	0.46	3	0.3	0.4	0.08	0	0	0	0	0
0.012	0.23	1.2	0.1	0.14	0.02	0	0	0	0	0

} h

t_5

h/r_c	m_1	m_2	m_3	m_4	m_5	x_1	x_2	x_3	x_4	x_5
0.1	65	8.2	1.7	15	1.1	0.08	0.06	0.49	2.8	0.8
0.05	15	2.5	0.8	7.9	0.47	0.05	0	0.21	1.6	0.35
0.033	6	1.2	0.5	4.9	0.3	0.03	0	0.12	0.7	0.2
0.02	1.7	0.45	0.2	2.2	0.13	0.01	0	0.04	0.3	0.08
0.016	1	0.3	0.16	1.5	0.086	0.008	0	0.03	0.2	0.05
0.012	0.25	0.07	0.06	0.6	0.04	0	0	0.01	0.08	0

t_{10}

h/r_c	m_1	m_2	m_3	m_4	m_5	x_1	x_2	x_3	x_4	x_5
0.1	200	51	0.1	1530	0	0.3	0	0.5	3.2	0.2
0.05	98	21	0.4	550	0	0.1	0	0.2	1.25	0
0.033	59	13	0.3	310	0	0	0	0.1	0.675	0
0.02	26	6	0.1	118	0	0	0	0	0.25	0
0.016	17	4	0.1	77	0	0	0	0	0	0
0.012	6	1.5	0.03	28	0	0	0	0	0	0

CHAPTER IV

DECOMPOSITION TECHNIQUE APPLIED IN PRESENCE OF NOISE

The decomposition technique cannot be applied directly to a system with disturbances, for the noises in the measurements introduce a new kind of correlation between the subsystems. If we consider one subsystem and the measurements related only to this subsystem we are neglecting in this subsystem the information brought by the data of the other subsystems. Hence, the computed control would not be the optimal control computed with all the data given by the measurements.

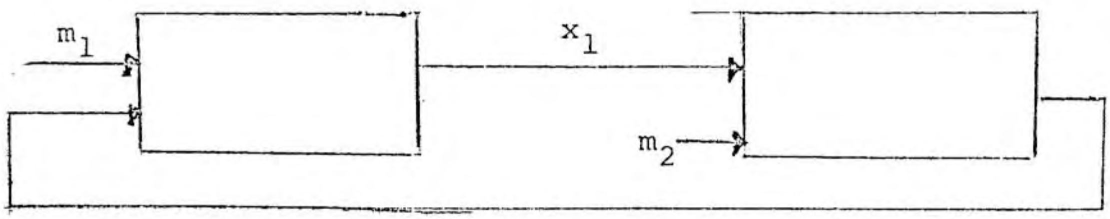
Nevertheless, in terms of the means and the variances of the state variables, it is possible to use this technique once deterministic equations of the global system are found. We then consider these equations as defining a new system, the state variables of which are the means and the variances of the previous state variables. The examples will show how this can be implemented.

4.1 Case of Linear System with Quadratic Performances

4.1.1 The Deterministic Problem

Part 1 The global method:

The system is the following one:



$$\begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} dt + \begin{bmatrix} \ell_1 & 0 \\ 0 & \ell_2 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} dt$$

or

$$dx = (C + C')\ x\ dt + L\ m\ dt.$$

$$\text{with } C = \begin{bmatrix} c_{11} & 0 \\ 0 & c_{22} \end{bmatrix} \quad C' = \begin{bmatrix} 0 & c_{12} \\ c_{21} & 0 \end{bmatrix} \quad L = \begin{bmatrix} \ell_1 & 0 \\ 0 & \ell_2 \end{bmatrix}$$

The performance criterion is:

$$\phi[x,m] = \int_0^{t_e} \{(r-x)(r-x)^T + m\ m^T\}\ dt = \int_0^{t_e} \{(r_1-x_1)^2 + (r_2-x_2)^2 + m_1^2 + m_2^2\}\ dt$$

This problem corresponds to something real:

$\int_0^{t_e} (m_1^2 + m_2^2)\, dt$ corresponds to the cost of the control action. $r_1(t)$, $r_2(t)$ corresponds to the desired trajectories of $x_1(t)$, $x_2(t)$. $\int_0^{t_e} \{(r_1-x_1)^2 + (r_2-x_2)^2\}dt$ is the cost of deviation from the desired output.

This problem is a classical problem of control which can be solved by well known techniques:

$$H = (r_1-x_1)^2 + (r_2-x_2)^2 + m_1^2 + m_2^2 + p_1[c_{11}\, x_1 + c_{12}\, x_2 + \ell_1\, m_1] + p_2[c_{21}\, x_1 + c_{22}\, x_2 + \ell_2\, m_2]$$

We have to solve the following T. P. B. V. P.:

$$\begin{cases} \dfrac{dx_1}{dt} = c_{11}\, x_1 + c_{12}\, x_2 + \ell_1\, m_1 & x_1(0) = x_{10} \\[2ex] \dfrac{dx_2}{dt} = c_{21}\, x_1 + c_{22}\, x_2 + \ell_2\, m_2 & x_2(0) = x_{20} \\[2ex] \dfrac{dp_1}{dt} = -Hx_1 & p_1(t_e) = 0 \\[2ex] \dfrac{dp_2}{dt} = -Hx_2 & p_2(t_e) = 0 \\[2ex] Hm_1 = 0 \qquad Hm_2 = 0 & \end{cases}$$

-Computation of a numerical example:

-Data:

$$c_{11}(t) = c_{22}(t) = c_{21}(t) = \ell_1(t) = \ell_2(t) = 1$$

$$r_1(t) = t \qquad r_2(t) = 2t \qquad t_e = 0.05.$$

-Computation:

$$\mathcal{D} = \mathcal{D}' = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \mathcal{E} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \quad w_1 = \begin{bmatrix} \frac{1}{2} & \delta Hm_1 \\ \frac{1}{2} & \delta Hm_2 \end{bmatrix}$$

$$\mathcal{L} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

-Results: see Appendix 5, Program 6.

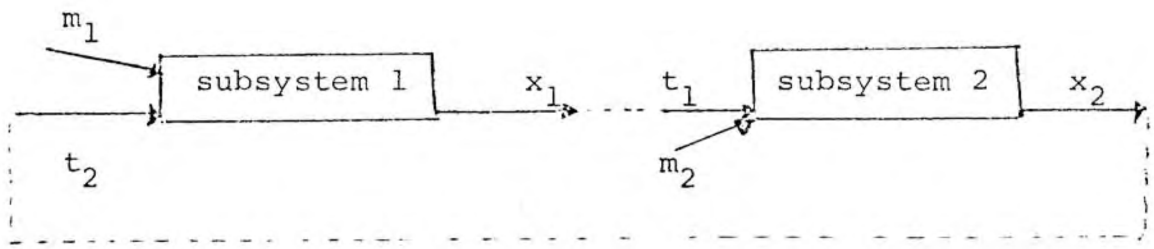
Part 2 The decomposition technique:

The system is decomposed in two subsystems (Appendix 2 and the thesis of S. Reich [3]). For this, the chosen system must have the two following fundamental properties:

a) It can be split into two subsystems such that the output of one system can be viewed as an input to the other subsystem.

b) The performance is additively separable.

In this case we have the following scheme:



We have:

<u>subsystem 1:</u>

$$\frac{dx_1}{dt} = c_{11}\, x_1 + c_{12}\, t_2 + \ell_1\, m_1$$

<u>subsystem 2:</u>

$$\frac{dx_2}{dt} = c_{21}\, t_1 + c_{22}\, x_2 + \ell_2\, m_2.$$

with the supplementary constraints: $x_2 = t_2$

$x_1 = t_1$

The Lagrangian for the whole system can be written as:

$$J = (r_1-x_1)^2 + (r_2-x_2)^2 + m_1^2 + m_2^2 + p_1[c_{11}\, x_1 + c_{12}\, t_2$$

$$+ \ell_1\, m_1 - \dot{x}_1] + p_2[c_{21}\, t_1 + c_{22}\, x_2 + \ell_2\, m_2 - \dot{x}_2]$$

$$+ k_1\, (x_1 - t_1) + k_2\, (x_2 - t_2).$$

This Lagrangian can be split, due to the fact that the performance is additively separable. This leads to the decomposition technique. It was shown [3] that the decomposition technique and the global method give the same results. The advantage of the decomposition technique is to break the problem in several subproblems easier to handle by the computer. In this case we have:

First Level

Given k_1, k_2 for the $2^{\underline{nd}}$ level we have the two T. P. B. V. P.:

-subproblem 1:

$$J = (r_1-x_1)^2 + m_1^2 + p_1[c_{11}\, x_1 + c_{12}\, t_2 + \ell_1\, m_1 - \dot{x}_1] + k_1\, x_1 - k_2\, t_2.$$

$$\begin{cases} \dfrac{dx_1}{dt} = c_{11}\, x_1 + \ell_1\, m_1 + c_{12}\, t_2 & x_1(0) = x_{10}. \\ \dfrac{dp_1}{dt} = -Hx_1 = -\{ 2\,(x_1-r_1) + p_1\, c_{11} + k_1 \} & p_1(t_e) = 0 \\ 2m_1 + p_1\, \ell_1 = 0 & \\ p_1\, c_{12} - k_2 = 0 & \end{cases}$$

-subproblem 2:

$$J = (r_2-x_2)^2 + m_2^2 + p_2[c_{21}\, t_1 + c_{22}\, x_2 + \ell_2\, m_2 - \dot{x}_2] - k_1\, t_1 + k_2\, x_2.$$

$$\begin{cases} \dfrac{dx_2}{dt} = c_{21}\, t_1 + c_{22}\, x_2 + \ell_2\, m_2 & x_2(0) = x_{20} \\[2ex] \dfrac{dp_2}{dt} = -\, Hx_2 = -\, \{2(x_2 - r_2) + p_2\, c_{22} + k_2\} & p_2(t_e) = 0 \\[2ex] 2m_2 + p_2\, \ell_2 = 0 & \\ p_2\, c_{21} - k_1 = 0 & \end{cases}$$

Second Level or Coordination Level

The 2nd level coordinates the two subproblems by varying k_1 and k_2.

$$[k_1]_{n+1} = [k_1]_n - \varepsilon\,(x_1 - t_1)$$

$$[k_2]_{n+1} = [k_2]_n - \varepsilon\,(x_2 - t_2)$$

The multilevel procedure consists in solving the first level and making one coordination step at the 2nd level. The procedure repeats until $|x_1 - t_1|$ and $|x_2 - t_2|$ are within a tolerance limit.

Computation of a numerical example:

-Data:

$$c_{11}(t) = c_{21}(t) = c_{22}(t) = \ell_1(t) = \ell_2(t) = 1$$

$$r_1(t) = t \qquad r_2(t) = 2t \qquad t_e = 0.05.$$

-Computation:

<u>subsystem 1:</u>

$$\mathcal{D} = \mathcal{D}' = 1 \qquad \mathcal{E} = -\frac{3}{2} \qquad w_1 = \frac{1}{2}\,\delta Hm_1 + \delta Ht_2$$

$$\mathcal{L} = -1 \qquad w_2 = 0$$

<u>subsystem 2:</u>

$$\mathcal{D} = \mathcal{D}' = 1 \qquad \mathcal{E} = -\frac{3}{2} \qquad w_1 = \frac{1}{2}\,\delta Hm_2 + \delta Ht_1$$

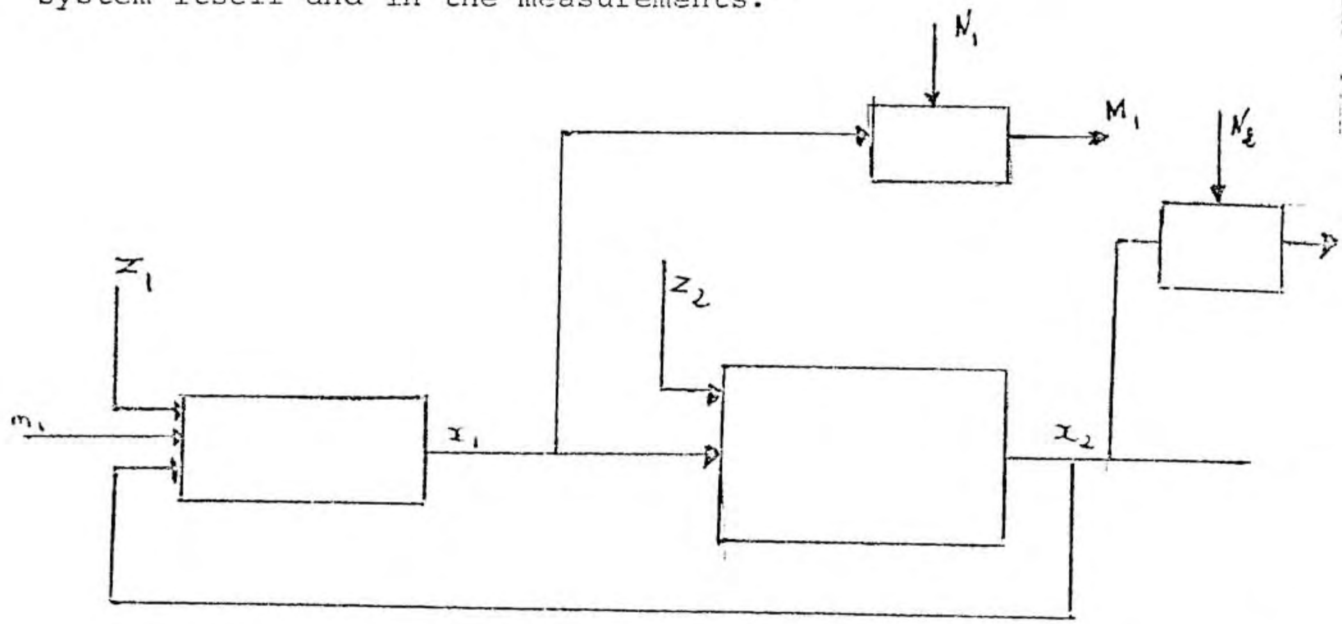
$$\mathcal{L} = -1 \qquad w_2 = 0$$

-Results: see Appendix 5, Program 7.

4.1.2 <u>The Stochastic Problem</u>

<u>A</u> <u>Formulation of the Problem:</u>

We take the same system but with noises in the system itself and in the measurements.



The control actions are m_1 and m_2.

-The system is:

$$\begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} dt + \begin{bmatrix} \ell_1 & 0 \\ 0 & \ell_2 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} dt + \begin{bmatrix} dz_1 \\ dz_2 \end{bmatrix}$$

or

$$dx = (C + C')\, x\, dt + L\, m\, dt + dz.$$

The measurements are:

$$\begin{bmatrix} dM_1 \\ dM_2 \end{bmatrix} = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} dt + \begin{bmatrix} dN_1 \\ dN_2 \end{bmatrix}$$

or

$$dM = G\, x\, dt + dN$$

-Assumptions on the noises:

dN and dz are uncorrelated vector Wiener processes, such that:

For any $t \in T$ and any $s \neq t$:

$$E[dN(t)] = E[dz(t)] = 0 \quad E[dN(t).dz^T(t)] = 0$$

$$E[dN(t) \,.\, dN^T(s)] = E[dN(t) \,.\, dz^T(s)] = E[dz(t) \,.\, dz^T(s)] = 0$$

$$E[dN(t) \,.\, dN^T(t)] = W\, dt$$

$$E[dz(t) \,.\, dz^T(t)] = Q\, dt$$

with

$$W = \begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix} \quad Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}$$

-Notations:

$$\bar{x} = E(x) = \begin{bmatrix} \bar{x}_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} E(x_1) \\ E(x_2) \end{bmatrix}$$

$$V = E\{(x - \bar{x}).(x - \bar{x})^T\} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

-Performance criterion:

We use the same performance criterion as in the deterministic case but with an expectation:

$$\phi = E\{\int_0^{t_e}[(r_1-x_1)^2 + (r_2-x_2)^2 + m_1^2 + m_2^2]\, dt$$

$$= \int_0^{t_e} \{(m_1^2 + (r_1-\bar{x}_1)^2 + v_{11}) + (m_2^2 + (r_2-\bar{x}_2)^2 + v_{22})\}\, dt$$

B <u>Solution of the global problem:</u>

To solve the global problem we need to know an expression of $\bar{x}_1$, v_1, $\bar{x}_2$, v_2 which are in the formulation of the performance criterion. To compute these values we shall use the method of Kalman [4]. This method will be explained in detail in this section. In the following demonstrations only the main steps of the calculation will be written.

We will solve the problem first with the vectorial notation, which will give $\bar{x}$ and V. Taking the elements of $\bar{x}$ and V we will get $\bar{x}_1$, v_1, $\bar{x}_2$, v_2.

Prior mean and variance of : $\bar{x}$ and V.

Posterior mean and variance of : $\underline{\bar{x}}$ and $\underline{V}$.

We approximate the continuous-time model by a finite time difference model:

$$\delta x = x(t + \delta t) - x(t) = (C + C') \, x\delta t + L \, m\delta t + \delta Z. \qquad (1)$$

$$\delta M = M(t + \delta t) - M(t) = G \, x\delta t + \delta N. \qquad (2)$$

Now we proceed in two stages: the first stage is the determination of the conditional probability density of δx just after an observation δM during the time δt, in terms of the conditional probability density of δx just before the observation δM. These conditional probabilities are known respectively as posterior and prior. In the second stage the effect of the dynamics of the process are taken in account and determine the transformation between the prior conditional probability density at time $t + \delta t$ and the posterior conditional probability density of x at time t.

It is assumed, and it can be shown inductively that the conditional probability densities of $x(t)$ are gaussian.

Stage 1: The effect of the observation:

Bayes rule gives:

$$f(x,\ t+\delta t\ /\ t+\delta t) = \frac{f[x,\ t+\delta t\ ;\ \delta M\ /\ t]}{f(\delta M/t)}$$

$$f(x,\ t+\delta t\ /\ t+\delta t) = \frac{f(\delta M,\ t+\delta t\ /\ x,\ t+\delta t\ ;\ t)}{f(\delta M/t)} \cdot f\ (x,\ t+\delta t/t).$$

↑ posterior probability density — ↑ "likelihood" — ↑ prior probability density

which gives:

$$\exp[-\tfrac{1}{2}\ (x-\bar{x})^T\ V^{-1}\ (x-\bar{x})] = k'\ \exp[-\tfrac{1}{2}\ (\delta M-G\ x\ \delta t)(W\delta t)^{-1}$$

$$(\delta M-G\ x\ \delta t)]\ x\ \exp[-\tfrac{1}{2}\ (x-\underline{\bar{x}})^T\ \underline{V}^{-1}\ (x-\underline{\bar{x}})] \qquad (4)$$

Where all the terms are evaluated at $t+\delta t$. By equating coefficients of x in (4), we get:

$$(5)\quad V = [\underline{V}^{-1} + G^T\ W^{-1}\ G\ \delta t]^{-1}$$

$$(6)\quad \bar{x} = V\ \underline{V}^{-1}\ \underline{\bar{x}} + V\ G^T\ W^{-1}\ \delta M$$

where all the terms are evaluated at $t+\delta t$.

Expanding (5) in a matrix Taylor's series, we get:

$$(7)\quad V = \underline{V} - \underline{V}\ G^T\ W^{-1}\ G\ \underline{V}\ \delta t + O(\delta t).$$

$$(6)\quad \bar{x} = V\ V^{-1}\ \underline{\bar{x}} + V\ G^T\ W^{-1}\ \delta M$$

Taking the conditional moments of (1), we get:

$$(8)\quad \underline{\bar{x}}(t+\delta t) = [I + (C + C')\ \delta t]\ \bar{x}(t) + L\ m\ \delta t$$

(9) $\underline{V}(t+\delta t) = [I + (C + C')\,\delta t]\, V(t)\, [I + (C + C')\,\delta t]^T$

$+ Q\,\delta t.$

Substituting (9) in (7), we get:

$$V(t+\delta t) = V(t) + \delta t\, \{ (C + C')\, V + V\, (C + C')^T + Q$$

$$- V\, G^T\, W^{-1}\, G\, V \} + 0(\delta t).$$

which gives:

(10) $$\frac{dV}{dt} = (C + C')\, V + V\, (C + C')^T + Q - V\, G^T\, W^{-1}\, G\, V$$

Similarly eliminating $\underline{\bar{x}}(t+\delta t)$ $\underline{V}(t+\delta t)$ from (6), we get:

(11) $$\frac{d\bar{x}}{dt} = (C + C')\, \bar{x} + L\, m - V\, G^T\, W^{-1}\, G\, \bar{x} + V\, G^T\, W^{-1}\, \frac{dM}{dt}$$

Taking the components of (10) and (11) we get the solution of the problem or more precisely its statement:

<u>-system:</u>

$$\frac{dv_{11}}{dt} = 2\, c_{11}\, v_{11} + 2\, c_{12}\, v_{12} + q_1 - \frac{v_{11}^2\, g_1^2}{w_1} - \frac{v_{12}^2\, g_2^2}{w_2}$$

$$\frac{dv_{22}}{dt} = 2\, c_{22}\, v_{22} + 2\, c_{21}\, v_{12} + q_2 - \frac{v_{12}^2\, g_1^2}{w_1} - \frac{v_{22}^2\, g_2^2}{w_2}$$

$$\frac{dv_{21}}{dt} = (c_{11} + c_{22})\, v_{12} + c_{12}\, v_{22} + c_{21}\, v_{11} - \frac{v_{11}\, v_{12}}{w_1}\, g_1^2$$

$$- \frac{v_{22}\, v_{12}}{w_2}\, g_2^2$$

$$\frac{d\bar{x}_1}{dt} = c_{11}\,\bar{x}_1 + c_{12}\,\bar{x}_2 + \ell_1\,m_1 - \frac{v_{11}\,g_1^2\,\bar{x}_1}{w_1} - \frac{v_{12}\,g_2^2\,\bar{x}_2}{w_2}$$

$$+ \frac{v_{11}\,g_1}{w_1}\,\frac{dM_1}{dt} + \frac{v_{12}\,g_2}{w_2}\,\frac{dM_2}{dt}$$

$$\frac{d\bar{x}_2}{dt} = c_{22}\,\bar{x}_2 + c_{21}\,\bar{x}_1 + \ell_2\,m_2 - \frac{v_{12}\,g_1^2}{w_1}\,\bar{x}_1 - \frac{v_{22}\,g_2^2}{w_2}\,\bar{x}_2$$

$$+ \frac{v_{12}\,g_1}{w_1}\,\frac{dM_1}{dt} + \frac{v_{22}\,g_2}{w_2}\,\frac{dM_2}{dt}$$

-Performance criterion:

$$\phi = \int_0^{t_e} \{ m_1^2 + m_2^2 + (r_1-\bar{x}_1)^2 + (r_2-\bar{x}_2)^2 + v_{11} + v_{22} \}\,dt$$

The endpoint conditions are:

$$\begin{cases} \bar{x}_1(0) = x_{10} \\ \bar{y}_1(0) = y_{10} \\ v_{11}(0) = 0 \\ v_{22}(0) = 0 \\ v_{12}(0) = 0 \\ v_{21}(0) = 0 \end{cases}$$

The problem is now formulated in terms of a deterministic problem. It can be solved easily.

[Note: p. 83 missing from original. Could just be mis-numbering. RC]

Computation of a numerical example:

-Data:

$c_{11}(t) = c_{12}(t) = c_{22}(t) = 0 \quad c_{21}(t) = 1 \quad \ell_1(t) = 1 \quad \ell_2(t) = 0$

$t_e = 0.05 \quad r_1(t) = t \quad r_2(t) = 2t \quad \frac{dM_1}{dt} = t+0.01 \text{ Sint}$

$\frac{dM_2}{dt} = 2t + 0.01 \text{ Sint} \quad w_1 = w_2 = 0.2 \quad q_1 = q_2 = 0.3.$

-Computation:

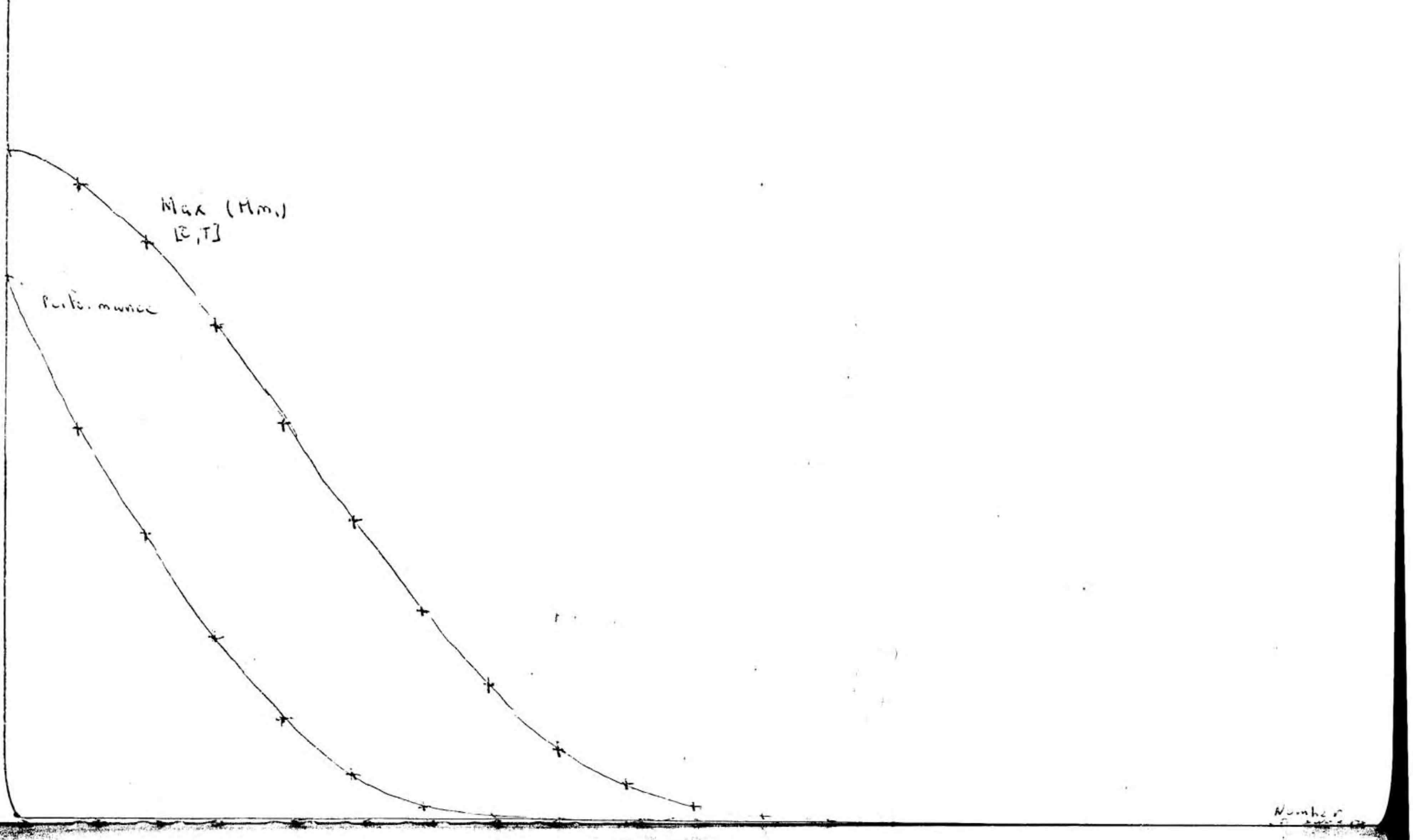
$$\mathcal{D} = \begin{bmatrix} 2(1-5v_{11}) & 0 & 2(1-5v_{12}) & 0 & 0 \\ 0 & 2(1-5v_{22}) & 2(1-5v_{12}) & 0 & 0 \\ 1-5v_{12} & 1-5v_{12} & 2(5v_{11}-5v_{22} & 0 & 0 \\ -5\bar{x}_1+5\,(t+0.01\mathrm{Sin}t) & 0 & -5\bar{x}_2+5\,(2t+0.01\mathrm{Sin}t) & 1-5v_{11} & 1-5v_{12} \\ 0 & -5\bar{x}_2+5\,(2t+0.01\mathrm{Sin}t) & -5\bar{x}_1+5\,(t+0.01\mathrm{Sin}t) & 1-5v_{12} & 1-5v_{22} \end{bmatrix}$$

$$\mathcal{E} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix} \qquad w_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \delta H x_{\frac{1}{2}} \\ \frac{\delta H x_2}{2} \end{bmatrix}$$

$$\mathcal{L} = \begin{bmatrix} 10p_1 & 0 & 5p_3 & 5p_4 & 0 \\ 0 & 10p_2 & 5p_3 & 0 & 5p_5 \\ 5p_3 & 5p_3 & 10(p_1+p_2) & 5p_5 & 5p_4 \\ 5p_4 & 0 & 5p_5 & -2 & 0 \\ 0 & 5p_5 & 5p_4 & 0 & -2 \end{bmatrix} \quad w_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

-Results: see Appendix 5, Program 8.

Evolution of the criterion of Performance (curve (1)) and of the derivative of the hamiltonian with respect to the control (curve (2)) versus the number of sweeps in the successive sweep method applied to the system with disturbances (integrated problem).



C Application of the decomposition technique:

If we consider the equations derived in the preceeding paragraph we can draw some conclusions:

-A linear system with noises will give a linear system.

-The noise increases the difficulty of the problem: if the former state vector was n-dimensional, the new one will be $[n + \frac{n^2+n}{2}]$-dimensional [n for the means, $\frac{n^2+n}{2}$ for the variances].

-The deterministic formulation of the stochastic problem shows that the two subproblems are not separated now: in particular when we compute x_1 we have to know the measurements of x_2 and vice-versa. So if we want to use all the data given by the measurements we have to deal with the whole system and transform it in a deterministic problem as we did in the preceeding paragraph and then apply the decomposition technique.

For that we introduce dummy variables:

$$\begin{cases} v'_{11} = v_{11} \\ v'_{22} = v_{22} \\ \bar{x}'_1 = \bar{x}_1 \\ \bar{x}'_2 = x_2 \end{cases}$$

$$\begin{cases} v'_{12} = v_{12} \\ v''_{12} = v_{12}. \end{cases}$$

Now we can state the problem:

$$\phi = \int_0^{t_e} \{\tfrac{1}{2}[r_1-\bar{x}_1]^2 + m_1^2 + v_{11} + \tfrac{1}{2}[r_2-\bar{x}'_2]^2 \} \, dt +$$

$$\int_0^{t_e} \{\tfrac{1}{2}[r_2-\bar{x}_2]^2 + m_2^2 + v_{22} + \tfrac{1}{2}[r_2-\bar{x}'_1]^2 \} \, dt.$$

With the following constraints:

Lagrange multiplier

$$\begin{cases} \dfrac{dv_{11}}{dt} = 2\, c_{11}\, v_{11} + 2\, c_{12}\, v_{12} + q_1 \\ \qquad - \dfrac{v_{11}^2\, g_1^2}{w_1} - \dfrac{v'^2_{12}\, g_2^2}{w_2} & p_1 \\ \dfrac{dv'_{12}}{dt} = (c_{11} + c_{22})\, v'_{12} + c_{12}\, v'_{22} + c_{21}\, v_{11} \\ \qquad - \dfrac{v_{11}\, v'_{12}\, g_1^2}{w_1} - \dfrac{v'_{22}\, v'_{12}\, g_2^2}{w_2} & p_2 \\ \dfrac{d\bar{x}_1}{dt} = c_{11}\, \bar{x}_1 + c_{12}\, \bar{x}'_2 + \ell_1\, m_1 - \dfrac{v_{11}\, g_1^2\, \bar{x}_1}{w_1} \\ \qquad - \dfrac{v'_{12}\, g_2^2\, \bar{x}'_2}{w_2} + \dfrac{v_{11}\, g_1}{w_1}\dfrac{dM_1}{dt} + \dfrac{v'_{12}\, g_2}{w_2}\dfrac{dM_2}{dt} & p_3 \end{cases}$$

$$\left\{\begin{array}{ll}
\dfrac{dv_{22}}{dt} = 2\ c_{22}\ v_{22} + 2\ c_{21}\ v''_{12} + q_2 - \dfrac{v''_{12}\ g_1^2}{w_1} & \\
\qquad - \dfrac{v_{22}^2\ g_2^2}{w_2} & q_1 \\
\dfrac{dv''_{12}}{dt} = (c_{11} + c_{22})\ v''_{12} + c_{12}\ v_{22} + c_{21}\ v'_{11} & \\
\qquad - \dfrac{v'_{11}\ v''_{12}\ g_1^2}{w_1} - \dfrac{v_{22}\ v''_{12}\ g_2^2}{w_2} & q_2 \\
\dfrac{d\bar{x}_2}{dt} = c_{22}\ \bar{x}_2 + c_{21}\ \bar{x}'_1 + \ell_2\ m_2 - \dfrac{v''_{12}\ g_1^2\ \bar{x}'_1}{w_1} & \\
\qquad - \dfrac{v_{22}\ g_2^2\ \bar{x}_2}{w_2} + \dfrac{v''_{12}\ g_1}{w_1}\dfrac{dM_1}{dt} + \dfrac{v_{22}\ g_2}{w_2}\dfrac{dM_2}{dt} & q_3
\end{array}\right.$$

$$\bar{x}_1 - \bar{x}'_1 = 0 \qquad R_1$$

$$\bar{v}_{11}^2 - \bar{v}'^2_{11} = 0 \qquad R_2$$

$$\bar{x}_2 - \bar{x}'_2 = 0 \qquad K_1$$

$$v_{22}^2 - v'^2_{22} = 0 \qquad K_2$$

Now we can apply the decomposition technique:

Level 1:

subproblem 1:

Given R_1, R_2, K_1, K_2 by the $2^{\underline{nd}}$ level, we have:

-Control variables : m_1, $\bar{x}'_2$, v'_{22}.

-State variables : $\bar{x}_1$, v_1, v'_{12}.

-Hamiltonian:

$$H = \frac{1}{2}\,[r_1-\bar{x}_1]^2 + m_1^2 + \frac{1}{2}\,[r_2-\bar{x}_2']^2 + R_1\,\bar{x}_1 + R_2\,v_{11}$$

$$- K_1\,\bar{x}_2' - K_2\,v_{22}' + p_1[2\,c_{11}\,v_{11} + 2\,c_{12}\,v_{12}' + q_1$$

$$- \frac{v_{11}^2\,g_1^2}{w_1} - \frac{v_{12}'^2\,g_2^2}{w_2}] + p_2[\,(c_{11} + c_{22})\,v_{12}' + c_{12}\,v_{22}'$$

$$+ c_{21}\,v_{11} - \frac{v_{11}\,v_{12}'\,g_1^2}{w_1} - \frac{v_{22}'\,v_{12}'\,g_2^2}{w_2} + p_3[c_{11}\,\bar{x}_1$$

$$+ c_{12}\,\bar{x}_2' + \ell_1\,m_1 - \frac{v_{11}\,g_1^2\,\bar{x}_1}{w_1} - \frac{v_{12}'\,g_2^2\,\bar{x}_2'}{w_2}$$

$$+ \frac{v_{11}\,g_1}{w_1} - \frac{dM_1}{dt} + \frac{v_{12}'\,g_2}{w_2}\,\frac{dM_2}{dt}]\,.$$

<u>subproblem 2:</u>

Given R_1, R_2, K_1, K_2 by the 2<u>nd</u> level, we have:

-Control variables : m_2, $\bar{x}_1'$, $\bar{v}_{11}'$.

-State variables : $\bar{x}_2$, v_{22}, v_{12}''.

-Hamiltonian:

$$H = \frac{1}{2}\,[r_2-\bar{x}_2]^2 + m_2^2 + v_{22} + \frac{1}{2}\,[r_1-\bar{x}_1']^2 - R_1\,\bar{x}_1'$$

$$- R_2\,v_{11}' + K_1\,\bar{x}_{22} + K_2\,v_{22} + q_1\,[2\,c_{22}\,v_{22}$$

$$+ 2\,c_{21}\,v_{12}'' + q_2 - \frac{v_{12}''^2\,g_1^2}{w_1} - \frac{v_{22}^2\,g_2^2}{w_2}]$$

$$+ q_2 \, [(c_{11} + c_{22}) \, v''_{12} + c_{12} \, v_{22} + c_{21} \, v'_{11}$$

$$- \frac{v'_{11} \, v''_{12} \, g_1^2}{w_1} - \frac{v_{22} \, v''_{12} \, g_2^2}{w_2}] + q_3 \, [c_{22} \, \bar{x}_2 + c_{21} \, \bar{x}'_1$$

$$+ \ell_2 \, m_2 - \frac{v''_{12} \, g_1^2 \, \bar{x}'_1}{w_1} - \frac{v_{22} \, g_2^2 \, \bar{x}_2}{w_2} + \frac{v''_{12} \, g_1}{w_1} \frac{dM_1}{dt}$$

$$+ \frac{v_{22} \, g_2}{w_2} \frac{dM_2}{dt}]$$

Level 2 or coordination level:

We vary K_1, K_2, R_1, R_2 according to:

$$[K_1]_{n+1} = [K_1]_n + \varepsilon * [\bar{x}_2 - \bar{x}'_2]$$

$$[K_2]_{n+1} = [K_2]_n + \varepsilon * [v_{22}^2 - v'^2_{22}]$$

$$[R_1]_{n+1} = [R_1]_n + \varepsilon * [\bar{x}_1 - \bar{x}'_1]$$

$$[R_2]_{n+1} = [R_2]_n + \varepsilon * [v_{11}^2 - v'^2_{11}]$$

Computation of a numerical example:

-Data:

$c_{11}(t) = c_{12}(t) = c_{22}(t) = 0 \quad c_{21}(t) = 1 \quad \ell_1(t) = 1 \quad \ell_2(t) = 0$

$t_e = 0.05 \quad r_1(t) = t \quad r_2(t) = 2t \quad \frac{dM_1}{dt} = t + 0.01\sin t$

$$\frac{dM_2}{dt} = 2t+0.01\text{Sin}t \qquad w_1 = w_2 = 0.2 \qquad q_1 = q_2 = 0.3$$

-Computation:

<u>subsystem 1:</u>

$$\mathcal{D} = \begin{bmatrix} -2\,\frac{v_{11}}{w_1} & -2\,\frac{v'_{12}}{w_2} & 0 \\ 1 - \frac{v'_{12}}{w_1} & -\frac{v_{11}}{w_1} - \frac{v'_{22}}{w_2} - \frac{v'_{12}\,p_2}{2K_2 w_2^2} & 0 \\ -\frac{\bar{x}_1}{w_1} + \frac{1}{w_1}\frac{dM_1}{dt} & -\frac{x'_2}{w_2} + \frac{1}{w_2}\frac{dM_2}{dt} - \frac{2p_3\,v'_{12}}{w_2^2} & -\frac{v_1}{w_1} \end{bmatrix}$$

$$\mathcal{E} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{v'^{2}_{12}}{2K_2 w_2^2} & 0 \\ 0 & 0 & -\frac{2v'^{2}_{12}}{w_2^2} - 1 \end{bmatrix} \quad W_1 = \begin{bmatrix} 0 \\ -\frac{v'_{12}}{2K_2 w_2}\ \delta H v'_{22} \\ \delta H m_1 - \frac{2v_{12}}{w_2}\ \delta H \bar{x}'_2 \end{bmatrix}$$

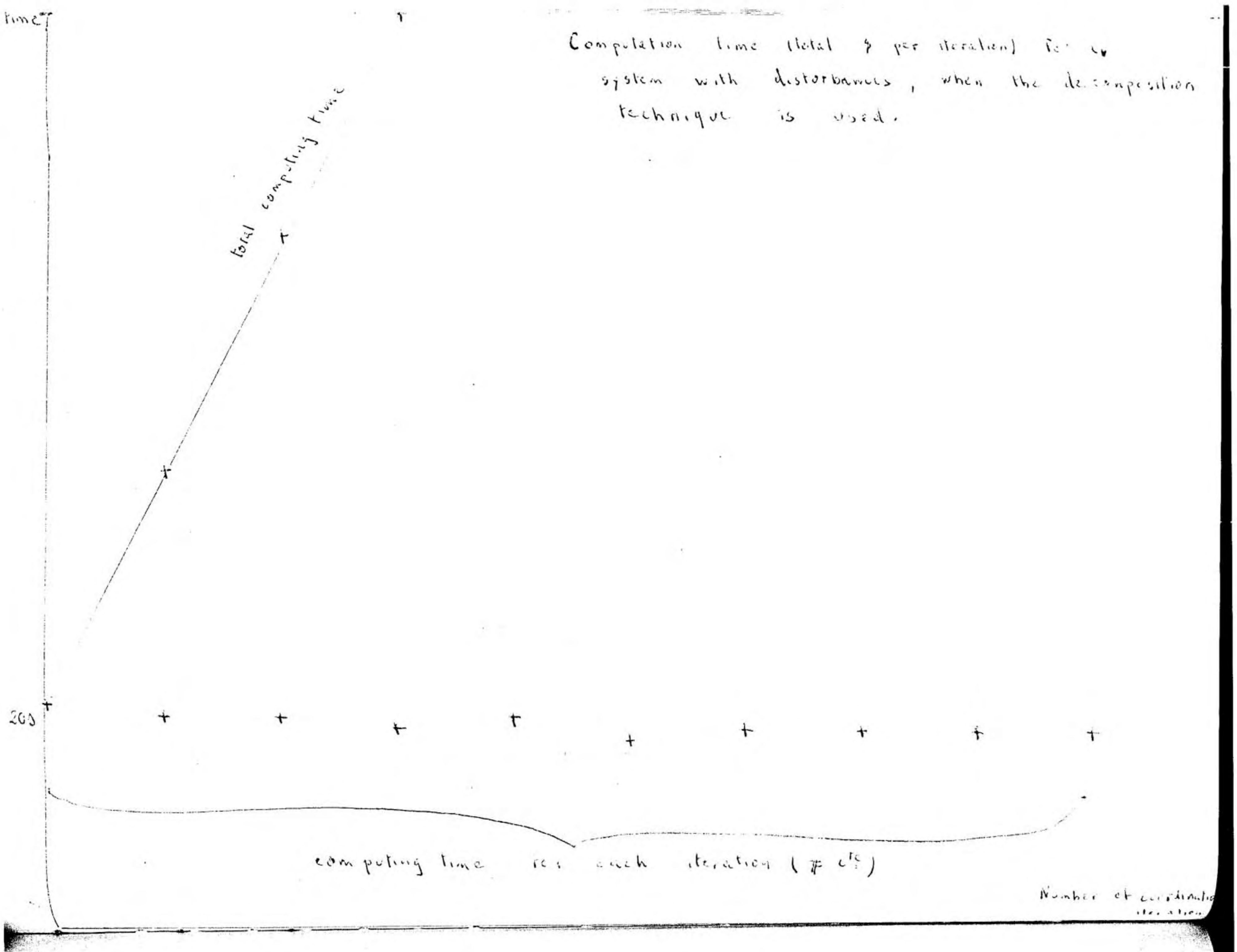
$$\mathcal{L} = \begin{bmatrix} \frac{2p_1}{w_1} - 2K_1 & \frac{p_2}{w_1} & \frac{p_3}{w_1} \\ \frac{p_2}{w_1} & \frac{2p_1}{w_2} + \frac{p_2^2}{2K_2 w_2^2} + \frac{2p_3}{w_2^2} & 0 \\ \frac{p_3}{w_1} & 0 & -0.5 \end{bmatrix} \quad W_2 = \begin{bmatrix} 0 \\ \frac{p_2}{2K_2 w_2} + \frac{2p_3}{w_2}\delta H \bar{x}'_2 \\ 0 \end{bmatrix}$$

subsystem 2:

$$\mathcal{D} = \begin{bmatrix} -2\,\frac{v_{22}}{w_2} & 2(1-\frac{v''_{12}}{w_1}) & 0 \\ -\frac{v''_{12}}{w_2} & -\frac{v'_{11}}{w_1} - \frac{v_{22}}{w_2} - \frac{q_2}{2K_1 w_1}\,(1-\frac{v''_{12}}{w_1}) & 0 \\ -\frac{\bar{x}_2}{w_2} + \frac{1}{w_2}\frac{dM_2}{dt} & -\frac{\bar{x}'_1}{w_1} + \frac{1}{w_1}\frac{dM_1}{dt} + (1-\frac{v''_{12}}{w_1})\,\frac{2q_3}{w_1} & -\frac{v_2}{w_2} \end{bmatrix}$$

$$\mathcal{E} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}K_1(1-\frac{v''_{12}}{w_1})^2 & 0 \\ 0 & 0 & -2(1-\frac{v''_{12}}{w_1})^2 \end{bmatrix} \quad w_1 = \begin{bmatrix} 0 \\ -\frac{1}{2}K_1(1-\frac{v''_{12}}{w_1})\,\delta H v'_{11} \\ 2(1-\frac{v''_{12}}{w_1})\,\delta H \bar{x}'_1 \end{bmatrix}$$

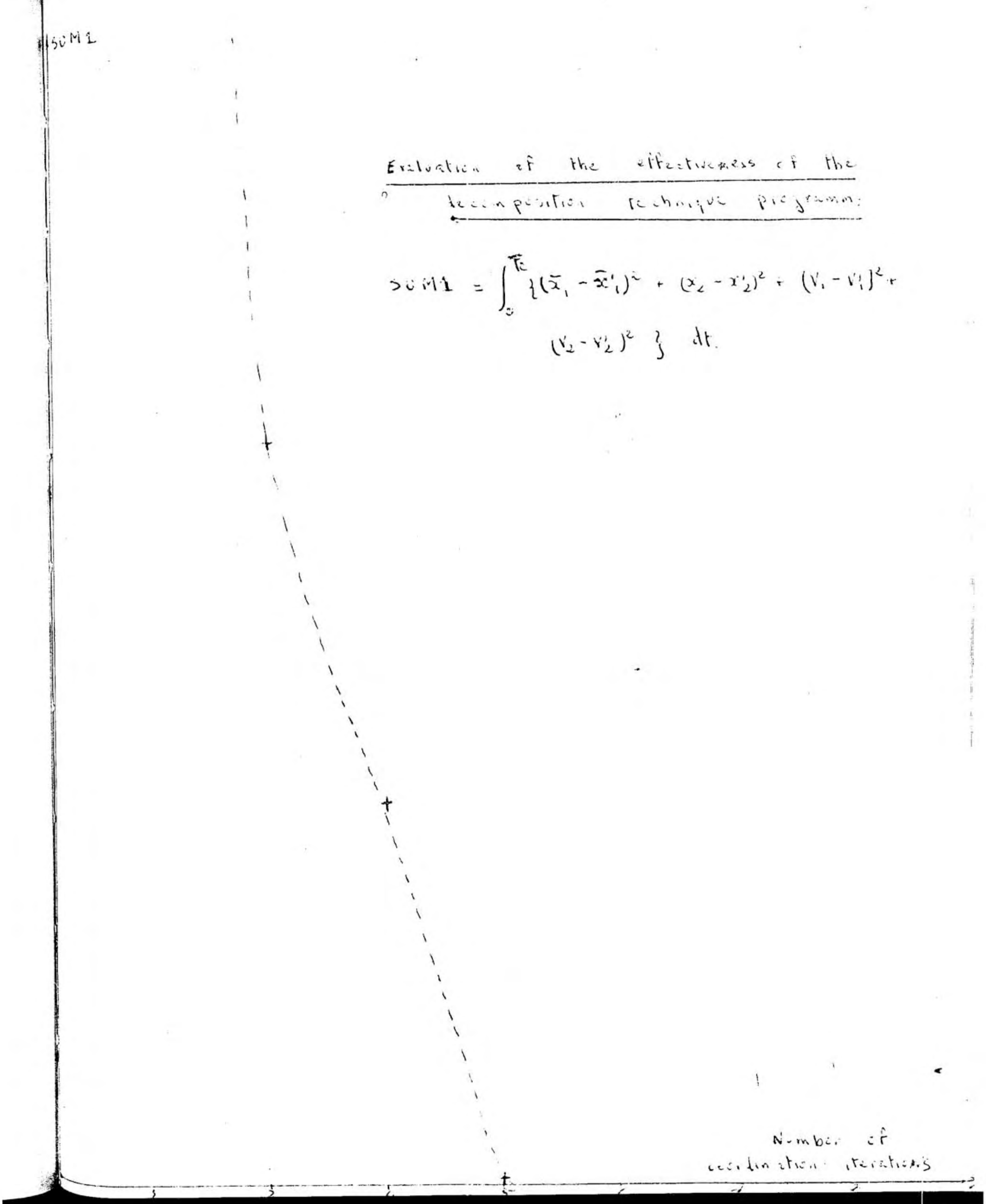
-Results: see Appendix 5, Program 9.



Computation time (total & per iteration) for the system with disturbances, when the decomposition technique is used.

Evaluation of the effectiveness of the decomposition technique programming

$$SUM1 = \int_0^{T_c} \{(\bar{x}_1 - \bar{x}'_1)^2 + (x_2 - x'_2)^2 + (y_1 - y'_1)^2 + (y_2 - y'_2)^2\} \, dt.$$



We can make the following observations:

-The decomposition technique for deterministic systems is based on a saddle value point argumentation. Since we are dealing with another kind of problem, i.e., a stochastic problem, it should be shown that the optimum point is a saddle value point. This argument is still true since we transformed the stochastic problem into a more complex but deterministic linear quadratic problem.

-The advantages of the decomposition technique are the same as in the deterministic case.

-To find the equations of the subsystems we had first to compute the equations of the global system. The set of equations of the two subsystems is equivalent to the set of equations of the global system. So the solutions will be the same. In particular, the information about the measurements of the global system are included in the equations of each subsystem.

4.2 A Case of Linear System With Non-Linear Coupling

In this section we deal with a numerical example of linear system with non-linear coupling. It is the case of slightly non-linear systems, the non-linearity of which can be expanded in a Taylor's series for which all the terms $\{ x^n / n \geq 3 \}$ are neglected.

4.2.1 The Deterministic Problem

Part 1: global method:

-system:

$$\dot{x}_1 = m$$

$$\dot{x}_2 = x_1 + 0.1\, x_1^2$$

-Control variable : m.

-State variables : $x_1 + x_2$.

-performance:

$$\phi = \int_0^{t_e} \frac{1}{2} [x_1^2 + x_2^2 + m_1^2]\, dt$$

-Computation:

$$\mathscr{D} = \begin{bmatrix} 0 & 0 \\ 1+0.2x_1 & 0 \end{bmatrix} \quad \mathscr{E} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \quad w_1 = \begin{bmatrix} \delta Hm \\ 0 \end{bmatrix}$$

$$\mathscr{L} = \begin{bmatrix} -(1+0.2p_2) & 0 \\ 0 & -1 \end{bmatrix} \quad w_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

-Results: see Appendix 5, Program 10.

Part 2: decomposition technique:

Level 1:

subsystem 1:

$$\dot{x}_1 = m \quad H = \frac{1}{2} m^2 + \frac{1}{2} x_1^2 + K x_1 (1+0.1 x_1) + p_1 m$$

subsystem 2:

$$\dot{x}_2 = t_1 \quad H = \frac{1}{2} x_2^2 - K t_1^2 + p_2 t_1$$

Level 2:

$$[K]_{n+1} = [K]_n + \varepsilon * [x_1 + 0.1 x_1^2 - t_1]$$

Computation:

subsystem 1:

$$\mathcal{D} = 0 \quad \mathcal{E} = -1 \qquad w_1 = \delta Hm \quad \mathcal{L} = -(1 \pm 0.2 K) \qquad w_2 = 0$$

subsystem 2:

$$\mathcal{D} = 0 \quad \mathcal{E} = -\frac{2 t_1^2}{p_2 - K} \qquad w_1 = -\frac{t_1 \delta Ht_1}{p_2 - K} \quad \mathcal{L} = -1 \qquad w_2 = 0$$

-Results: see Appendix 5, Program 11.

4.2.2 The Problem With Noises

A Statement of the problem:

System:

$$\left\{ dx_1 = x_1 \, dt + dz_1 \right.$$

$$dx_2 = (x_1 + 0.1\, x_1^2)\, dt + dz_2$$

Measurements:

$$\begin{cases} dM_1 = x_1\, dt + dN_1 \\ dM_2 = x_2\, dt + dN_2 \end{cases}$$

dz, dz_2, dN_1, dN_2 are uncorrelated noises with gaussian probability such that: $\forall t$, $\forall\, s \neq t$, we have

$$E[dz_1(t)] = E[dz_2(t)] = E[dN_1(t)] = E[dN_2(t)] = 0$$

$$E[dz_1(t)^2] = q_1\, dt \qquad E[dN_1^2(t)] = w_1\, dt$$

$$E[dz_2(t)^2] = q_2\, dt \qquad E[dN_2^2(t)] = w_2\, dt$$

We are assuming too that the variables x_1, x_2 have a two dimensional normal distribution, which, in fact, is not true, because of the non-linear term. But we shall consider that the coefficients of this term is small enough and does not greatly affect the distribution of x, the density function of which is given by:

$$F(x_1, x_2) = \frac{1}{2\pi \sqrt{v_{11}\, v_{22} - v_{12}^2}}\ \mathrm{Exp}\left\{ \frac{-1}{2\left(1 - \frac{v_{12}^2}{v_{11}\, v_{22}}\right)} \left[\frac{(x_1-\bar{x}_1)^2}{v_{11}} - \frac{2(x_1-\bar{x}_1)(x_2-\bar{x}_2)\, v_{12}}{v_{11}\, v_{22}} + \frac{(x_2-\bar{x}_2)^2}{v_{22}}\right] \right\}.$$

which means:

$$E[(x_1-\bar{x}_1)^2\ (x_2-\bar{x}_2)] = E\ \{\ [x_2-\bar{x}_2]^2\ [x_1-\bar{x}_1]\ \}$$

$$= E\ \{\ [x_1-\bar{x}_1]^{2K+1}\ \} = E\ \{\ [x_2-\bar{x}_2]^{2K+1}\ \} = 0$$

$$E\ \{\ (x_1-\bar{x}_1)^2\ (x_2-\bar{x}_2)^2\ \} = \frac{v_{11}\ v_{22}\ (v_{11}\ v_{22} + 2\ v_{12}^2)}{2(v_{11}\ v_{22} - v_{12}^2)}$$

B <u>The global method.</u>

The method is the same as in the linear case. We have the following steps:

$$\underline{\bar{x}}_1(t+\delta t) = \bar{x}_1(t) + m_1(t)\ \delta t$$

$$\underline{\bar{x}}_2(t+\delta t) = \bar{x}_2(t) + \delta t[\ddot{x}_1(t) + 0.1\ v_1(t) + 0.1\ x_1(t)^2]$$

$$\underline{v}_1(t+\delta t) = v_1(t) + q_1\ \delta t$$

$$\underline{v}_2(t+\delta t) = v_2(t) + \delta t\ [2.4\ v_{12}(t) + q_2]$$

$$\underline{v}_{12}(t+\delta t) = v_{12}(t) + v_1(t)\ (1 + 0.2\ \bar{x}_1(t))$$

The assumption of a normal distribution gives:

$$\Bigg\{ v_{11} = \frac{v_{11}}{} - \frac{v_{11}^2}{w_1}\,\delta t - \frac{v_{12}^2}{w_2}\,\delta t$$

with all the variables evaluated at $t+\delta t$.

$$
\left\{
\begin{aligned}
v_{12} &= v_{12} - \frac{v_{11}\, v_{12}}{w_1}\,\delta t - \frac{v_{22}\, v_{12}}{w_2}\,\delta t \\
v_{22} &= v_{22} - \frac{v_{12}^2}{w_1}\,\delta t - \frac{v_{22}^2}{w_2}\,\delta t \\
\bar{x}_1 &= \bar{x}_1 = \frac{\bar{x}_1\, v_{11}}{w_1}\,\delta t - \frac{\bar{x}_2\, v_{12}}{w_2}\,\delta t + \frac{v_{11}}{w_1}\,\delta M_1 \\
&\qquad + \frac{v_{12}}{w_2}\,\delta M_2 . \\
\bar{x}_2 &= \bar{x}_2 - \frac{\bar{x}_1\, v_{12}}{w_1}\,\delta t - \frac{\bar{x}_2\, v_{22}}{w_2}\,\delta t + \frac{v_{12}}{w_1}\,\delta M_1 \\
&\qquad + \frac{v_{22}}{w_2}\,\delta M_2 .
\end{aligned}
\right.
$$

Finally the equations of the system are:

$$
\left\{
\begin{aligned}
\frac{dv_{11}}{dt} &= q_1 - \frac{v_{11}^2}{w_1} - \frac{v_{12}^2}{w_2} \\
\frac{dv_{12}}{dt} &= v_{11}\,(1 + 0.2\,\bar{x}_1) - \frac{v_{11}\, v_{12}}{w_1} - \frac{v_{22}\, v_{12}}{w_2} \\
\frac{dv_{22}}{dt} &= 2.4\, v_{12} + q_2 - \frac{v_{12}^2}{w_1} - \frac{v_{22}^2}{w_2} \\
\frac{d\bar{x}_1}{dt} &= m_1 - \frac{\bar{x}_1\, v_{11}}{w_1} - \frac{\bar{x}_2\, v_{12}}{w_2} + \frac{v_{11}}{w_1}\frac{dM_1}{dt} + \frac{v_{12}}{w_2}\frac{dM_2}{dt} .
\end{aligned}
\right.
$$

$$\frac{d\bar{x}_2}{dt} = \bar{x}_1 + 0.1\ v_{11} + 0.1\ \bar{x}_1^2 - \frac{\bar{x}_1\ v_{12}}{w_1} - \frac{\bar{x}_2\ v_{22}}{w_2} + \frac{v_{12}}{w_1}\frac{dM_1}{dt} + \frac{v_{22}}{w_2}\frac{dM_2}{dt}$$

Criterion of performance:

$$\phi = \int_0^{t_e} \frac{1}{2}\ [\ \bar{x}_1^2 + \bar{x}_2^2 + v_{11} + v_{22} + m_1^2]\ dt$$

Computation of the solution

$$\mathcal{D} = \begin{bmatrix} -\frac{2\,v_{11}}{w_1} & -\frac{2\,v_{12}}{w_2} & 0 & 0 & 0 \\ 1 + 0.2\,\bar{x}_1 - \frac{v_{12}}{w_1} & -\frac{v_{11}}{w_1} - \frac{v_{22}}{w_2} & -\frac{v_{12}}{w_2} & 0.2\,v_{11} & 0 \\ 0 & 2.4 - \frac{2\,v_{12}}{w_1} & -\frac{2\,v_{22}}{w_2} & 0 & 0 \\ -\frac{\bar{x}_1}{w_1} + \frac{1}{w_1}\frac{dM_1}{dt} & -\frac{\bar{x}_2}{w_2} + \frac{1}{w_2}\frac{dM_2}{dt} & 0 & -\frac{v_{11}}{w_1} & -\frac{v_{12}}{w_2} \\ 0.1 & -\frac{\bar{x}_1}{w_1} + \frac{1}{w_1}\frac{dM_1}{dt} & -\frac{\bar{x}_2}{w_2} + \frac{1}{w_2}\frac{dM_2}{dt} & 1+0.2\bar{x}_1 - \frac{v_{12}}{w_1} & -\frac{v_{22}}{w_2} \end{bmatrix}$$

Computation of the solution (con't)

$$\mathcal{E} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad w_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \delta Hm_1 \\ 0 \end{bmatrix}$$

$$\mathcal{L} = \begin{bmatrix} \frac{2p_1}{w_1} & \frac{p_2}{w_1} & 0 & \frac{p_4}{w_1} - 0.2\,p_2 & 0 \\ \frac{p_2}{w_1} & \frac{2p_1}{w_2} + \frac{2p_3}{w_1} & \frac{p_2}{w_2} & \frac{p_5}{w_1} & \frac{p_4}{w_2} \\ 0 & \frac{p_2}{w_2} & \frac{2p_3}{w_2} & 0 & \frac{p_5}{w_2} \\ 0 & \frac{p_4}{w_2} & \frac{p_5}{w_2} & 0 & -1 \end{bmatrix}$$

$$w_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

C <u>The decomposition technique:</u>

As before we take the dummy variables:

$$\begin{cases} \bar{x}_1' = \bar{x}_1 \\ \bar{x}_2' = \bar{x}_2 \\ v_{11}' = v_{11} \\ v_{22}' = v_{22} \\ v_{12}' = v_{12}'' = v_{12}. \end{cases}$$

we get:

Level 1:

*subsystem 1:

-Control variables : m_1, v_{22}', $\bar{x}_2'$.

-performance:

$$\int_0^{t_e} \{\frac{1}{2}[\frac{1}{2}\,\bar{x}_1^2 + \frac{1}{2}\,\bar{x}_2'^2 + v_{11} + m_1^2] + R_1\,\bar{x}_1 + R_2\,v_{11}^2 - K_1\,\bar{x}_2' - K_2\,v_{22}'^2\}\,dt$$

-constraints: Lagrange multipliers

$$\begin{cases} \dfrac{dv_{11}}{dt} = -\dfrac{v_{11}^2}{w_1} - \dfrac{v_{12}'^2}{w_2} & p_1 \\ \dfrac{dv_{12}'}{dt} = v_{11}\,(1 + 0.2\,\bar{x}_1) - \dfrac{v_{11}\,v_{12}'}{w_1} - \dfrac{v_{22}'\,v_{12}'}{w_2} & p_2 \end{cases}$$

$$\left\{ \frac{d\bar{x}_1}{dt} = m_1 - \frac{\bar{x}_1 \, v_{11}}{w_1} - \frac{\bar{x}_2' \, v_{12}'}{w_2} + \frac{v_{11}}{w_1} \frac{dM_1}{dt} + \frac{v_{12}'}{w_2} \frac{dM_2}{dt} \right. \qquad p_3$$

subsystem 2:

-Control variables : v_{11}', $\bar{x}_1'$

-performance:

$$\int_0^{t_e} \{ \tfrac{1}{2} [\tfrac{1}{2} \bar{x}_1'^2 + \tfrac{1}{2} \bar{x}_2^2 + v_{22}] + K_1 \bar{x}_2 + K_2 v_{22}^2 - R_1 \bar{x}_1' - R_2 v_{11}'^2$$

-constraints: Lagrange multipliers

$$\frac{dv_{22}}{dt} = 2.4 \, v_{12}'' + q_2 - \frac{v_{12}''^2}{w_1} - \frac{v_{22}^2}{w_2} \qquad q_1$$

$$\frac{dv_{12}''}{dt} = v_{11}' \, (1 + 0.2 \, \bar{x}_1') - \frac{v_{11}' \, v_{12}''}{w_1} - \frac{v_{22} \, v_{12}''}{w_2} \qquad q_2$$

$$\frac{d\bar{x}_2}{dt} = \bar{x}_1' + 0.1 \, v_{11}' + 0.1 \, \bar{x}_1'^2 - \frac{\bar{x}_1' \, v_{12}''}{w_1} - \frac{\bar{x}_2 \, v_{22}}{w_2} + \frac{v_{12}''}{w_1} \frac{dM_1}{dt} + \frac{v_{22}}{w_2} \frac{dM_2}{dt} \qquad q_3$$

Level 2 or coordination level:

We vary R_1, R_2, K_1, K_2 according to:

$$[R_1]_{n+1} = [R_1]_n + \varepsilon * [\bar{x}_1 - \bar{x}_1']$$

$$[R_2]_{n+1} = [R_2]_n + \varepsilon * [v_{11}^2 - v_{11}'^2]$$

$$[K_1]_{n+1} = [K_1]_n + \varepsilon * [\bar{x}_2 - \bar{x}_2']$$

$$[K_2]_{n+1} = [K_2]_n + \varepsilon * [v_{22}^2 - v_{22}'^2]$$

Computation of the solution:

subsystem 1:

$$\mathcal{D} = \begin{bmatrix} -\dfrac{2\,v_{11}}{w_1} & -\dfrac{2\,v_{12}'}{w_2} & 0 \\ 1+0.2\bar{x}_1 - \dfrac{v_{12}'}{w_1} & -\dfrac{v_{11}}{w_1} - \dfrac{v_{22}'}{w_2} + \dfrac{p_2\, v_{12}'}{2K_2 w_2^2} & 0.2\,v_{11} \\ -\dfrac{\bar{x}_1}{w_1} + \dfrac{1}{w_1}\dfrac{dM_1}{dt} & -\dfrac{\bar{x}_2'}{w_2} + \dfrac{1}{w_2}\dfrac{dM_2}{dt} - \dfrac{2p_3\, v_{12}'}{w_2^2} & -\dfrac{v_{11}}{w_1} \end{bmatrix}$$

$$\mathcal{E} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{v_{12}'^2}{2K_2 w_2^2} & 0 \\ 0 & 0 & -1-\frac{2\ v_{12}'^2}{w_2^2} \end{bmatrix} \qquad w_1 = \begin{bmatrix} 0 \\ \frac{v_{12}'}{2K_2 w_2^2}\ \delta Hv_{22}' \\ \delta Hm_1 - \frac{2\ v_{12}'}{w_2}\ \delta Hx_2' \end{bmatrix}$$

$$\mathcal{L} = \begin{bmatrix} -2p_2 + \frac{2p_1}{w_1} & \frac{p_3}{w_1} & \frac{p_3}{w_1} - 0.2\ p_2 \\ \frac{p_2}{w_1} & \frac{2p_1}{w_2} - \frac{p_2^2}{2K_2 w_2^2} + \frac{2p_3^2}{w_2^2} & 0 \\ -0.2p_2 + \frac{p_3}{w_1} & 0 & -0.5 + 2\ K_1 \end{bmatrix}$$

$$w_2 = \begin{bmatrix} 0 \\ -\frac{p_2}{2K_2 w_2}\ \ \delta Hv_{22}' + \frac{2p_3}{w_2}\ \delta H\bar{x}_2' \\ 0 \end{bmatrix}$$

<u>subsystem 2:</u>

The equations are far more complex. Only the equations were derived:

$$\delta\bar{x}_1' = \frac{1}{1+0.4\ q_3}\ [-0.4\ q_2\ \delta v_{11}' + \frac{2q_3}{w_1}\ \delta v_{12}'' - 0.4\ v_{11}'\ \delta q_2$$

$$- 2(1 + 0.2\ \bar{x}_{11}' - \frac{v_{12}''}{w_1})\ \delta q_3 + 2\delta Hx_1']$$

$$\delta v_{11}' = \frac{1}{2R_2}\ [\ - \frac{q_2}{w_1}\ \delta v_{12}'' + (1 + 0.2\ \bar{x}_1' - \frac{v_{12}''}{w_1})\ \delta q_2 + 0.1\ \delta q_3$$

$$- \delta Hv_{11}' + 0.2\ q_2\ \delta\bar{x}_1']$$

$$\left\{\begin{aligned}
\delta\dot{v}_{22} &= - \frac{2\ v_{22}}{w_2}\ \delta v_{22} + (2.4 - \frac{2\ v_{12}''}{w_1})\ \delta v_{12}'' \\
\delta\dot{v}_{12}'' &= - \frac{v_{12}''}{w_2}\ \delta v_{22} + (- \frac{v_{22}}{w_2} - \frac{v_{11}'}{w_1})\ \delta v_{12}'' + (1 + 0.2\ \bar{x}_1' \\
&\qquad - \frac{v_{12}''^{2}}{w_1}\ \delta v_{11}' + 0.2\ v_{11}'\ \delta\bar{x}_1' \\
\delta\dot{\bar{x}}_2 &= (\ - \frac{\bar{x}_2}{w_2} + \frac{1}{w_2}\ \frac{dM_2}{dt})\ \delta v_{22} + (\ - \frac{\bar{x}_1'}{w_1} + \frac{1}{w_1}\ \frac{dM_1}{dt})\ \delta v_{12}'' \\
&\quad - \frac{v_{22}}{w_2}\ \delta\bar{x}_2 + 0.1\ \delta v_{11}' + (1.\ 0.2\ \bar{x}_1' - \frac{v_{12}''}{w_1})\ \delta\bar{x}_1'.
\end{aligned}\right.$$

$$\left\{\begin{aligned}
\delta\dot{q}_1 &= - \{(2K_2 - \frac{2q_1}{w_2})\ \delta v_{22} - \frac{q_2}{w_2}\ \delta v_{12}'' - \frac{2\ v_{22}}{w_2}\ \delta q_1 \\
&\quad - \frac{v_{12}''}{w_2}\ \delta q_2 + (- \frac{x_2}{w_2} + \frac{1}{w_2}\ \frac{dM_2}{dt})\ \delta q_3 - \frac{q_3}{w_2}\ \ \delta\bar{x}_2\}
\end{aligned}\right.$$

$$\left\{\begin{aligned}
\delta\dot{q}_2 &= -\left\{-\frac{q_2}{w_2}\;\delta v_{22} - \frac{2q_1}{w_1}\,\delta v''_{12} + \left(2.4 - \frac{2\;v''_{12}}{w_1}\right)\delta q_1\right.\\
&\quad \left. + \left(-\frac{v'_{11}}{w_1} - \frac{v_{22}}{w_2}\right)\;\delta q_2 + \left(-\frac{\bar{x}'_1}{w_1} + \frac{1}{w_1}\,\frac{dM_1}{dt}\right)\;\delta q_3 - \frac{q_3}{w_1}\,\delta\bar{x}'_1\right\}\\
\delta\dot{q}_3 &= -\left\{\,-\frac{q_3}{w_2}\,\delta v_{22} + 0.5\;\delta\bar{x}_2\right\}.
\end{aligned}\right.$$

4.3 Generalization

Let us consider a general system with additive noise:

system:

$$\begin{cases} \dot{x}_1 = f_1\ (x_1,\ x_2,\ m_1,\ m_2,\ t) + \dfrac{dZ_1}{dt} \\ \dot{x}_2 = f_2\ (x_1,\ x_2,\ m_1,\ m_2,\ t) + \dfrac{dZ_2}{dt} \end{cases}$$

measurements:

$$\begin{cases} \dfrac{dM_1}{dt} = x_1 + \dfrac{dN_1}{dt} \\ \dfrac{dM_2}{dt} = x_2 + \dfrac{dN_2}{dt} \end{cases}$$

performance:

$$\int_0^{t_e} \varphi\ [v_{11},\ v_{22},\ v_{12},\ \bar{x}_1,\ \bar{x}_2]$$

controls : m_1, m_2.

state variables : x_1, x_2.

We shall call

$$g_1\ [v_{11},\ v_{22},\ v_{12},\ \bar{x}_1,\ \bar{x}_2,\ m_1,\ m_2,\ t] = E\ \{f_1\}.$$

$$g_2\ [v_{11},\ v_{22},\ v_{12},\ \bar{x}_1,\ \bar{x}_2,\ m_1,\ m_2,\ t] = E\ \{f_2\}.$$

$$g_3\ [v_{11},\ v_{22},\ v_{12},\ \bar{x}_1,\ \bar{x}_2,\ m_1,\ m_2,\ t] = E\ \{(f_1-g_1)(x_1-\bar{x}_1)\}.$$

$$g_4\ [v_{11},\ v_{22},\ v_{12},\ \bar{x}_1,\ \bar{x}_2,\ m_1,\ m_2,\ t] = E\ \{(f_2-g_2)(x_2-\bar{x}_2)\}.$$

$$g_5\ [v_{11},\ v_{22},\ v_{12},\ \bar{x}_1,\ \bar{x}_2,\ m_1,\ m_2,\ t] = E\ \{(f_2-g_2)(x_1-\bar{x}_1)\}.$$

$$g_6\ [v_{11},\ v_{22},\ v_{12},\ \bar{x}_1,\ \bar{x}_2,\ m_1,\ m_2,\ t] = E\ \{(f_1-g_1)(x_2-\bar{x}_2)\}.$$

Two strong assumptions have to be done on the system:

-The possibility of using the decomposition technique for the deterministic problem, i.e., we must check the argumentation based on a saddle value point at the optimal.

-We assume during all the demonstration that the variable (x_1, x_2) has a two dimensional normal distribution which is unlikely if the system is strongly non-linear.

We have:

$$x_1(t+\delta t) = x_1(t) + f_1 \delta t + \delta Z_1$$

$$x_2(t+\delta t) = x_2(t) + f_2 \delta t + \delta Z_2.$$

Taking the expectations, we get:

$$\begin{cases} \underline{\bar{x}}_1(t+\delta t) = \bar{x}_1(t) + g_1 \ \delta t \\ \underline{\bar{x}}_2(t+\delta t) = \bar{x}_2(t) + g_2 \ \delta t \end{cases}$$

$$\begin{cases} \underline{v}_1(t+\delta t) = v_1(t) + \delta t(2 \ g_3 + q_1) \\ \underline{v}_2(t+\delta t) = v_2(t) + \delta t(2 \ g_4 + q_2) \\ \underline{v}_{12}(t+\delta t) = v_{12}(t) + \delta t \ (g_5 + g_6). \end{cases}$$

As before, the assumption of a normal distribution gives us:

$$\begin{cases} v_1 = \underline{v_1} - \dfrac{\underline{v_1^2}}{w_1} \delta t - \dfrac{\underline{v_{12}^2}}{w_2} \delta t \\ v_{12} = \underline{v_{12}} - \dfrac{\underline{v_{11}} \ \underline{v_{12}}}{w_1} \delta t - \dfrac{\underline{v_{22}} \ \underline{v_{12}}}{w_2} \delta t \\ v_{22} = \underline{v_{22}} - \dfrac{\underline{v_{12}^2}}{w_1} \delta t - \dfrac{\underline{v_{22}^2}}{w_2} \delta t \end{cases}$$

$$\begin{cases} \bar{x}_1 = \underline{\bar{x}}_1 - \dfrac{\bar{x}_1 \; v_{11}}{w_1} \delta t - \dfrac{\bar{x}_2 \; v_{12}}{w_2} \delta t + \dfrac{v_{11}}{w_1} \delta M_1 + \dfrac{v_{12}}{w_2} \delta M_2 . \\ \underline{\bar{x}}_2 = \underline{\bar{x}}_2 - \dfrac{\bar{x}_1 \; v_{12}}{w_1} \delta t - \dfrac{\bar{x}_2 \; v_{12}}{w_1} \delta t - \dfrac{\bar{x}_2 \; v_2}{w_2} \delta t + \dfrac{v_{12} \; \delta M_1}{w_1} \\ \qquad + \dfrac{v_{22} \; \delta H_2}{w_2} \end{cases}$$

Combining the two systems of equations, we get:

$$\begin{cases} \dfrac{dv_{11}}{dt} = 2 \; g_3 + q_1 - \dfrac{v_{11}^2}{w_1} g_1^2 - \dfrac{v_{12}^2}{w_2} g_2^2 \\ \dfrac{dv_{12}}{dt} = g_5 + g_6 - \dfrac{v_{11} \; v_{12}}{w_1} \; g_1^2 - \dfrac{v_{22} \; v_{12}}{w_2} \; g_2^2 \\ \dfrac{dv_{22}}{dt} = 2 \; g_4 + q_2 - \dfrac{v_{12}^2 \; g_1^2}{w_1} - \dfrac{v_{22} \; v_{12}}{w_2} g_2^2 \end{cases}$$

$$\begin{cases} \dfrac{d\bar{x}_1}{dt} = g_1 - \dfrac{\bar{x}_1 \; v_{11}}{w_1} - \dfrac{\bar{x}_2 \; v_{12}}{w_2} + \dfrac{v_{11}}{w_1} \dfrac{dM_1}{dt} + \dfrac{v_{12}}{w_2} \dfrac{dM_2}{dt} \\ \dfrac{d\bar{x}_2}{dt} = g_2 - \dfrac{\bar{x}_1 \; v_{12}}{w_1} - \dfrac{\bar{x}_2 \; v_{22}}{w_2} + \dfrac{v_{12}}{w_1} \dfrac{dM_1}{dt} + \dfrac{v_{22}}{w_2} \dfrac{dM_2}{dt} \end{cases}$$

We succeeded in transforming the problem with stochastic variables in a deterministic problem. So we can apply the decomposition technique:

Level 1:

*subproblem 1:

-State variables : $\bar{x}_1$, v_{11}, v'_{12}

-Controls : m_1, $\bar{x}'_2$, v'_{22}

system:

$$\begin{cases} \dfrac{dv_{11}}{dt} = 2\ g_3 + q_1 - \dfrac{v_{11}^2}{w_1}\ g_1^2 - \dfrac{v_{12}'^2\ g_2^2}{w_2} \\[2ex] \dfrac{dv_{12}'}{dt} = g_5 + g_6 - \dfrac{v_{11}\ v_{12}'}{w_1}\ g_1^2 - \dfrac{v_{22}'\ v_{12}'}{w_2}\ g_2^2 \\[2ex] \dfrac{d\bar{x}_1}{dt} = g_1 - \dfrac{\bar{x}_1\ v_{11}}{w_1} - \dfrac{x_2\ v_{12}'}{w_2} + \dfrac{v_{11}}{w_1}\dfrac{dM_1}{dt} + \dfrac{v_{12}'}{w_2}\dfrac{dM_2}{dt} \end{cases}$$

performance:

$$\phi = \int_0^{t_e}\{\tfrac{1}{2}\ \varphi\ [v_{11},\ v_{22}, v_{12}',\ \bar{x}_1,\ \bar{x}_2'] + R_1\ x_1 + R_2\ v_{11}$$

$$- K_1\ \bar{x}_2' - K_2\ v_{22}'\}\ dt$$

subproblem 2:

-State variables : $\bar{x}_2$, v_{22}, v_{12}''

-Controls : m_2, v_1', $\bar{x}_1'$

system:

$$\begin{cases} \dfrac{dv_{22}}{dt} = 2\ g_4 + q_2 - \dfrac{v_{12}''^2\ g_1^2}{w_1} - \dfrac{v_{22}\ v_{12}''}{w_2}\ g_2^2 \\[2ex] \dfrac{dv_{12}''}{dt} = g_5 + g_6 - \dfrac{v_{11}'\ v_{12}''}{w_1}\ g_1^2 - \dfrac{v_{22}\ v_{12}''}{w_2}\ g_2^2 \\[2ex] \dfrac{d\bar{x}_2}{dt} = g_2 - \dfrac{\bar{x}_1'\ v_{12}''}{w_1} - \dfrac{\bar{x}_2\ v_{22}}{w_2} + \dfrac{v_{12}''}{w_1}\dfrac{dM_1}{dt} + \dfrac{v_{22}}{w_2}\dfrac{dM_2}{dt} \end{cases}$$

performance:

$$\phi = \int_0^{t_e} \{ \frac{1}{2} \varphi [v'_{11}, v_{22}, v''_{12}, \bar{x}'_1, \bar{x}_2] - R_1 \bar{x}'_1 - R_2 v'_{11} + K_1 \bar{x}_2 + K_2 v_{22} \} dt.$$

Level 2 or coordination level:

We vary K_1, K_2, R_1, R_2 according to:

$$[R_1]_{n+1} = [R_1]_n + \varepsilon * [\bar{x}_1 - \bar{x}'_1]$$

$$[R_2]_{n+1} = [R_2]_n + \varepsilon * [v_{11} - v'_{11}]$$

$$[K_1]_{n+1} = [K_1]_n + \varepsilon * [\bar{x}_2 - \bar{x}'_2]$$

$$[K_2]_{n+1} = [K_2]_n + \varepsilon * [v_{22} - v'_{22}].$$

CHAPTER V

CONCLUSIONS

The application of the satisfaction approach of Takahara to linear quadratic systems was easy to implement, once it was shown that the set of internal disturbances could be reduced to one point. To extend this technique to non linear systems, a way of formulating the mathematical models of the first level and of implementing the coordination must be found first.

A heuristic dealing with discrete linear systems was studied in Chapter 3. The main advantage of this heuristic lies in the fact that the programming of the computation of the optimal control is much easier to do than with the general method. This heuristic dealing with separate subsystems, should be of some interest from a computing time point of view. But this has to be proved. One can expect, too, that, with some more assumptions, this heuristic converges for linear continuous systems.

In Chapter 4, the decomposition technique was applied to linear systems with disturbances, with the help of the Kalman technique. No significant reduction in the computing time was given by the application of the technique, but it should be remembered that the technique is efficient for large scale systems, which was not the case here. For non linear systems, the state variable is not likely to have a gaussian distribution, and the Kalman technique cannot be used. So the problem of application of the decomposition technique to non linear systems with disturbances has still to be solved.

APPENDIX I

A COMPUTATIONAL TECHNIQUE USED TO SOLVE OPTIMAL PROGRAMMING PROBLEMS: THE SUCCESSIVE SWEEP METHOD.

A complete study of the successive sweep method can be found in reference [6]. The method will be explained briefly here in order to define the notations which will be used later.

Consider the system:

$$\dot{x} = f\ (x, m, t) \qquad x(0) = x_0$$

The performance can be written as:

$$\phi = \int_0^{t_e} [H\ (p, x, m) - p^T \dot{x}]\ dt$$

with:

x(t): n-component state vector.

m(t): m-component control vector.

p(t): n-component Lagrange multiplier vector.

The necessary conditions for an extremal path are:

$$\dot{p} = -H_x \tag{1}$$

$$H_m = 0 \tag{2}$$

$$p(t_e) = 0 \tag{3}$$

If some arbitrary control function m(t) is chosen, then equation (2) will not be satisfied. We consider a perturbation around m(t): we get:

$$\begin{cases} \delta\dot{x} = f_x\, \delta_x + f_m\, \delta_m & (4) \\ \delta\dot{p} = -\, H_{xx}\, \delta_x = f_x^T\, \delta_p - H_{xm}\, \delta_m & (5) \\ \delta H_m = H_{mx}\, \delta_x + H_{mm}\, \delta_m + f_m^T\, \delta_p & (6) \end{cases}$$

Solving (6) for $\delta m(t)$ gives:

$$\delta m(t) = -\, H_{mm}^{-1}\, [\, -\, \delta H_m + H_{mx}\, \delta_x + f'_m\, \delta_p] \qquad (7)$$

We can now write:

$$\delta\dot{x} = \mathscr{D}\, \delta_x + \mathscr{E}\, \delta_p + w_1 \qquad (8)$$

$$\delta\dot{p} = \mathscr{L}\, \delta_y - \mathscr{D}^T\, \delta_p + w_2 \qquad (9)$$

with:

$$\mathscr{D} = f_x - f_m\, H_{mm}^{-1}\, H_{mx}$$

$$\mathscr{E} = -f_m\, H_{mm}^{-1}\, f_m^T$$

$$\mathscr{L} = -H_{xx} + H_{xm}\, H_{mm}^{-1}\, H_{mx}$$

$$w_1 = f_m\, H_{mm}^{-1}\, \delta H_m$$

$$w_2 = -H_{xm}\, H_{mm}^{-1}\, \delta H_m$$

To solve this problem we use the usual matrix Riccati transformation: we express δ_p as a function of δ_x such that (8) and (9) are satisfied.

$$\delta p(t) = T(t)\, \delta x(t) + h(t) \qquad (10)$$

The result of this transformation yields the following equations:

$$\dot{T} = -\mathcal{D}^T T - T \mathcal{D} - T \mathcal{E} T + \mathcal{L} \qquad (11)$$

$$T(t_e) = 0$$

$$\dot{h} = -(\mathcal{D}^T + T\mathcal{E})\, h - T w_1 + w_2 \qquad (12)$$

$$h(t_e) = 0$$

Substituting (10) into (7) gives an equation for δm which will produce a change δH_m in H_m as required.

$$\delta m(t) = -H_{mm}^{-1} \{[H_{mx} + f_m^T T]\, \delta x + [-\delta H_m + f_m^T h]\} \qquad (13)$$

Because of the boundary conditions, $\dot{x}(t)$ is integrated forward in time while $\dot{p}(t)$, $\dot{T}(t)$, and $\dot{h}(t)$ are integrated backward in time.

By repeating this forward-backward sweep N times we can bring H_m to zero. A reasonable policy is to choose for each step:

$$\delta H_m^{(j)}(t) = -\frac{J}{N} H_m^{(j-1)}(t)$$

where j is the step number.

In this way larger and larger reductions are made with each step, and with the last step the whole remaining correction is made.

APPENDIX II

THE DECOMPOSITION TECHNIQUE

This brief summary and explaination of the decomposition technique for linear quadratic deterministic problems is done in order to precise the method and the notations used in Chapter 4. For more details on the decomposition technique, refer to [3].

The decomposition technique can be presented using a saddle value argument on the variational form of a control problem.

Consider the following system:

$$\begin{cases} \dot{x}_1 = c_{11}\, x_1 + c_{12}\, t_2 + \ell_1\, m_1 \\ \dot{x}_2 = c_{21}\, t_1 + c_{22}\, x_2 + \ell_2\, m_2 \end{cases}$$

or

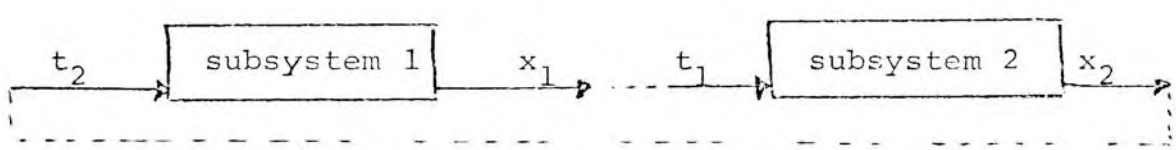
$$\dot{x} = C\, x + C'\, t + L\, m$$

with the following constraints:

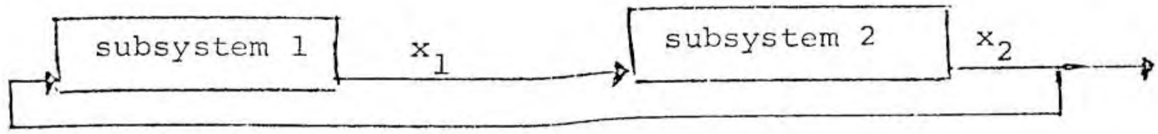
$$x_2 = t_2 \qquad\qquad x_1 = t.$$

This system can be represented in two ways:

1)



2)



The cost function is:

$$\int_0^{t_e} \{ A_1 (r_1-x_1)^2 + A_2 (r_2-x_2)^2 + B_1 m_1^2 + B_2 m_2^2 \} dt$$

Proceeding directly to the solution the Lagrangian form for the integrated system is given by:

$$J (x_1, x_2, t_1, t_2, m_1, m_2, p_1, P_2, K_1, K_2) = A_1 (r_1-x_1)^2$$

$$+ A_2 (r_2-x_2)^2 + B_1 m_1^2 + B_2 m_2^2, + K_1 (x_1-t_1) + K_2 (x_2-t_2)$$

$$+ p_1 [c_{11} x_1 + c_{12} t_2 + \ell_1 m_1 - \dot{x}_1] + p_2 [c_{21} t_1 + c_{22} x_2$$

$$+ \ell_2 m_2 - \dot{x}_2].$$

For this problem it is known that:

$$J (x_1, x_2, t_1, t_2, m_1, m_2, P_1, P_2, K_1, K_2)$$

has a saddle value as follows:

$$J^0 = J\,[x_1^0,\ x_2^0,\ t_1^0,\ t_2^0,\ m_1^0,\ m_2^0,\ p_1^0,\ p_2^0,\ K_1^0,\ K_2^0] = \underset{p_1 p_2 K_1 K_2}{\text{Max}} \quad \underset{x_1 x_2 t_1 t_2 m_1 m_2}{\text{Min}} \quad J\,[x_1,\ x_2,\ t_1,\ t_2,\ m_1,\ m_2,\ p_1,\ p_2,\ K_1,\ K_2].$$

This last equation, together with the separable nature of the problem directly suggests a two-level procedure for the solution:

Level 1:

Given the arbitrary bounded continuous functions $K_1(t)$, $K_2(t)$, $0 \le t \le t_e$, find the optimal solutions $x_1^0(t, K_1, K_2)$, $x_2^0(t, K_1, K_2)$, $t_1^0(t, K_1, K_2)$, $t_2^0(t, K_1, K_2)$, $m_1^0(t, K_1, K_2)$, $m_2^0(t, K_1, K_2)$ which minimize the parametric subproblems subject to the independent subsystems equations.

$$\min_{x_1 x_2 t_1 t_2 m_1 m_2} J[x_1, x_2, t_1, t_2, m_1, m_2] \equiv \min_{x_1, t_2, m_1} J_1 + \min_{x_2, t_1, m_2} J_2.$$

with

$$J_1 = A_1 (r_1 - x_1)^2 + B_1 m_1^2 + K_1 x_1 - K_2 t_2 + p_1 [c_{11} x_1 + c_{12} t_2 + \ell_1 m_1 - \dot{x}_1]$$

and

$$J_2 = A_2(r_2-x_2)^2 + B_2\, m_2^2 + K_2\, x_2 - K_1\, t_1 + p_2\, [c_{21}\, t_1$$

$$+ c_{22}\, x_2 + \ell_2\, m_2 - \dot{x}_1].$$

Level 2:

Given the optimal solutions from level 1: $x_1^0(t,K_1,K_2)$, $x_2^0(t,K_1,K_2)$, $m_1^0(t,K_1,K_2)$, $m_2^0(t,K_1,K_2)$, $t_1^0(t,K_1,K_2)$, $t_2^0(t,K_1,K_2)$, find the optimal coordinating functions $K_1^0(t)$, $K_2^0(t)$ which maximize

$$J(K_1,K_2) = J[x_1^0(t,K_1,K_2),\ x_2^0(t,K_1,K_2),\ t_1^0(t,K_1,K_2)$$

$$t_2^0(t,K_1,K_2),\ m_1^0(t,K_1,K_2),\ m_2^0(t,K_1,K_2),\ K_1,\ K_2].$$

The computation technique for this second level is easy to do:

Consider a perturbation in $K_1(t)$ to $K_1(t) + \delta K_1(t)$, in $K_2(t)$ to $K_2(t) + \delta K_2(t)$.

$$\delta J(K_1,K_2) = \frac{\delta J}{\delta m_1}\,\delta m_1 + \frac{\delta J}{\delta m_2}\,\delta m_2 + \frac{\delta J}{\delta x_1}\,\delta x_1 + \frac{\delta J}{\delta x_2}\,\delta x_2$$

$$+ \frac{\delta J}{\delta t_1}\,\delta t_1 + \frac{\delta J}{\delta t_2}\,\delta t_2 + \frac{\delta J}{\delta K_1}\,\delta K_1 + \frac{\delta J}{\delta K_2}\,\delta K_2 + \text{higher order}$$

term where $\frac{\delta J}{\delta m_1}$, ... denotes the functional derivatives or first derivations.

But when we solve for the optimal subproblems (1^{st} level) the following conditions are satisfied:

$$\frac{\delta J}{\delta m_1} = \frac{\delta J}{\delta m_2} = \frac{\delta J}{\delta x_1} = \frac{\delta J}{\delta x_2} = \frac{\delta J}{\delta t_1} = \frac{\delta J}{\delta t_2} = 0$$

Therefore, we get:

$$\delta J \ (K_1 \ K_2) = \frac{\delta J}{\delta K_1} \ \delta K_1 + \frac{\delta J}{\delta K_2}$$

$$= (x_1 - t_1) \ \delta K_1 + (x_2 - t_2) \ \delta K_2 .$$

Therefore, a gradient method of adjustment for $K_1(t)$, $K_2(t)$ to give steepest ascent is given by:

$$[K_1]_{n+1} = [K_1]_n + \varepsilon \ [x_1 - t_1]$$

$$[K_2]_{n+1} = [K_2]_n + \varepsilon \ [x_2 - t_2].$$

where ε is a number chosen small enough to insure correctness of the first order expansion.

APPENDIX III, PROGRAM 1

```
B0                                REAL V,TF,T,I,J,F,X10,X20,X30,X40,X50,N,[illegible]
BLOCK 1     LEVEL 1
                                  REAL ARRAY C11,C12,C13,C14,C15,C21,C22,C23,C24,C25,
                                  M1,M2,[illegible],M4,[illegible], Z,X1I,X2I,X3I,X4I,X5I,X1,X2,
                                  C31,C32,C34,C35,C41,C42,C43,C45,C44,    C73,C1,C2,H0,[illegible]
                                  C51,C52,C53,C54,C55,A1,A2,A3,A4,A5,B1,B2,B3,B4,B5,
                                                 [illegible]2,Y22,Y32,Y42,Y52,[illegible]
                                  R11,R22,R33,R44,R55,                 [illegible]
                                  HG11,H211,H311,H411,H511,H611,H711,H811,H911,X3,X4,X5,[illegible]
                                  DX1,DX2,DX3,DX4,DX5,DELC,X311,X411,X511,
                                  X1R,X2R,X3R,X4R,X5R,C,DELR(-1..101);
                                  PROCEDURE OPTI (C,A,F,X,[illegible],TF,[illegible])
BLOCK 2     LEVEL 2
                                  VALUE C,A,B,X0,[illegible],V,N,H ;
                                  REAL ARRAY C,A,B,X,H,R ;
                                  REAL X0,TF,[illegible] ;
B1                                BEGIN REAL U,E,T ;
                                  REAL ARRAY HA,[illegible],F,T2,H2,XS(-1..101) ;
                                  FOR T=(0,1,TF) DO X(T)=0;
                                  FOR I=(1,1,N+1) DO BEGIN
B2                                F=-U/N ;
                                  COMMENT FORWARD INTEGRATION ;
                                  FOR T=(0,1,TF) DO BEGIN
B3                                X(0)=X0;
                                  X(T+1)=X(T)+V*(-C(T)*X(T)+U(T)) ;
E3                                END;
                                  COMMENT BACKWARDS INTEGRATION ;
B4                                FOR T=(TF,-1,0) DO BEGIN   F(TF)=0;
                                  P(T-1)=P(T)+V*(2*A(T)*(X(T)-R(T))+F(T)*C(T)) ;
                                  HA(T)=2*H(T)*A(T)+F(T) ;
E8                                IF I EQL (N+1) THEN GOTO  B5;
                                  T2(TF)=0*H2(TF)=0;
                                  T2(T-1)=T2(T)-V*(-2*C(T)*T2(T)+H(T)*(2(T)/(2*B(T))
                                  +E*HA(T)/(2*B(T))              );
                                  H2(T-1)=H2(T)-V*(-(C(T)-T2(T)/(2*B(T)))*H2(T)-E*HA(T)*T2(T)/(2*B(T))),
                                  U(T)=U(T)+E*HA(T)/(2*B(T))-(T2(T)*(X(T)-X2(T))+H2(T))/(2*B(T))
                                  X2(T)=X(T);
E4                                B2,,END;
                                  FOR T=(0,1,TF) DO BEGIN
B5                                DEL(0)=0;
                                  DEL(T+1)=DEL(T)+V*(A(T)*(X(T)-R(T))*(A(T)*X(T)-R(T))+B(T)*U(T)*U(T)) ;
                                  IF T EQL TF THEN
                                  WRITE (DEL(T),[illegible],DEL(T),U,T);
E5                                END;
```

```
E2                          ENDS
                            FOR T=(0,1,TF) DO WRITE (*HM(T),DEL(T),T,*,HM(T),DEL(T),T)S
E1                          ENDS
END BLOCK 2   LEVEL 2
                            PROCEDURE TAK (D0,D2,D3,D4,D5,D6,D7,D8,D9,Y1,Y2,Y3,R1,R2,R3,C11,C12,C13,
BLOCK 3       LEVEL 2
                            D21,D22,D23,D31,D32,D33,F1,F2,F3,M11,M21,M31,B11,B21,B31,
                            Y10,Y20,Y30,TF,V,N)S
                            VALUE D0,D2,D3,D4,D5,D6,D7,D8,D9,R1,R2,R3,D11,D12,D13,D21,D22,D23,
                            D31,D32,D33,F1,F2,F3,P11,B21,B31,Y10,Y20,Y30,TF,V,N
                            REAL Y10,Y20,Y30,TF,V,N
                            REAL ARRAY D0,D2,D3,D4,D5,D6,D7,D8,D9,Y1,Y2,Y3,R1,R2,R3,C11,C12,C13,
                            D21,D22,D23,D31,D32,D33,F1,F2,F3,M11,M21,M31,B11,B21,B31S
B6                          BEGIN REAL J,K,I,E,T
                            REAL ARRAY HM11,HM21,HM31,H1,H2,H3,DEL1,K1,K2,K3,Y1S,Y2S,
                            Y3S(-1,,101),T1,D1,E1,
                            L1(-1,,101,1,,3,1,,3)S
B7                          FOR T=(0,1,TF) DO BEGIN M11(T)=0SM21(T)=0SM31(T)=0S
E7                          ENDS
                            FOR       L=(1,1,N+1) DO BEGIN
B8                          E=-L/NS
                            COMMENT FORWARDS INTEGRATION  S
                            FOR T=(0,1,TF) DO BEGIN
B9                          Y1(0)=Y10SY2(0)=Y20SY3(0)=Y30S
                            Y1(T+1)=Y1(T)+V*(C11(T)*Y1(T)+D12(T)*Y2(T)+C13(T)*Y3(T)+F1(T)+[illegible]
                            Y2(T+1)=Y2(T)+V*(D21(T)*Y1(T)+D22(T)*Y2(T)+D23(T)*Y3(T)+F2(T)+[illegible]
                            Y3(T+1)=Y3(T)+V*(D31(T)*Y1(T)+D32(T)*Y2(T)+D33(T)*Y3(T)+F3(T)+[illegible]
E9                          ENDS
                            COMMENT BACKWARDS   INTEGRATIONS
                            FOR T=(TF,-1,0)          DO BEGIN
B10                         K1(TF)=0SK2(TF)=0SK3(TF)=0S
                            K1(T-1)=K1(T)+V*(2*D0(T)*(Y1(T)-R1(T))+2*D4(T)*(Y2(T)-R2(T))+
                            2*D5(T)*(Y3(T)-R3(T))+K1(T)*C11(T)+K2(T)*D21(T)+[illegible]
                            K2(T-1)=K2(T)+V*(2*D2(T)*(Y2(T)-R2(T))+2*D4(T)*(Y1(T)-R1(T))+
                            2*D6(T)*(Y3(T)-R3(T))+  K1(T)*C12(T)+K2(T)*D22(T)+K3(T)*[illegible]
                            K3(T-1)=K3(T)+V*(2*D3(T)*(Y3(T)-R3(T))+2*D5(T)*(Y1(T)-R1(T))+
                            2*D6(T)*(Y2(T)-R2(T))+K1(T)*C13(T)+K2(T)*D23(T)+[illegible]
                            HM11(T)=K1(T)+2*B11(T)*M11(T)S
                            HM21(T)=K2(T)+2*B21(T)*M21(T)S
                            HM31(T)=K3(T)+2*B31(T)*M31(T)S
FR                          IF L EQL  (N+1) THEN  GOTO R1S
                            FOR J=(1,1,3) DO BEGIN
B11                         FOR I=(1,1,3) DO BEGIN
B12   E12   E11             T1(TF,I,J)=0SENDSENDS
                            H1(TF)=0SH2(TF)=0SH3(TF)=0S
                            L1(T,1,1)=-2*D0(T)SL1(T,1,2)=-2*D4(T)SL1(T,1,3)=-2*D5(T)S
                            L1(T,2,1)=-2*D4(T)SL1(T,2,2)=-2*D2(T)SL1(T,2,3)=-2*D6(T)S
                            L1(T,3,1)=-2*D5(T)SL1(T,3,2)=-2*D6(T)SL1(T,3,3)=-2*D3(T)S
                            D1(T,1,1)=C11(T)SD1(T,1,2)=C12(T)S  D1(T,1,3)=C13(T)S
                            D1(T,2,1)=D21(T)SD1(T,2,2)=D22(T)SD1(T,2,3)=D23(T)S
                            D1(T,3,1)=D31(T)SD1(T,3,2)=D32(T)SD1(T,3,3)=D33(T)S
                            E1(T,1,1)=-1/(2*B11(T))SE1(T,2,2)=-1/(2*B21(T))SE1(T,3,3)=-1/(2*B31(T))S
                            FOR J=(1,1,3) DO BEGIN
                            FOR I=(1,1,3) DO BEGIN
```

```
B14        T1(T-1,I,J)=T1(T,I,J)  +V*(D1(T,1,I)*T1(T,1,J)+D1(T,2,I)*T1(T,2,J)+
           D1(T,3,I)*T1(T,3,J)+T1(T,I,1)*D1(T,1,J)+T1(T,I,2)*D1(T,2,J)+
           T1(T,I,3)*D1(T,3,J)+(T1(T,I,1)*E1(T,1,1)+T1(T,I,2)*E1(T,2,1)+
           T1(T,I,3)*E1(T,3,1))*T1(T,1,J)+(T1(T,I,1)*E1(T,1,2)+T1(T,I,2)*E1(T,2,2)+
           +T1(T,I,3)*E1(T,3,2)     )*T1(T,2,J)+(T1(T,I,1)*E1(T,1,3)        +
           T1(T,I,2)*E1(T,2,3)+T1(T,I,3)*E1(T,3,3))*T1(T,3,J)-E1(T,I,J))$
E14  E13   END$END$
           H1(T-1)=H1(T)+V*((D1(T,1,1)+T1(T,1,1)*E1(T,1,1))*H1(T)+
           D1(T,2,1)*H2(T)+D1(T,3,1)*H3(T)+T1(T,1,1)*E*HM11(T)/(2*R11(T))
           +T1(T,1,2)*E*HM21(T)/(2*R21(T))+T1(T,1,3)*E*HM31(T)/(2*R31(T))   )$
           H2(T-1)=H2(T)+V*(D1(T,1,2)*H1(T)+(D1(T,2,2)+T1(T,2,2)*E1(T,2,2))*H2(T)
           +D1(T,3,2)*H3(T)+T1(T,2,1)*E*HM11(T)/(2*R11(T))+T1(T,2,2)*E*HM21(T)/(2*R      21(T))
N          21(T))+T1(T,2,3)*E*HM31(T)/(2*R31(T))      )$
           H3(T-1)=H3(T)+V*(D1(T,1,3)*H1(T)+D1(T,2,3)*H2(T)+
           (D1(T,3,3)+T1(T,3,3)*E1(T,3,3))*H3(T)+
           +T1(T,3,1)*E*HM11(T)/(2*R11(T))+T1(T,3,2)*E*HM21(T)/(2*R21(T))
           +T1(T,3,3)*E*HM31(T)/(2*R31(T))     )$
           M11(T)=M11(T)+E*HM11(T)/(2*R11(T))-( T1(T,1,1)*(Y1(T)-Y1S(T))+
           T1(T,1,2)*(Y2(T)-Y2S(T))+T1(T,1,3)*(Y3(T)-Y3S(T))+H1(T))/(2*R11(T))+
           M21(T)=M21(T)+E*HM21(T)/(2*R21(T)) -(T1(T,2,1)*(Y1(T)-Y1S(T))+
           T1(T,2,2)*(Y2(T)-Y2S(T))+T1(T,2,3)*(Y3(T)-Y3S(T))+H2(T))/(2*R21(T))+
           M31(T)=M31(T)+E*HM31(T)/(2*R31(T))-(T1(T,3,1)*(Y1(T)-Y1S(T))
           +T1(T,3,2)*(Y2(T)-Y2S(T))+T1(T,3,3)*(Y3(T)-Y3S(T))+H3(T))/(2*R31(T)).
           Y1S(T)=Y1(T)$Y2S(T)=Y2(T)$Y3S(T)=Y3(T)$
E10        P1..END$
           FOR T=(0,1,IF) DO BEGIN
B15        DEL1(T+1)=DEL1(T)+V*(D0(T)*((Y1(T)-R1(T))**2)+
           D2(T)*((Y2(T)-R2(T))**2)+D3(T)*((Y3(T)-R3(T))**2))
           2*D4(T)*(Y1(T)-R1(T))*(Y2(T)-R2(T))+2*D5(T)*(Y1(T)-R1(T))*(Y3(T)-R3(T))
           +2*D6(T)*(Y2(T)-R2(T))*(Y3(T)-R3(T))+R11(T)*M11(T)*M11(T)+
           B21(T)*M21(T)*M21(T)+R31(T)*M31(T)*M31(T)  -2*C7(T)*Y1(T)-2*C8(T)*Y2(T)
           -2*D9(T)*Y3(T))+    <--   WRITE  (" L,T, Y1 Y2 Y3,  LT  Y1 Y2 Y3
           IF T EQL TF THEN WRITE
           ("DEL1(T)",L,T,",DEL1(T),L,T)$
E15        END$
E8         END$
           FOR T=(0,1,IF) DO WRITE
           ("T",L,HM11(T),HM21(T),HM31(T)",",T,L,HM11(T),HM21(T),HM31(T))$
E6         END$
END BLOCK 3    LEVEL 2
                      V=0.01$X10=2$X20=3$X40=0$X50=6$              TF=10$
           N=10$
           FOR T=(0,1,IF) DO BEGIN
B16        A1(T)=1$A2(T)=1$A3(T)=1$A4(T)=1$A5(T)=1$B1(T)=2$B2(T)=2$B3(T)=1$
           B4(T)=1$B5(T)=1$
           C11(T)=1$C12(T)=2$C13(T)=0$C14(T)=5$       C15(T)=0.1$
           C21(T)=0.1$C22(T)=2$C23(T)=-1$C24(T)=-8$C25(T)=0$
           C31(T)=0.1$C32(T)=0.2$C33(T)=3$C34(T)=0.4$C35(T)=0.5$
           C41(T)=0.8$C42(T)=0.7$C43(T)=0.6$C44(T)=0.9$C45(T)=0$
           C51(T)=0$       C52(T)=0$C53(T)=1$C54(T)=1$C55(T)=1$
           R11(T)=0$R22(T)=0$R33(T)=5$R44(T)=0$R55(T)=0$
           Z(T)=0$
E16        END$
           COMMENT STEP II COMPUTATION OF A REFERENCE CONTROL    $
```

```
      OPTI (C11,A1,B1,X1I,M1R,P11,X10,TF,V,A)*
      OPTI (C22,A2,B2,X2I,M2R,P22,X20,TF,V,A)*
      OPTI (C33,A3,B3,X3I,M3R,P33,X30,TF,V,A)*
      OPTI (C44,A4,B4,X4I,M4R,P44,X40,TF,V,A)*
      OPTI (C55,A5,B5,X5I,M5R,P55,X50,TF,V,A)*
      COMMENT STEP II COMPUTATION OF A REFERENCE TRAJECTORY *
      FOR T=(0,1,TF) DO BEGIN
b17   X1R(0)=X10*X2R(0)=X20*X3R(0)=X30*X4R(0)=X40*X5R(0)=X50*
      X1R(T+1)=X1R(T)+V*(C11(T)*X1R(T)+C12(T)*X2R(T)+C13(T)*X3R(T)+
      C14(T)*X4R(T)+C15(T)*X5R(T)+M1R(T))*
      X2R(T+1)=X2R(T)+V*(C21(T)*X1R(T)    +C22(T)*X2R(T)+
      C23(T)*X3R(T)+C24(T)*X4R(T)+C25(T)*X5R(T)+M2R(T))*
      X3R(T+1)=X3R(T)+V*(C31(T)*X1R(T)+C32(T)*X2R(T)+C33(T)*X3R(T)+
      C34(T)*X4R(T)+C35(T)*X5R(T)+M3R(T))*
      X4R(T+1)=X4R(T)+V*(C41(T)*X1R(T)+C42(T)*X2R(T)+C43(T)*X3R(T)+
      C44(T)*X4R(T)+C45(T)*X5R(T)+M4R(T))*
      X5R(T+1)=X5R(T)+V*(C51(T)*X1R(T)+C52(T)*X2R(T)+
      C53(T)*X3R(T)+C54(T)*X4R(T)+C55(T)*X5R(T)+M5R(T))*
      DELR(0)=0*
      DELR(T+1)=DELR(T)+V*(A1(T)*((X1R(T)-Z11(T))**2)+[illegible]
      2)+A3(T)*((X3R(T)-Z33(T))**2)+A4(T)*((X4R(T)-Z44(T))**2)+
      A5(T)*((X5R(T)-Z55(T))**2)+[illegible]
      [illegible]
v     WRITE    (T,X1R(T),X2R(T),X3R(T),X4R(T),X5R(T),[illegible]
      M4R(T),M5R(T),DELR(T),*,
      T,X1R(T),X2R(T),X3R(T),X4R(T),X5R(T),[illegible]
      DELR(T))*
e17   END*
      FOR T=(0,1,TF) DO BEGIN
b18   H0(T)=A1(T)+B1(T)*C11(T)*C21(T)+C4(T)*C41(T)*C51(T)+
      H5(T)*C51(T)*C55(T)    *
      H2(T)=[illegible]
      H4(T)=[illegible]
      H7(T)=[illegible]
      (L3(T),C31(T)*[illegible]
      -H11(T))+ (H3(T)*C31(T)*[illegible]
      (X2R(T)-Z22(T))    *
      H8(T)=[illegible]
      (L3(T),C31(T)*[illegible]

      (T))+(H7(T)*C32(T)*C32(T)+[illegible]
      X2R(T)-Z22(T))    *
      G1(T)=C13(T)*X3R(T)+C14(T)*X4R(T)+C15(T)*X5R(T)*
      G2(T)=C23(T)*X3R(T)+C24(T)*X4R(T)+C25(T)*X5R(T)    *
e18   END*
      TAK (H0,H2,H4,H5,H7,H8,[illegible]
      Z,G1,G2,Z,[illegible],F1,H7,F3,X10,X20,D,TF,V,A)*
      FOR T=(0,1,TF) DO BEGIN
b19   H11(T)=A3(T)+[illegible]*C13(T)*C13(T)+[illegible]
      H21(T)=A4(T)+[illegible]*C14(T)*C14(T)+[illegible]*C24(T)*C24(T)*
      H31(T)=A5(T)+[illegible]*C15(T)*C15(T)+[illegible]*C25(T)*C25(T)*
      H41(T)=H1(T)*C13(T)*C14(T)+[illegible]*C23(T)*C24(T)*
      H51(T)=H1(T)*C13(T)*C15(T)+[illegible]*C23(T)*C25(T)*
      H61(T)=H1(T)*C14(T)*C15(T)+[illegible]*C24(T)*C25(T)*
```

```
        H711(T)=B1(T)*C13(T)*M1(T)      +B2(T)*C23(T)*M2(T)+
        (B1(T)*C13(T)*C13(T)+B2(T)*C23(T)*C23(T))*(X3R(T)-R33(T))+
        (B1(T)*C13(T)*C14(T)+B2(T)*C23(T)*C24(T))*(X4R(T)-R44(T))+
        (B1(T)*C13(T)*C15(T)+B2(T)*C23(T)*C25(T))*(X5R(T)-R55(T))     $
        H811(T)=B1(T)*C14(T)*M1(T)+B2(T)*C24(T)*M2(T)+(B1(T)*C13(T)*C14(T)
        +B2(T)*C23(T)*C24(T))*(X3R(T)-R33(T))+
        (B1(T)*C14(T)*C14(T)+B2(T)*C24(T)*C24(T))*(X4R(T)-R44(T))+
        (B1(T)*C14(T)*C15(T)+B2(T)*C24(T)*C25(T))*(X5R(T)-R55(T))     $
        H911(T)=B1(T)*C15(T)*M1(T)+B2(T)*C25(T)*M2(T)+
        (B1(T)*C13(T)*C15(T)+B2(T)*C23(T)*C25(T))*(X3R(T)-R33(T))
        +(B1(T)*C14(T)*C15(T)+B2(T)*C24(T)*C25(T))*(X4R(T)-R44(T))+
        (B1(T)*C15(T)*C15(T)+B2(T)*C25(T)*C25(T))*(X5R(T)-R55(T))        $
        G111(T)=C31(T)*X1(T)+C32(T)*X2(T)   $
        G211(T)=C41(T)*X1(T)+C42(T)*X2(T)      $
        G311(T)=C51(T)*X1(T)+C52(T)*X2(T)        $
E19     END$
        TAK (H011,H211,H311,H411,H511,H611,H711,H811,H911,X3,X4,X5,
        R33,R44,R55,C33,C34,C35,C43,C44,C45,C53,C54,C55,G111,G211,G311,
        N311,N411,N511,B3,B4,B5,X30,X40,X50,TF,V,A)$
        FOR T=(0,1,TF) DO WRITE (*X1(T),X2(T),X3(T),X4(T),X5(T),*,X1(T),
        X2(T),X3(T),X4(T),X5(T)   )$
        COMMENT COMPUTATION OF M1N,...,M5N    $
        FOR T=(0,1,TF-1) DO BEGIN
B20     DX1(T)=(X1(T+1)-X1(T))/V$    DX2(T)=(X2(T+1)-X2(T))/V$
        DX3(T)=(X3(T+1)-X3(T))/V$    DX4(T)=(X4(T+1)-X4(T))/V$
E20     DX5(T)=(X5(T+1)-X5(T))/V$END$
        DX1(TF)=DX1(TF-1)$DX2(TF)=DX2(TF-1)$DX3(TF)=DX3(TF-1)$
        DX4(TF)=DX4(TF-1)$DX5(TF)=DX5(TF-1)$
        FOR T=(0,1,TF) DO BEGIN
B21     M1N(T)=DX1(T)-C11(T)*X1(T)-C12(T)*X2(T)-C13(T)*X3(T)-C14(T)*X4(T)-
        C15(T)*X5(T)         $
        M2N(T)=DX2(T)-C21(T)*X1(T)-C22(T)*X2(T)-C23(T)*X3(T)-C24(T)*X4(T)-
        C25(T)*X5(T)      $
        M3N(T)=DX3(T)-C31(T)*X1(T)-C32(T)*X2(T)-C33(T)*X3(T)-C34(T)*X4(T)-
        C35(T)*X5(T)            $
        M4N(T)=DX4(T)-C41(T)*X1(T)-C42(T)*X2(T)-C43(T)*X3(T)-C44(T)*X4(T)-
        C45(T)*X5(T)         $
        M5N(T)=DX5(T)-C51(T)*X1(T)-C52(T)*X2(T)-C53(T)*X3(T)-C54(T)*X4(T)-
        C55(T)*X5(T)         $
        DELF(T+1)=DELF(T)+V*(A1(T)*((X1(T)-R11(T))**2)+A2(T)*((X2(T)-R22(T))**2)
        +A3(T)*((X3(T)-R33(T))**2)+A4(T)*((X4(T)-R44(T))**2)+
        A5(T)*((X5(T)-R55(T))**2)+B1(T)*M1N(T)*M1N(T)+
        B2(T)*M2N(T)*M2N(T)+B3(T)*M3N(T)*M3N(T)+B4(T)*M4N(T)*M4N(T)
        +B5(T)*M5N(T)*M5N(T))           $
        WRITE (*M1N(T),M2N(T),M3N(T),M4N(T),M5N(T),DELF(T),*,
        M1N(T),M2N(T),M3N(T),M4N(T),M5N(T),DELF(T)       )$
E21     END$
END BLOCK 1    LEVEL 1
```

APPENDIX IV, PROGRAM 2

```
B0                                REAL ARRAY C11,C12,C21,C22,A1,A2,B1,B2,X1,X2,X10,X20,M1,M2,M10,M20,R1,
BLOCK 1     LEVEL 1
                                  R2,Z(-1,,101)$
                                  REAL V,TF,T,I,J,E,Y10,Y20                                                $
                                  PROCEDURE GLOBAL (C11,C12,C21,C22,A1,A2,B1,B2,R1,R2,Y10,Y20,TF,V,I,X1,X2
BLOCK 2     LEVEL 2
                                  ,M1,M2)     $
                                  VALUE C11,C12,C21,C22,A1,A2,B1,B2,R1,R2,Y10,Y20,TF,V,I                   $
                                  REAL ARRAY C11,C12,C21,C22,A1,A2,B1,B2,X1,X2,M1,M2,R1,R2                 $
                                  REAL Y10,Y20,TF,V,I        $
B1                                BEGIN REAL J,E,T      $
                                  REAL ARRAY HM1,HM2,DEL,P1,P2,T11,T12,T21,T22,H1,H2,X1S,X2S(-1,,101)$
                                  FOR T=(0,1,TF) DO M1(T)=0$M2(T)=0$
                                  FOR J=(1,1,I+1) DO BEGIN
B2                                E=-J/I$
                                  COMMENT FORWARDS INTEGRATIONS
                                  FOR T=(0,1,TF) DO BEGIN
B3                                X1(0)=Y10$X2(0)=Y20$
                                  X1(T+1)=X1(T)+V*(C11(T)*X1(T)+C12(T)*X2(T)+M1(T))   $
                                  X2(T+1)=X2(T)+V*(C21(T)*X1(T)+C22(T)*X2(T)+M2(T))        $
E3                                END$
                                  COMMENT BACKWARDS INTEGRATION   $
                                  FOR T=(TF,-1,0) DO BEGIN
B4                                P1(TF)=0$P2(TF)=0$
                                  P1(T-1)=P1(T)-V*(-2*A1(T)*(X1(T)-R1(T))-P1(T)*C11(T)-P2(T)*C21(T))        $
                                  P2(T-1)=P2(T)-V*(-2*A2(T)*(X2(T)-R2(T))-P1(T)*C12(T)-P2(T)*C22(T))        $
                                  HM1(T)=2*B1(T)*M1(T)+P1(T)      $
                                  HM2(T)=2*B2(T)*M2(T)+P2(T)            $
FR                                IF J EQL (I+1) THEN GOTO B3$
                                  T11(TF)=0$T12(TF)=0$T21(TF)=0$T22(TF)=0$H1(TF)=0$H2(TF)=0$
                                  T11(T-1)=T11(T)-V*(-2*C11(T)*T11(T)-C21(T)*T12(T)-C21(T)*T21(T)+
                                  T11(T)*T11(T)/(2*B1(T))   +T12(T)*T21(T)/(2*B2(T))-2*A1(T))       $
                                  T12(T-1)=T12(T)-V*(-C12(T)*T11(T)-(C11(T)+C22(T))*T12(T)-
                                  C21(T)*T22(T)+T11(T)*T12(T)/(2*B1(T))+T12(T)*T22(T)/(2*B2(T))       )$
                                  T21(T-1)=T21(T)-V*(-C12(T)*T11(T)-(C11(T)+C22(T))*T21(T)-
                                  C21(T)*T22(T)+T21(T)*T11(T)/(2*B1(T))+T22(T)*T21(T)/(2*B2(T))        )$
                                  T22(T-1)=T22(T)-V*(-C12(T)*T12(T)-C12(T)*T21(T)-2*C22(T)*T22(T)+
                                  T12(T)*T21(T)/(2*B1(T))    +T22(T)*T22(T)/(2*B2(T))  -2*A2(T))     $
                                  H1(T-1)=H1(T)-V*((T11(T)/(2*B1(T))-C11(T))*H1(T)+
                                  (T12(T)/(2*B2(T))-C21(T))*H2(T)-T11(T)*E*HM1(T)/(2*B1(T))-
                                  T12(T)*E*HM2(T)/(2*B2(T))         )$
                                  H2(T-1)=H2(T)-V*((T21(T)/(2*B1(T))-C12(T))*H1(T)+
                                  (T22(T)/(2*B2(T))-C22(T))*H2(T)-T21(T)*E*HM1(T)/(2*B1(T))
                                  -T22(T)*E*HM2(T)/(2*B2(T))        )$
```

```
          M1(T)=M1(T)+E*HM1(T)/(2*B1(T))-(T11(T)*(X1(T)-X1S(T))+
          T12(T)*(X2(T)-X2S(T))+H1(T))/(2*B1(T))          $
          M2(T)=M2(T)+E*HM2(T)/(2*B2(T))-(T21(T)*(X1(T)-X1S(T))+
          T22(T)*(X2(T)-X2S(T))+H2(T))/(2*B2(T))          $
          X1S(T)=X1(T)$X2S(T)=X2(T)$
E4        B3..END$
          FOR T=(0,1,TF) DO BEGIN
B5        DEL(0)=0$
          DEL(T+1)=DEL(T)+V*(A1(T)*((X1(T)-R1(T))**2)+
          A2(T)*((X2(T)-R2(T))**2)+B1(T)*M1(T)*M1(T)+B2(T)*M2(T)*M2(T)     )$
          IF T EQL TF THEN
          WRITE (,DEL(T),J,T,,,DEL(T),J,T)        $
E5        END$
E2        END$
          FOR T=(0,1,TF) DO
          WRITE (,HM1(T),HM2(T),DEL(T),T,,,HM1(T),HM2(T),DEL(T),T)        $
E1        END$
END BLOCK 2   LEVEL 2
          Y10=5$ Y20=2$
          TF=10$V=0.01$I=20$
          FOR T=(0,1,TF) DO BEGIN
B6        Z(T)=0$
          C11(T)=2$
          C12(T)=2$C21(T)=2$C22(T)=3$
          A1(T)=10$
          A2(T)=10$
          B1(T)=1$
          B2(T)=1$
          R1(T)=0$R2(T)=0$
E6        END$
          GLOBAL (C11,C12,C21,C22,A1,A2,B1,B2,R1,R2,Y10,Y20,TF,V,I,X10,X20,M10,M20
          )$
          FOR T=(0,1,TF) DO WRITE
          (,X10(T),X20(T),M10(T),M20(T),T,,,X10(T),X20(T),M10(T),M20(T),T)$
END BLOCK 1   LEVEL 1

COMPILATION COMPLETED IN     8.81 SECONDS
```

APPENDIX IV, PROGRAM 3

```
b0
BLOCK 1      LEVEL 1
             REAL ARRAY C11(-1..101),C12(-1..101),C21(-1..101),C22(-1..101),
             A1(-1..101),A2(-1..101),B1(-1..101),B2(-1..101),P1(-1..101),
             P2(-1..101),
             Z(-1..101),
             X10(-1..101),X20(-1..101),X1R(-1..101),X2R(-1..101),X1(-1..101),
             X2(-1..101),X(-1..101),
             M1R(-1..101),M2R(-1..101),M1(-1..101),M2(-1..101),
             F(-1..101),G(-1..101),
             P(-1..101),H1(-1..101),T1(-1..101),
             DEL(-1..101),DEL0(-1..101)
             Q2R,Q2,
             DX1,DX2,M1N,M2N,DELF,
             D12R,Q1,Q1R,
             F1,F2,C1,C2,G1,G2,DX1R,DX2R,DEL3(-1..101)
             $
             REAL V,TF,T,I,J,E,Y10,Y20                                      $
             PROCEDURE OPTIMIZATION (C,F,A,B,R,D,G,Y0,TF,V,I,X,M)$
BLOCK 2      LEVEL 2
             VALUE C,F,A,B,R,D,G,Y0,TF,V,I      $
             REAL ARRAY C,F,A,B,R,D,G,X,M   $
             REAL Y0,TF,V,I      $
b1           BEGIN REAL J,E,T       $
             REAL ARRAY HM,DEL,P,T1,H1,XS(-1..101)       $
             FOR T=(0,1,TF) DO M(T)=0$
             FOR J=(1,1,I+1) DO BEGIN
b2           E=-J/I  $
             COMMENT FORWARDS INTEGRATION   $
             FOR T=(0,1,TF) DO BEGIN
b3           X(0)=Y0$
E3           X(T+1)=X(T)+V*(C(T)*X(T)+M(T)+F(T))   $END$
             COMMENT BACKWARDS INTEGRATION  $
b4           FOR T=(TF,-1,0)    DO BEGIN P(TF)=0$
             P(T-1)=P(T)-V*(-2*A(T)*(X(T)-R(T))-2*D(T)*X(T)-G(T)-C(T)*P(T))        $
             HM(T)=2*B(T)*M(T)+P(T)      $
FR           IF J EQL (I+1) THEN GOTO B3$
             T1(TF)=0$H1(TF)=0$
             T1(T-1)=T1(T)-V*(-2*C(T)*T1(T)+T1(T)*T1(T)/(2*B(T))-2*(A(T)+D(T)))    $      $
             H1(T-1)=H1(T)-V*(-(C(T)-T1(T)/(2*B(T)))*H1(T)-E*HM(T)*T1(T)/(2*B(T)))  $
             M(T)=M(T)+E*HM(T)/(2*B(T))-(T1(T)*(X(T)-XS(T))+H1(T))/(2*B(T))        $
             XS(T)=X(T)$
E4           B3..END$
b5           FOR T=(0,1,TF) DO BEGIN DEL(0)=0$
             DEL(T+1)=DEL(T)+V*(A(T)*(X(T)-R(T))*(X(T)-R(T))+B(T)*M(T)*M(T)+
```

```
          D(T)*X(T)*X(T)+C(T)*X(T)         )$
          IF T EQL TF THEN WRITE ('DEL(T),J,T,','DEL(T),J,T)      $
B5        END$
E2        END$
          FOR T=(0,1,TF) DO
          WRITE ('HM(T),DEL(T),T','HM(T),DEL(T),T)       $
          IF T EQL TF THEN WRITE ('DEL(T),T,','DEL(T),T)$
E1        END$
END BLOCK 2   LEVEL 2
          Y10=5$Y20=2$
          TF=10$V=0.01$
          I=7$
          FOR T=(0,1,TF) DO BEGIN
B6        Z(T)=0$
          C11(T)=2$
          C21(T)=2$C12(T)=2$
          C22(T)=7$
          A1(T)=10$
          A2(T)=10$
          B1(T)=1$
          B2(T)=1$
          R1(T)=0$R2(T)=0$
E6        END$
          COMMENT COMPUTATION OF M1R,M2R    $
          FOR T=(0,1,TF) DO BEGIN
B7        M1R(T)=0$M2R(T)=0$
E7        END$
          COMMENT COMPUTATION OF X1R,X2R     $
          FOR T=(0,1,TF)  DO BEGIN
B8        X1R(0)=Y10$
          X2R(0)=Y20  $
          X1R(T+1)=X1R(T)+V*(C11(T)*X1R(T)+C12(T)*X2R(T)+M1R(T))   $
          X2R(T+1)=X2R(T)+V*(C21(T)*X1R(T)+C22(T)*X2R(T)+M2R(T))          $
          DELR(0)=0$
          DELR(T+1)=DELR(T)+V*(A1(T)*(X1R(T)-R1(T))*(X1R(T)-R1(T))+
          A2(T)*(X2R(T)-R2(T))*(X2R(T)-R2(T))    +B1(T)*M1R(T)*M1R(T)+
          B2(T)*M2R(T)*M2R(T)     )      $
          WRITE ('X1R(T),X2R(T),M1R(T),M2R(T),DELR(T),T,',
          X1R(T),X2R(T),M1R(T),M2R(T),DELR(T),T   )$
E8        END$
          FOR T=(0,1,TF) DO BEGIN
B9        F1(T)=C12(T)*X2R(T)$
          D1(T)=C12(T)*C21(T)*C21(T)     $
          IF T LSS TF THEN
          DX2R(T)=(X2R(T+1)-X2R(T))/V       $
          IF T EQL TF THEN
          DX2R(TF)=DX2R(TF-1)     $
          G1(T)=-2*B2(T)+C21(T)*(DX2R(T)-C22(T)*X2R(T))          $
E9        END$
          OPTIMIZATION (C11,F1,A1,B1,R1,D1,G1,Y10,TF,V,I,X1,M1)     $
          FOR  T=(0,1,TF) DO WRITE
          ('DX2R(T),M1(T),X1(T),T,',DX2R(T),M1(T),X1(T),T)$
          FOR T=(0,1,TF) DO BEGIN
B10       F2(T)=C21(T)*X1(T)$
```

```
      D2(T)=B1(T)*C12(T)*C12(T)    $
      IF T LSS TF THEN
      DX1(T)=(X1(T+1)-X1(T))/V    $
      DX1(TF)=DX1(TF-1)    $
      G2(T)=-2*B1(T)*C12(T)*(DX1(T)-C11(T)*X1(T))    $
E10   ENDS
      OPTIMIZATION (C22,F2,A2,B2,R2,D2,G2,Y20,TF,V,I,X2,M2)$
      FOR T=(0,1,TF) DO WRITE
      ('DX1R(T),X2(T),M2(T),T,',DX1R(T),X2(T),M2(T),T)$
      FOR T=(0,1,TF)  DO BEGIN
B11   DEL3(0)=0$
      DEL3(T+1)=DEL3(T)
      +V*(A1(T)*(X1(T)-R1(T))*(X1(T)-R1(T))+
      A2(T)*(X2(T)-R2(T))*(X2(T)-R2(T))
      +B1(T)*M1(T)*M1(T)+B2(T)*M2(T)*M2(T)    )$
      WRITE ('X1(T),X2(T),M1(T),M2(T),DEL3(T),',
      X1(T),X2(T),M1(T),M2(T),DEL3(T)    )$
E11   ENDS
      COMMENT COMPUTATION OF MIN,M2N$
      FOR T=(0,1,TF)  DO BEGIN
B12   IF T LSS TF THEN
      DX1(T)=(X1(T+1)-X1(T))/V     $
      DX1(TF)=DX1(TF-1)$
      IF T LSS TF THEN
      DX2(T)=        (X2(T+1)-X2(T))/V$
      DX2(TF)=DX2(TF-1)$
      MIN(T)=DX1(T)-C11(T)*X1(T)-C12(T)*X2(T)    $
      M2N(T)=DX2(T)-C22(T)*X2(T)-C21(T)*X1(T)$
      DELF(0)=0$
      DELF(T+1)=DELF(T)+      V*(A1(T)*(X1(T)-R1(T))*(X1(T)-R1(T))   +
      A2(T)*(X2(T)-R2(T))*(X2(T)-R2(T))          +B1(T)*MIN(T)*MIN(T)+
      B2(T)*M2N(T)*M2N(T)    )$
      WRITE ('MIN(T),M2N(T),X1(T),X2(T),DELF(T),T,',
      MIN(T),M2N(T),X1(T),X2(T),DELF(T),T)   $
E12   ENDS
      FOR T=(0,1,TF) DO BEGIN
B13   D12R(0)=0$
      D12R(T+1)=D12R(T)+V*(A1(T)*(X1(T)-R1(T))*(X1(T)-R1(T))+
      A2(T)*(X2R(T)-R2(T))*(X2R(T)-R2(T))    +B1(T)*((DX1(T)-C11(T)*X1(T)-
      C12(T)*X2R(T))**2)+B2(T)*((DX2R(T)-C21(T)*X1(T)-C22(T)*X2R(T))**2)    )$
      Q1(0)=0$
      Q2(0)=0$
      Q2(T+1)=Q2(T)        +V*(A2(T)*((X2(T)-R2(T))**2)+B1(T)*(
      C12(T)*C12(T)*X2(T)*X2(T)-2*C12(T)*X2(T)*(DX1(T)-C11(T)*X1(T)))
         +B2(T)*((DX2(T)-C22(T)*X2(T)-C21(T)*X1(T))**2))     $
      Q2R(0)=0$
      Q2R(T+1)=Q2R(T)+V*(A2(T)*((X2R(T)-R2(T))**2)+
      B1(T)*(C12(T)*C12(T)*X2R(T)*X2R(T)-
      2*C12(T)*X2R(T)*(DX1(T)-C11(T)*X1(T)))+B2(T)*((
      DX2R(T)-C22(T)*X2R(T)-C21(T)*X1(T))**2) )     $
      Q1R(0)=0$
      Q1(T+1)=Q1(T)+        V*(A1(T)*((X1(T)-R1(T))**2)+
      B2(T)*(C21(T)*C21(T)*X1(T)*X1(T)-
      2*C21(T)*(DX2R(T)-C22(T)*X2R(T))*X1(T))+ B1(T)*((DX1(T)-
```

```
              C11(T)*X1(T)-C12(T)*X2R(T))**2)    )$
              Q1R(T+1)=Q1R(T)+V*(A1(T)*((X1R(T)-R1(T))**2)+
              B2(T)*(C21(T)*C21(T)*X1R(T)*X1R(T)-2*C21(T)*(DX2R(T)-C22(T)*X2R(T))*X1R(        T)
              T))   +R1(T)*(M1R(T)**2))$
E13           ENDS
              FOR T=(0,1,IF) DO WRITE
              ('D12R(T),Q1R(T),Q1(T),Q2R(T),Q2(T),T,','D12R(T),Q1R(T),Q1(T),Q2R(T),Q2(T
END BLOCK 1   LEVEL 1   ))$
```

APPENDIX IV, PROGRAM 4

```
B0
BLOCK 1     LEVEL 1
            REAL ARRAY  C11,C12,C21,C22,D11,D12,D21,D22,A1,A2,B1,B2,R1,R2,Z,X10,X20,
            M10,
            X1R,X2R,X1,X2,X,M1R,M2R,M1,M2,M1N,M2N,F,G,D1,D2,F1,F2,K11,K12,K21,K22,
            K31,K32,K41,K42,G1,G2,DX1,DX2,DELF,DEL0,Q1R,Q2R,DX1R,DX2R(-1..101)      $
            REAL V,TF,T,I,J,E,Y10,Y20          $
            PROCEDURE OPTIMIZATION (C,D,F,A,B,R,K1,K2,K3,K4,G,Y0,TF,V,I,X,M,M10)$
BLOCK 2     LEVEL 2
            VALUE C,D,F,A,B,R,K1,K2,K3,K4,G,YU,TF,V,I,M10$
            REAL ARRAY  C,D,F,A,B,R,K1,K2,K3,K4,G,X,M,M10     $
            REAL Y0,TF,V,I       $
B1          BEGIN REAL J,E,T     $
            REAL ARRAY  HM,DEL,P,T1,H1,XS(-1..101)   $
            FOR T=(0,1,TF) DO M(T)=M10(T)   $
            FOR J=(1,1,I+1) DO BEGIN
B2          E=-J/I$
            COMMENT FORWARDS INTEGRATION $
            FOR T=(0,1,TF) DO BEGIN
B3          X(0)=Y0$
            X(T+1)=X(T)+V*(C(T)*X(T)+D(T)*X(T)*X(T)+M(T)+F(T))   $
E3          ENDS
            COMMENT BACKWARDS INTEGRATION $
B4          FOR T=(TF,-1,0) DO BEGIN P(TF)=0$
            P(T-1)=P(T)- V*(-2*A(T)*(X(T)-R(T))-K1(T)-2*K2(T)*X(T)-3*K3(T)*X(T)*X(T)
            -4*K4(T)*X(T)*X(T)*X(T)-P(T)*C(T)-2*P(T)*D(T)*X(T))        $
            HM(T)=2*B(T)*M(T)+P(T)$
FR          IF J EQL (I+1) THEN GOTO B3    $
            T1(TF)=0$H1(TF)=0$
            T1(T-1)=T1(T)-V*(-2*(C(T)+2*D(T)*X(T))*T1(T)+T1(T)*T1(T)/(2*B(T))
            -2*A(T)-2*K2(T)-6*K3(T)*X(T)- 12*K4(T)*X(T)*X(T)-2*P(T)*D(T) )   $
            H1(T-1)=H1(T)-V*(-(C(T)+2*D(T)*X(T)-T1(T)/(2*B(T)))*H1(T)-
            T1(T)*E*HM(T)/(2*B(T)))          $
            M(T)=M(T)+(E*HM(T)-(T1(T)*(X(T)-XS(T))+H1(T)))/(2*B(T))          $
            XS(T)=X(T)$
E4          B3..ENDS
B5          FOR T=(0,1,TF) DO BEGIN DEL(0)=0$
            DEL(T+1)=DEL(T)+V*(A(T)*((X(T)-R(T))**2)+B(T)*M(T)*M(T)+K1(T)*X(T)+
            K2(T)*X(T)*X(T)+K3(T)*X(T)*X(T)*X(T)+K4(T)*X(T)*X(T)*X(T)*X(T)+G(T))   $
            IF T EQL TF THEN
            WRITE (,DEL(T),J,T,,DEL(T),J,T)        $
E5          ENDS
E2          ENDS
            FOR T=(0,1,TF) DO
            WRITE (,HM(T),DEL(T),T,,HM(T),DEL(T),T)        $
```

```
E1          ENDS
END BLOCK 2    LEVEL 2
            Y10=5$Y20=2$
            TF=10$T=10$V=0.01$
            FOR T=(0,1,TF) DO BEGIN
B6          Z(T)=0$
            C11(T)=2$
            C12(T)=10$
            C21(T)=15$
            C22(T)=3$
            A1(T)=10$
            A2(T)=10$
            B1(T)=1$
            B2(T)=1$
            R1(T)=0$R2(T)=0$
            D11(T)=0$
            D12(T)=0$
            D21(T)=0$
            D22(T)=0$
E6          ENDS
            COMMENT COMPUTATION OF M1R,M2R     $
            FOR T=(0,1,TF) DO BEGIN
B7          M1R(T)=100$M2R(T)=100$
E7          ENDS
            COMMENT COMPUTATION OF X1R,X2R     $
            FOR T=(0,1,TF) DO BEGIN
B8          X1R(0)=Y10$
            X2R(0)=Y20$
            X1R(T+1)=X1R(T)+V*(C11(T)*X1R(T)+C12(T)*X2R(T)+M1R(T)+
            D11(T)*X1R(T)*X1R(T)+D12(T)*X2R(T)*X2R(T))          $
            X2R(T+1)=X2R(T)+V*(C21(T)*X1R(T)+C22(T)*X2R(T)+M2R(T)+
            D21(T)*X1R(T)*X1R(T)+D22(T)*X2R(T)*X2R(T))          $
            DELO(0)=0$
            DELO(T+1)=DELO(T)+V*(A1(T)*((X1R(T)-R1(T))**2)+
            A2(T)*((X2R(T)-R2(T))**2)+B1(T)*M1R(T)*M1R(T)+B2(T)*M2R(T)*M2R(T)   )$
            WRITE (,X1R(T),X2R(T),M1R(T),M2R(T),DELO(T),T,,
            X1R(T),X2R(T),M1R(T),M2R(T),DELO(T),T)        $
E8          ENDS
            FOR T=(0,1,TF) DO BEGIN
B9          F1(T)=C12(T)*X2R(T)+D12(T)*X2R(T)*X2R(T)     $
            IF T LSS TF THEN
            DX2R(T)=(X2R(T+1)-X2R(T))/V     $
            IF T EQL TF THEN
            DX2R(TF)=DX2R(TF-1)     $
            K11(T)=-2*B2(T)+C21(T)*(DX2R(T)-D22(T)*X2R(T)*X2R(T)-C22(T)*X2R(T))     $
            K21(T)=B2(T)*(C21(T)*C21(T)-2*D21(T)*(DX2R(T)-D22(T)*X2R(T)*X2R(T)
            -C22(T)*X2R(T)))          $
            K31(T)=2*B2(T)*C21(T)*D21(T)$
            K41(T)=B2(T)*D21(T)*D21(T)     $
            G1(T)=       ((DX2R(T)-D22(T)*X2R(T)*X2R(T)-C22(T)*X2R(T))**2)     $
E9          ENDS
            FOR T=(0,1,TF)  DO WRITE
            ('D1''),F1(T),K11(T),K21(T),K31(T),K41(T),G1(T),',
              D11(*, F1(T),K11(T),K21(T),K31(T),K41(T),G1(T)      )$
```

```
        OPTIMIZATION (C11,D11,F1,A1,B1,R1,K11,K21,K31,K41,G1,Y10,TF,V,I,X1,M1,M1
        R)$
        FOR T =(0,1,TF) DO WRITE
        ('DX2R(T),M1(T),X1(T),T',DX2R(T),M1(T),X1(T),T )   $
        FOR T=(0,1,TF) DO BEGIN
B10     F2(T)=C21(T)*X1(T)+D21(T)*X1(T)*X1(T)    $
        IF T LSS TF THEN
        DX1(T)=(X1(T+1)-X1(T))/V    $
        IF T EQL TF THEN
        DX1(TF)=DX1(TF-1)   $
        K12(T)=-2*B1(T)+C12(T)*(DX1(T)-D11(T)*X1(T)*X1(T)-C11(T)*X1(T))       $
        K22(T)=B1(T)*(C12(T)*C12(T)-2*D12(T)*( X1(T)-D11(T)*X1(T)*X1(T)-
        C11(T)*X1(T)))          $
        K32(T)=2*B1(T)*C12(T)*D12(T)    $
        K42(T)=B1(T)*D12(T)*D12(T)    $
        G2(T)=((DX1(T)-D11(T)*X1(T)*X1(T)-C11(T)*X1(T))**2)       $
E10     ENDS
        OPTIMIZATION (C22,D22,F2,A2,B2,R2,K12,K22,K32,K42,G2,Y20,TF,V,I,X2,M2,M2
N       R)$
        FOR T=(0,1,TF) DO WRITE
        ('DX1(T),X2(T),M2(T),T',DX1(T),X2(T),M2(T),T)    $
        COMMENT COMPUTATION OF M1N,M2N$
        FOR T=(0,1,TF) DO BEGIN
B11     IF T LSS TF THEN
        DX1(T)=(X1(T+1)-X1(T))/V      $
        IF T EQL TF THEN
        DX1(TF)=DX1(TF-1)$
        IF T LSS TF THEN
        DX2(T)=(X2(T+1)-X2(T))/V    $
        IF T EQL TF THEN
        DX2(TF)=DX2(TF-1)      $
        WRITE ('X1(T),DX1(T),X2(T),DX2(T)',X1(T),DX1(T),X2(T),DX2(T) )$
        M1N(T)=DX1(T)-C11(T)*X1(T)-C12(T)*X2(T)-D11(T)*X1(T)*X1(T)
        -D12(T)*X2(T)*X2(T)          $
        M2N(T)=DX2(T)-C21(T)*X1(T)-C22(T)*X2(T)-D21(T)*X1(T)*X1(T)
        -D22(T)*X2(T)*X2(T)      $
        DELF(0)=0$
        DELF(T+1)=DELF(T)+V*(A1(T)*((X1(T)-R1(T))**2)+A2(T)*((X2(T)-R2(T))**2)
        +B1(T)*M1N(T)*M1N(T)+B2(T)*M2N(T)*M2N(T)   )$
        WRITE ('M1N(T),M2N(T),X1(T),X2(T),DELF(T),T','
        M1N(T),M2N(T),X1(T),X2(T),DELF(T),T   )$
E11     ENDS
        FOR T=(0,1,TF) DO BEGIN
B12     Q1R(0)=0$Q2R(0)=0$
        Q1R(T+1)=Q1R(T)+V*(A1(T)*((X1R(T)-R1(T))**2) +
        B1(T)*M1R(T)*M1R(T)+        K11(T)*X1R(T)*X1R(T)+K21(T)*X1R(T)*X1R(T)
        +K31(T)*X1R(T)*X1R(T)*X1R(T)+K41(T)*X1R(T)*X1R(T)*X1R(T)*X1R(T)
        +G1(T))      $
        Q2R(T+1)=Q2R(T)+V*(A2(T)*((X2R(T)-R2(T))**2)+B2(T)*M2R(T)*M2R(T)
        +K12(T)*X2R(T)+K22(T)*X2R(T)*X2R(T)+K32(T)*X2R(T)*X2R(T)*X2R(T)
        +K42(T)*X2R(T)*X2R(T)*X2R(T)*X2R(T)+   G2(T))   $
        WRITE ('Q1R(T),Q2R(T),T',Q1R(T),Q2R(T),T)      $
E12     ENDS
END BLOCK 1   LEVEL 1
```

APPENDIX IV, PROGRAM 5

```
B0
BLOCK 1     LEVEL 1
                REAL  V,TF,T,I,J,E,X10,X20,X30,X40,X50,N,W
                COM1,COM2,COM3,COM4,COM5,COMT,E0,ES$
                REAL ARRAY      C11,C12,C13,C14,C15,C21,C22,C23,C24,C25,
                S11,
                M1,M2,M3,M4,M5,
                C31,C32,C33,C34,C35,C41,C42,C43,C44,C45,
                C51,C52,C53,C54,C55,A1,A2,A3,A4,A5,B1,B2,B3,B4,B5,
                R1,R2,R3,R4,R5,X12,X22, X32,X42,X52,M1N,M2N,M3N,M4N,M5N,
                X11,X21,X31,X41,X51,X1I,X2I,X3I,X4I,X5I,M1R,M2R,M3R,M4R,M5R,
                HME1,HME2,HME3,HME4,HME5,
                X1R,X2R,X3R,X4R,X5R,Q,DELR(-1..101)$
                PROCEDURE SVP (A11,A21,A31,A41,A51,B11,B21,B31,B41,B51,C111,C121,C131,
BLOCK 2     LEVEL 2
                C141,C151,C211,C221,C231,C241,C251,C311,C321,C331,C341,C351,C411,
                C421,C431,C441,C451,C511,C521,C531,C541,C551,R11,R21,R31,R41,R51,Y1,
                Y2,Y3,Y4,Y5,M11,M21,M31,M41,M51,Y10,Y20,Y30,Y40,Y50,TF,V,I,S11)$
                VALUE   A11,A21,A31,A41,A51,B11,B21,B31,B41,B51,
                C111,C121,C131,C141,C151,C211,C221,C231,C241,C251,C311,C321,C331,
                C341,C351,C411,C421,C431,C441,C451,C511,C521,C531,C541,C551,
                R11,R21,R31,R41,R51,Y2,Y3,Y4,Y5,Y10,Y20,Y30,Y40,Y50,S11,
                TF,V,I $
                REAL ARRAY
                        A11,A21,A31,A41,A51,B11,B21,B31,B41,B51,
                C111,C121,C131,C141,C151,C211,C221,C231,C241,C251,C311,C321,C331,
                C341,C351,C411,C421,C431,C441,C451,C511,C521,C531,C541,C551,
                R11,R21,R31,R41,R51,Y2,Y3,Y4,Y5,
                S11,
                Y1,M11,M21,M31,M41,M51$
                REAL TF,V,I,Y10,Y20,Y30,Y40,Y50                     $
B1              BEGIN REAL J,E,T                  $
                REAL ARRAY  DY2,DY3,DY4,DY5,HM1,T1,H1,DEL1,P1,Y1S(-1..101)$
                FOR T=(0,1,TF) DO M11(T)=S11(T)$
                FOR T=(0,1,TF-1) DO BEGIN
B2              DY2(T)=(Y2(T+1)-Y2(T))/V$DY3(T)=(Y3(T+1)-Y3(T))/V$
E2              DY4(T)=(Y4(T+1)-Y4(T))/V$DY5(T)=(Y5(T+1)-Y5(T))/V$END$
                DY2(TF)=DY2(TF-1)$DY3(TF)=DY3(TF-1)$DY4(TF)=DY4(TF-1)$DY5(TF)=DY5(TF-1)$
                FOR J=(1,1,I+1) DO BEGIN
B3              E=-J/I$
                COMMENT FORWARDS INTEGRATION$
                FOR T=(0,1,TF) DO BEGIN
B4              Y1(0)=Y10$
                Y1(T+1)=Y1(T)+V*(C111(T)*Y1(T)+C121(T)*Y2(T)+C131(T)*Y3(T)+
                C141(T)*Y4(T)+C151(T)*Y5(T)+M11(T))$
```

```
        M21(T)=DY2(T)-C211(T)*Y1(T)-C221(T)*Y2(T)-C231(T)*Y3(T)
        -C241(T)*Y4(T)-C251(T)*Y5(T)            $
        M31(T)=DY3(T)-C311(T)*Y1(T)-C321(T)*Y2(T)-C331(T)*Y3(T)
        -C341(T)*Y4(T)-C351(T)*Y5(T)        $
        M41(T)=DY4(T)-C411(T)*Y1(T)-C421(T)*Y2(T)-C431(T)*Y3(T)
        -C441(T)*Y4(T)-C451(T)*Y5(T)            $
        M51(T)=DY5(T)-C511(T)*Y1(T)-C521(T)*Y2(T)-C531(T)*Y3(T)
        -C541(T)*Y4(T)-C551(T)*Y5(T)            $
E4      END$
        COMMENT BACKWARDS INTEGRATION $
        FOR T=(TF,-1,0) DO BEGIN
B5      P1(TF)=0$
        P1(T-1)=P1(T)+V*(2*A11(T)*(Y1(T)-R11(T))-2*B21(T)*C211(T)*M21(T)
        -2*B31(T)*C311(T)*M31(T)-2*B41(T)*C411(T)*M41(T)-
        2*B51(T)*C511(T)*M51(T)+P1(T)*C111(T))     $
        HM1(T)=2*B11(T)*M11(T)+P1(T)$
FR      IF J EQL (I+1) THEN GOTO K2 $
        T1(TF)=0$H1(TF)=0$
        T1(T-1)=T1(T)-V*(-2*C111(T)*T1(T)+T1(T)*T1(T)/(2*B11(T))-
        (2*A11(T)+2*B21(T)*C211(T)*C211(T)+2*B41(T)*C411(T)*C411(T)+
        2*B31(T)*C311(T)*C311(T)+2*B51(T)*C511(T)*C511(T)))    $
        H1(T-1)=H1(T)-V*(-(C11(T)-T1(T)/(2*B11(T)))*H1(T)-T1(T)*E*HM1(T)/(2*B11(
        T)))$
        M11(T)=M11(T)+E*HM1(T)/(2*B11(T))-(T1(T)*(Y1(T)-Y1S(T))+H1(T))/(2*B11(T)
        )$
        Y1S(T)=Y1(T)$
E5      K2..END$
        FOR T=(0,1,TF) DO BEGIN
B6      DEL1(0)=0$
        DEL1(T+1)=DEL1(T)+V*(A11(T)*((Y1(T)-R11(T))**2)+A21(T)*((Y2(T)-
        R21(T))**2)+A31(T)*((Y3(T)-R31(T))**2)+A41(T)*((Y4(T)-R41(T))**2)
        +A51(T)*((Y5(T)-R51(T))**2) +B11(T)*M11(T)*M11(T)+
        B21(T)*M21(T)*M21(T)+B31(T)*M31(T)*M31(T)+B41(T)*M41(T)*M41(T)+
        B51(T)*M51(T)*M51(T))$
        IF T EQL TF THEN WRITE
        ('DEL1(T),T,J,',DEL1(T),T,J)$
E6      END$
E3      END$
        FOR T=(0,1,TF) DO WRITE
        ('HM1(T),J,T',HM1(T),J,T)$
E1      END$
END BLOCK 2   LEVEL 2
        I=10$
        V=0.001$X10=2$X20=3$X40=0$X50=6$TF=10$
        FOR T=(0,1,TF) DO BEGIN
B7      A1(T)=1$A2(T)=1$A3(T)=1$A4(T)=1$A5(T)=1$B1(T)=2$B2(T)=2$B3(T)=1$
        B4(T)=1$B5(T)=1$
        C11(T)=1$C12(T)=2$C13(T)=0$C14(T)=5$          C15(T)=0.1$
        C21(T)=0.1$C22(T)=2$C23(T)=-1$C24(T)=-8$C25(T)=0$
        C31(T)=0.1$C32(T)=0.2$C33(T)=3$C34(T)=0.4$C35(T)=0.5$
        C41(T)=0.8$C42(T)=0.7$C43(T)=0.6$C44(T)=0.9$C45(T)=0$
        C51(T)=10$C52(T)=-12$C53(T)=4$C54(T)=-8$C55(T)=2$
        R1(T)=10$R2(T)=20$R3(T)=30$R4(T)=0$R5(T)=50$
E7      END$
```

```
        COMMENT STEP 0 COMPUTATION OF A REFERENCE CONTROL    $
        FOR T=(0,1,TF) DO BEGIN
B8      M1R(T)=100$M2R(T)=-100$M3R(T)=100$M4R(T)=-100$M5R(T)=100$
E8      ENDS
        COMMENT STEP 0 COMPUTATION OF A REFERENCE TRAJECTORY   $
        FOR T=(0,1,TF) DO BEGIN
B9      X1R(0)=X10$X2R(0)=X20$X3R(0)=X30$X4R(0)=X40$X5R(0)=X50$
        X1R(T+1)=X1R(T)+V*(C11(T)*X1R(T)+C12(T)*X2R(T)+C13(T)*X3R(T)+
        C14(T)*X4R(T)+C15(T)*X5R(T)+M1R(T))$
        X2R(T+1)=X2R(T)+V*(C21(T)*X1R(T)        +C22(T)*X2R(T)+
        C23(T)*X3R(T)+C24(T)*X4R(T)+C25(T)*X5R(T)+M2R(T))          $
        X3R(T+1)=X3R(T)+V*(C31(T)*X1R(T)+C32(T)*X2R(T)+C33(T)*X3R(T)+
        C34(T)*X4R(T)+C35(T)*X5R(T)+M3R(T))$
        X4R(T+1)=X4R(T)+V*(C41(T)*X1R(T)+C42(T)*X2R(T)+C43(T)*X3R(T)+
        C44(T)*X4R(T)+C45(T)*X5R(T)+M4R(T))$
        X5R(T+1)=X5R(T)+V*(C51(T)*X1R(T)+C52(T)*X2R(T)+
        C53(T)*X3R(T)+C54(T)*X4R(T)+C55(T)*X5R(T)+M5R(T))          $
        DELR(0)=0$
        DELR(T+1)=DELR(T)+V*(A1(T)*((X1R(T)-R1(T))**2)+A2(T)*((X2R(T)-R2(T))**2)
        +A3(T)*((X3R(T)-R3(T))**2)+A4(T)*((X4R(T)-R4(T))**2)
        +A5(T)*((X5R(T)-R5(T))**2)+B1(T)*M1R(T)*M1R(T)+B2(T)*M2R(T)*M2R(T)+
U       B3(T)*M3R(T)*M3R(T)+B4(T)*M4R(T)*M4R(T)+B5(T)*M5R(T)*M5R(T))$
        WRITE      ('T,X1R(T),X2R(T),X3R(T),X4R(T),X5R(T),M1R(T),M2R(T),M3R(T),
        M4R(T),M5R(T),DELR(T)','
        T,X1R(T),X2R(T),X3R(T),X4R(T),X5R(T),M1R(T),M2R(T),M3R(T),M4R(T),M5R(T),
E9      DELR(T))$
        ENDS
        W=5$
        FOR T=(0,1,TF) DO BEGIN
B10     X11(T)=X1R(T)$X21(T)=X2R(T)$X31(T)=X3R(T)$X41(T)=X4R(T)$X51(T)=X5R(T)$
E10     M1(T)=M1R(T)$M2(T)=M2R(T)$M3(T)=M3R(T)$M4(T)=M4R(T)$M5(T)=M5R(T)$
        ENDS
        FOR N=(1,1,*) DO BEGIN
B11     WRITE ('N,STEP1;','N)$
        FOR T=(0,1,TF) DO  S11(T)=M1(T)$
        E0=CLOCK$
        SVP (A1,A2,A3,A4,A5,B1,B2,B3,B4,B5,C11,C12,C13,C14,C15,C21,C22,C23,C24,
        C25,C31,C32,C33,C34,C35,C41,C42,C43,C44,C45,C51,C52,C53,C54,C55,
        R1,R2,R3,R4,R5,X12,X21,X31,X41,X51,M1,M2,M3,M4,M5,
        X10,X20,X30,X40,X50,TF,V,N,S11)$
        E5=CLOCK$
        COM1=E5-E0$
        FOR T=(0,1,TF) DO WRITE ('M1(T),M2(T),M3(T),M4(T),M5(T)','
        M1(T),M2(T),M3(T),M4(T),M5(T))$
        WRITE ('N,STEP2,','N)$
        FOR T=(0,1,TF) DO  S11(T)=M2(T)$
        E0=CLOCK$
        SVP (A2,A3,A4,A5,A1,B2,B3,B4,B5,B1,C22,C23,C24,C25,C21,
        C32,C33,C34,C35,C31,C42,C43,C44,C45,C41,C52,C53,C54,C55,C51,C12,C13,C14,
        C15,C11,R2,R3,R4,R5,R1,X22,X31,X41,X51,X12,M2,M3,M4,M5,
        M1,X20,X30,X40,X50,X10,TF,V,I,S11)$
        E5=CLOCK$
        COM2=E5-E0$
        FOR T=(0,1,TF) DO WRITE ('M1(T),M2(T),M3(T),M4(T),M5(T)','
```

```
          M1(T),M2(T),M3(T),M4(T),M5(T))$
          WRITE ('N,STEP 3,','N)$
          FOR T=(0,1,TF) DO  S11(T)=M3(T)$
          E0=CLOCK$
          SVP (A3,A4,A5,A1,A2,B3,B4,B5,B1,B2,C33,C34,C35,C31,C32,
          C43,C44,C45,C41,C42,C53,C54,C55,C51,C52,C13,C14,C15,C11,C12,C23,C24,C25,
          C21,C22,R3,R4,R5,R1,R2,X32,X41,X51,X12,X22,M3,M4,M5,M1,M2,X30,X40,
          X50,X10,X20,TF,V,I,S11)$
          E5=CLOCK$
          COM3=E5-E0$
          FOR T=(0,1,TF) DO WRITE ('M1(T),M2(T),M3(T),M4(T),M5(T),',
          M1(T),M2(T),M3(T),M4(T),M5(T))$
          WRITE ('N,STEP 4,','N)$
          FOR T=(0,1,TF) DO  S11(T)=M4(T)$
          E0=CLOCK$
          SVP (A4,A5,A1,A2,A3,B4,B5,B1,B2,B3,C44,C45,C41,C42,C43,
          C54,C55,C51,C52,C53,C14,C15,C11,C12,C13,C24,C25,C21,C22,C23,
          C34,C35,C31,C32,C33,R4,R5,R1,R2,R3,X42,X51,X12,X22,X32,
          M4,M5,M1,M2,M3,X40,X50,X10,X20,X30,TF,V,I,S11)$
          E5=CLOCK$
          COM4=E5-E0$
          WRITE ('N,STEP 5,','N)$
          FOR T=(0,1,TF) DO  S11(T)=M5(T)$
          E0=CLOCK$
          SVP (A5,A1,A2,A3,A4,B5,B1,B2,B3,B4,C55,C51,C52,C53,C54,
          C15,C11,C12,C13,C14,C25,C21,C22,C23,C24,
          C35,C31,C32,C33,C34,
          C45,C41,C42,C43,C44,R5,R1,R2,R3,R4,X52,
          X12,X22,X32,X42,     M5N,M1N,M2N,M3N,M4N,X50,X10,X20,X30,X40,TF,V,I,S11)$
          E5=CLOCK$
          COM5=E5-E0$
          FOR T=(0,1,TF) DO BEGIN
B12       X21(T)=X22(T)$X31(T)=X32(T)$X41(T)=X42(T)$X51(T)=X52(T)$
E12       END$
          FOR T=(0,1,TF) DO WRITE
U         ('T,X12(T),X22(T),X32(T),X42(T),X52(T),M1N(T),M2N(T),M3N(T),M4N(T),M5N(T
          ),',
            T,X12(T),X22(T),X32(T),X42(T),X52(T),M1N(T),M2N(T),M3N(T),M4N(T),M5N(T
          ))$
          COMT=COM1+COM2+COM3+COM4+COM5$
          WRITE ('1,2,3,4,5,TOT,',COM1,COM2,COM3,COM4,COM5,COMT)$
E11       END$
 END BLOCK 1   LEVEL 1
```

APPENDIX V, PROGRAM 6

```
B0                     REAL T,ET,BT,TIME,TF,V,I,J,E                                        $
BLOCK 1    LEVEL 1
                       REAL ARRAY X1,X2,X1S,X2S,P1,P2,M1,M2,HM1,HM2,T11,T12,T21,T22,H1,
                       H2,DEL(-1..101)    $
                       COMMENT DETERMINISTIC PROBLEM  GLOBAL METHOD  $
                       FOR I =(1,1,6)  DO BEGIN
B1                     TF=50$V=0.001$
B2    E2               FOR T=(0,1,TF) DO BEGIN M1(T)=0$M2(T)=0$END$
                       WRITE (CLOCK)$
                       BT=CLOCK$
                       FOR J=(1,1,I+1) DO BEGIN
B3                     E=-J/I$
                       COMMENT FORWARDS INTEGRATION $
                       FOR T=(0,1,TF) DO BEGIN
B4                     X1(0)=5$X2(0)=2$
                       X1(T+1)=X1(T)+V*(X1(T)+X2(T)+M1(T))$
                       X2(T+1)=X2(T)+V*(X1(T)+X2(T)+M2(T))$
                       DEL(0)=0$
                       DEL(T+1)=DEL(T)+V*(((T/1000-X1(T))**2)+((2*T/1000-X2(T))**2)+
                       M1(T)*M1(T)+M2(T)*M2(T) )$
                       IF T EQL TF AND J EQL (I+1) THEN WRITE
                       ('DEL(T),I,T'',DEL(T),I,T)$
E4                     END$
                       COMMENT BACKWARDS INTEGRATION  $
                       FOR T=(TF,-1,0) DO BEGIN
B5                     P1(TF)=0$P2(TF)=0$
                       P1(T-1)=P1(T)+V*(2*(X1(T)-T/1000)+P1(T)+P2(T))    $
                       P2(T-1)=P2(T)+V*(2*(X2(T)-2*T/1000)+P1(T)+P2(T))    $
                       HM1(T)=2*M1(T)+P1(T)   $
                       HM2(T)=2*M2(T)+P2(T)    $
E8                     IF J EQL (I+1) THEN GOTO B3$
                       T11(TF)=0$T12(TF)=0$T21(TF)=0$T22(TF)=0$H1(TF)=0$H2(TF)=0$
                       T11(T-1)=T11(T)+V*(2*T11(T)+T12(T)+T21(T)-0.5*T11(T)*T11(T)
                       -0.5*T12(T)*T21(T)+2)         $
                       T12(T-1)=T12(T)+V*(2*T12(T)+T22(T)+T11(T)-0.5*T11(T)*T12(T)
                       -0.5*T12(T)*T22(T)    )$
                       T21(T-1)=T21(T)+V*(2*T21(T)+T11(T)+T22(T)-0.5*T21(T)*T11(T)
                       -0.5*T22(T)*T21(T))        $
                       T22(T-1)=T22(T)+V*(2*T22(T)+T12(T)+T21(T)-0.5*T21(T)*T12(T)
                       -0.5*T22(T)*T22(T)+2         )$
                       H1(T-1)=H1(T)+V*((1-0.5*T11(T))*H1(T)+(1-0.5*T12(T))*H2(T)
                       +T11(T)*E*HM1(T)/2+0.5*T12(T)*E*HM2(T))          $
                       H2(T-1)=H2(T)+V*((1-0.5*T21(T))*H1(T)+(1-0.5*T22(T))*H2(T)+0.5*T21(T)*E*
                       HM1(T)+T22(T)*E*HM2(T)/2)        $
```

```
        M1(T)=M1(T)+0.5*F*HM1(T)-0.5*(T11(T)*(X1(T)-X1S(T))+
        T12(T)*(X2(T)-X2S(T)) +H1(T))           $
        M2(T)=M2(T)+0.5*F*HM2(T)-0.5*(T21(T)*(X1(T)-X1S(T))
        +T22(T)*(X2(T)-X2S(T)) +H2(T))          $
        X1S(T)=X1(T)$        X2S(T)=X2(T)$
E5      B3..END$
        IF ( I EQL 6) AND ( J EQL 7 ) THEN FOR  T=(0,1,TF) DO WRITE
        ('X1(T),X2(T),M1(T),M2(T),DEL(T),HM1(T),HM2(T),T,',
        X1(T),X2(T),M1(T),M2(T),DEL(T),HM1(T),HM2(T),T)$
E3      END$
        WRITE (CLOCK)$
        ET=CLOCK$
        TIME=ET-BT$
        WRITE ('COMPUTING TIME',TIME )$
E1      END$
END BLOCK 1    LEVEL 1
```

APPENDIX V, PROGRAM 7

```
                        COMMENT DETERMINISTIC PROBLEM  DECOMPOSITION METHOD   $
B0                      REAL T,FT,BT,TIME,TF,V,I,J,E,N,R,EPS      $
BLOCK 1    LEVEL 1
                        REAL ARRAY X1,X2,T1,T2,M1,M2,X1S,X2S,HM1,HM2,HT1,HT2,P1,P2,T11,T22,
                        H1,H2,K1,K2:DEL(-1..101)$
                        N=10$EPS=0.6$
                        I=7$
                        TF=50$V=0.001$
                        WRITE (CLOCK)$
                        BT=CLOCK$
                        FOR T=(0,1,TF) DO BEGIN
B1    E1                K1(T)=K2(T)=X1(T)=X2(T)=M1(T)=M2(T)=T1(T)=T2(T)=0$END$
                        FOR R=(1,1,N) DO BEGIN
B2                      WRITE ('STEP')$
                        COMMENT COORDINATUION      $
                        FOR T=(0,1,TF) DO BEGIN
B3                      K1(T)=K1(T)+EPS*(X1(T)-T1(T))        $
                        K2(T)=K2(T)+EPS*(X2(T)-T2(T))             $
E3                      END$
                        FOR J=(1,1,I+1) DO BEGIN
B4                      E=-J/I$
                        COMMENT     SUBSYSTEM1     $
                        COMMENT FORWARDS INTEGRATION $
                        FOR T=(0,1,TF) DO BEGIN
B5                      X1(0)=5$
                        X1(T+1)=X1(T)+V*(X1(T)+T2(T)+M1(T))$
E5                      END$
                        COMMENT BACKWARDS INTEGRATION  $
                        FOR T=(TF,-1,0) DO BEGIN
B6                      P1(TF)=0$
                        P1(T-1)=P1(T)+V*(X1(T)-T/1000+P1(T)+K1(T))            $
                        HM1(T)=2*M1(T)+P1(T)$
                        HT2(T)=T2(T)-2*T/1000+P1(T)-K2(T)      $
FR                      IF J EQL (I+1) THEN GOTO B3$
                        T11(TF)=0$H1(TF)=0$
                        T11(T-1)=T11(T)+V*(2*T11(T)-3*T11(T)*T11(T)/2+1)          $
                        H1(T-1)=P1(T)+V*((1-3*T11(T)/2)*H1(T)+(0.5*E*HM1(T)+E*HT2(T))*T11(T))$
                        M1(T)=M1(T)+0.5*E*HM1(T)-0.5*(T11(T)*(X1(T)-X1S(T))+H1(T))        $
                        T2(T)=T2(T)+E*HT2(T)-(T11(T)*(X1(T)-X1S(T))+H1(T))             $
                        X1S(T)=X1(T)$
E6                      B3..END$
                        COMMENT SUB SYSTEM 2$
                        FOR T=(0,1,TF) DO BEGIN
B7                      X2(0)=2$
```

```
X2(T+1)=X2(T)+V*(T1(T)+X2(T)+M2(T))$
END$
COMMENT BACKWARDS INTEGRATION  $
FOR T=(TF,-1,0) DO BEGIN
P2(TF)=0$
P2(T-1)=P2(T)+V*(X2(T)-2*T/1000+P2(T)+K2(T))        $
HM2(T)=2*M2(T)+P2(T)$
HT1(T)=T1(T)-T/1000+P2(T)-K1(T)         $
IF J EQL (I+1) THEN GOTO B4$
T22(TF)=0$H2(TF)=0$
T22(T-1)=T22(T)+V*(2*T22(T)-3*T22(T)*T22(T)/2+1)     $
H2(T-1)=H2(T)+V*((1-3*T22(T)/2)*H2(T)+T22(T)*(0.5*E*HM2(T)+E*HT1(T)))$
M2(T)=M2(T)+0.5*E*HM2(T)-0.5*(T22(T)*(X2(T)-X2S(T))+H2(T))       $
T1(T)=T1(T)+E*HT1(T)-(T22(T)*(X2(T)-X2S(T))+H2(T))        $
X2S(T)=X2(T)$
B4..END$
IF ( R EQL N )AND ( J EQL 7) THEN
FOR T=(0,1,TF) DO BEGIN
WRITE
('HM1(T),HM2(T),HT1(T),HT2(T),X1(T),T1(T),X2(T),T2(T),M1(T),M2(T),',
HM1(T),HM2(T),HT1(T),HT2(T),X1(T),T1(T),X2(T),T2(T),M1(T),M2(T) )$
END$
END$
WRITE (CLOCK)$
ET=CLOCK$
TIME=ET-BT$
WRITE ('COMPUTING TIME',TIME )$
END$
```

LOCK 1 LEVEL 1

APPENDIX V, PROGRAM 8

```
B0                  REAL  TF,I,J,E,T,ET,BT,TIME,Q1,Q2,W1,W2 ,V,E1,E2,E3
BLOCK 1    LEVEL 1
                    REAL ARRAY T11(-1..111),T12(-1..111),T13(-1..111),T14(-1..111),
                    DEL,
                    T15(-1..111),T21(-1..111),T22(-1..111),     T23(-1..111),T24(-1..111),
                    T25(-1..111),T31(-1..111),T32(-1..111),T33(-1..111),T34(-1..111),
                    T35(-1..111),T41(-1..111),T42(-1..111),T43(-1..111),T44(-1..111),
                    T45(-1..111),T51(-1..111),T52(-1..111),T53(-1..111),T54(-1..111),
                    T55(-1..111),H1(-1..111),H2(-1..111),H3(-1..111),H4(-1..111),
                    H5(-1..111),V1(-1..111),V2(-1..111),V12(-1..111),M1(-1..111),
                    M2(-1..111),P1(-1..111),P2(-1..111),P3(-1..111),P4(-1..111),P5(-1..111),
                    X1(-1..111),HX1(-1..111),S1(-1..111),S2(-1..111),L1(-1..111),
                    L2(-1..111),V10(-1..111),V20(-1..111),V120(-1..111),M10(-1..111),
                    M20(-1..111),F2(-1..111)
                    $
                    I=20$
                    WRITE  ('NUMBER OF CONVERGENCE ITERATIONS',I)           $
                    COMMENT INTEGRATED PROBLEM        $
                    TF=50$
                    V=0.001$
                    W1=0.1$W2=0.1$Q1=0.1$Q2=0.1$
                    FOR T=(0,1,TF) DO F2(T)=T     $
B1    E1            FOR T=(0,1,TF) DO BEGIN L1(T)=0.0004$L2(T)=0.0006*0.01$   END$
                    FOR T=(0,1,TF) DO  X1(T)=10$
                    BT=CLOCK $WRITE('BT',BT) $
                    E2=BT$
                    FOR J =(1,1,I+1) DO BEGIN
B2                  E=-J/I          $
                    COMMENT FORWARD INTEGRATION OF V1,V2,V12,M1,M2        $
                    FOR T=(0,1,TF) DO BEGIN
B3                  V1(0)=0 $V2(0)=0$V12(0)=0$M1(0)=0$M2(0)=0 $
                    V1(T+1)=V1(T)+V*(Q1-V1(T)*V1(T)/W1-V12(T)*V12(T)/W2)$
                    V2(T+1)=V2(T)+V*(2*V12(T)+Q2-V12(T)*V12(T)/W1-V2(T)*V2(T)/W2  )$
                    V12(T+1)=V12(T)+V*(V1(T)-V1(T)*V12(T)/W1-V2(T)*V12(T)/W2)$
                    M1(T+1)=M1(T)+V*(X1(T)-V1(T)*M1(T)/W1-V12(T)*M2(T)/W2+V1(T)*L1(T)/W1
                    +V12(T)*L2(T)/W2          )$
                    M2(T+1)=M2(T)+V*(M1(T)-V12(T)*M1(T)/W1-V2(T)*M2(T)/W2
                    +V12(T)*L1(T)/W1+V2(T)*L2(T)/W2    )$
E3                  END$
                    COMMENT BACKWARD INTEGRATION OF P1,P2,P3,P4,P5                $
                    FOR T=(TF,-1,0) DO BEGIN
B4                  P1(TF)=P2(TF)=P3(TF)=P4(TF)=P5(TF)=H1(TF)=H2(TF)=H3(TF)=H4(TF)=H5(TF)=0$
                    P1(T-1)=P1(T)+V*(0.5-2*P1(T)*V1(T)/W1+P3(T)-P3(T)*V12(T)/W1
                    -P4(T)*M1(T)/W1+P4(T)*L1(T)/W1    )$
```

```
      P2(T+1)=P2(T)+V*(0.5-2*P2(T)*V2(T)/W2-P3(T)*V12(T)/W2-P5(T)*M2(T)/W2
      +P5(T)*L2(T)/W2      )$
      P3(T+1)=P3(T)+V*(-2*P1(T)*V12(T)/W2+2*P2(T)-2*P2(T)*V12(T)/W1
      -P3(T)*V1(T)/W1-P3(T)*V2(T)/W2-P4(T)*M2(T)/W2+P4(T)*L2(T)/W2
      -P5(T)*M1(T)/W1+P5(T)*L1(T)/W1      )$
      P4(T+1)=P4(T)+V*(M1(T)-P4(T)*V1(T)/W1-P5(T)*V12(T)/W1      )$
      P5(T+1)=P5(T)+V*(M2(T)-F2(T)-P4(T)*V12(T)/W2-P5(T)*V2(T)/W2      )$
      HX1(T)=X1(T)+P1(T)      $
FR    IF J EQL (I+1) THEN GOTO 81   $
      T11(TF)=0$T12(TF)=0$T13(TF)=0$T14(TF)=0$T15(TF)=0    $
      T21(TF)=0$T22(TF)=0$T23(TF)=0$T24(TF)=0$T25(TF)=0$
      T31(TF)=0$T32(TF)=0$T33(TF)=0$T34(TF)=0$T35(TF)=0$
      T41(TF)=0$T42(TF)=0$T43(TF)=0$T44(TF)=2$T45(TF)=0$
      T51(TF)=0$T52(TF)=0$T53(TF)=0$T54(TF)=0$T55(TF)=2$
      H1(TF)=H2(TF)=H3(TF)=H4(TF)    =H5(TF)=0$
      S1(T)=L1(T)/W1-M1(T)/W1      $
      S2(T)=L2(T)/W2-M2(T)/W2      $
      T11(T+1)=T11(T)+V*(
      -2*V1(T)*T11(T)/W1 +(1-V12(T)/W1)*T31(T)+S1(T)*T41(T)
      -2*V1(T)*T11(T)/W1+(1-V12(T)/W1)*T13(T)+S1(T)*T14(T)
      -T14(T)*T41(T)-2*P1(T)/W1   )$
      T12(T+1)=T12(T)+V*(
      -2*V1(T)*T12(T)/W1 +(1-V12(T)/W1)*T32(T)+S1(T)*T42(T)
      -2*V2(T)*T12(T)/W2-V12(T)*T13(T)/W2+S2(T)*T15(T)
      -T14(T)*T42(T)      )$
      T13(T+1)=T13(T)+V*(
      -2*V1(T)*T13(T)/W1 +(1-V12(T)/W1)*T33(T)+S1(T)*T43(T)
      -2*V12(T)*T11(T)/W2+2*(1-V12(T)/W1)*T12(T)+(-V1(T)/W1-V2(T)/W2)*T13(T)
      +S2(T)*T14(T)+S1(T)*T15(T)
      -T14(T)*T43(T)-P3(T)/W1      )$
      T14(T+1)=T14(T)+V*(
      -2*V1(T)*T14(T)/W1 +(1-V12(T)/W1)*T34(T)+S1(T)*T44(T)
      -V1(T)*T14(T)/W1+(1-V12(T)/W1)*T15(T)
      -T14(T)*T44(T)-P4(T)/W1      )$
      T15(T+1)=T15(T)+V*(
      -2*V1(T)*T15(T)/W1 +(1-V12(T)/W1)*T35(T)+S1(T)*T45(T)
      -V12(T)*T14(T)/W2-V2(T)*T15(T)/W2
      -T14(T)*T45(T)      )$
      T21(T+1)=T21(T)+V*(
      -2*V2(T)*T21(T)/W2 -V12(T)*T31(T)/W2 +S2(T)*T51(T)
      -2*V1(T)*T21(T)/W1+(1-V12(T)/W1)*T23(T)+S1(T)*T24(T)
      -T24(T)*T41(T)      )$
      T22(T+1)=T22(T)+V*(
      -2*V2(T)*T22(T)/W2 -V12(T)*T32(T)/W2 +S2(T)*T52(T)
      -2*V2(T)*T22(T)/W2-V12(T)*T23(T)/W2+S2(T)*T25(T)
      -T24(T)*T42(T)-2*P2(T)/W2      )$
      T23(T+1)=T23(T)+V*(
      -2*V2(T)*T23(T)/W2 -V12(T)*T33(T)/W2 +S2(T)*T53(T)
      -2*V12(T)*T21(T)/W2+2*(1-V12(T)/W1)*T22(T)+(-V1(T)/W1-V2(T)/W2)*T23(T)
      +S2(T)*T24(T)+S1(T)*T25(T)
      -T24(T)*T43(T)-P3(T)/W2      )$
      T24(T+1)=T24(T)+V*(
      -2*V2(T)*T24(T)/W2 -V12(T)*T34(T)/W2 +S2(T)*T54(T)
      -V1(T)*T24(T)/W1+(1-V12(T)/W1)*T25(T)
```

```
-T24(T)*T44(T)      )$
T25(T-1)=T25(T)+V*(
-2*V2(T)*T25(T)/W2 -V12(T)*T35(T)/W2 +S2(T)*T55(T)
-V12(T)*T24(T)/W2-V2(T)*T25(T)/W2
-T24(T)*T45(T)-P5(T)/W2      )$
T31(T-1)=T31(T)+V*(
-2*V12(T)*T11(T)/W2+2*(1-V12(T)/W1)*T21(T)+(-V1(T)/W1-V2(T)/W2)*T31(T)
+S2(T)*T41(T)+S1(T)*T51(T)
-2*V1(T)*T31(T)/W1+(1-V12(T)/W1)*T33(T)+S1(T)*T34(T)
-T34(T)*T41(T)-P3(T)/W1      )$
T32(T-1)=T32(T)+V*(
-2*V12(T)*T12(T)/W2+2*(1-V12(T)/W1)*T22(T)+(-V1(T)/W1-V2(T)/W2)*T32(T)
+S2(T)*T42(T)+S1(T)*T52(T)
-2*V2(T)*T32(T)/W2-V12(T)*T33(T)/W2+S2(T)*T35(T)
-T34(T)*T42(T)-P3(T)/W2       )$
T33(T-1)=T33(T)+V*(
-2*V12(T)*T13(T)/W2+2*(1-V12(T)/W1)*T23(T)+(-V1(T)/W1-V2(T)/W2)*T33(T)
+S2(T)*T43(T)+S1(T)*T53(T)
-2*V12(T)*T31(T)/W2+2*(1-V12(T)/W1)*T32(T)+(-V1(T)/W1-V2(T)/W2)*T33(T)
+S2(T)*T34(T)+S1(T)*T35(T)
-T34(T)*T43(T)-2*P1(T)/W2       )$
T34(T-1)=T34(T)+V*(
-2*V12(T)*T14(T)/W2+2*(1-V12(T)/W1)*T24(T)+(-V1(T)/W1-V2(T)/W2)*T34(T)
+S2(T)*T44(T)+S1(T)*T54(T)
-V1(T)*T34(T)/W1+(1-V12(T)/W1)*T35(T)
-T34(T)*T44(T)-P5(T)/W1       )$
T35(T-1)=T35(T)+V*(
-2*V12(T)*T15(T)/W2+2*(1-V12(T)/W1)*T25(T)+(-V1(T)/W1-V2(T)/W2)*T35(T)
+S2(T)*T45(T)+S1(T)*T55(T)
-V12(T)*T34(T)/W2-V2(T)*T35(T)/W2
-T34(T)*T45(T)-P4(T)/W2        )$
T41(T-1)=T41(T)+V*(
-V1(T)*T41(T)/W1+(1-V12(T)/W1)*T51(T)
-2*V1(T)*T41(T)/W1+(1-V12(T)/W1)*T43(T)+S1(T)*T44(T)
-T44(T)*T41(T)-P4(T)/W1        )$
T42(T-1)=T42(T)+V*(
-V1(T)*T42(T)/W1+(1-V12(T)/W1)*T52(T)
-2*V2(T)*T42(T)/W2-V12(T)*T43(T)/W2+S2(T)*T45(T)
-T44(T)*T42(T)            )$
T43(T-1)=T43(T)+V*(
-V1(T)*T43(T)/W1+(1-V12(T)/W1)*T53(T)
-2*V12(T)*T41(T)/W2+2*(1-V12(T)/W1)*T42(T)+(-V1(T)/W1-V2(T)/W2)*T43(T)
+S2(T)*T44(T)+S1(T)*T45(T)
-T44(T)*T43(T)-P5(T)/W1       )$
T44(T-1)=T44(T)+V*(
-V1(T)*T44(T)/W1+(1-V12(T)/W1)*T54(T)
-V1(T)*T44(T)/W1+(1-V12(T)/W1)*T45(T)
-T44(T)*T44(T)+1         )$
T45(T-1)=T45(T)+V*(
-V1(T)*T45(T)/W1+(1-V12(T)/W1)*T55(T)
-V12(T)*T44(T)/W2-V2(T)*T45(T)/W2
-T44(T)*T45(T)           )$
T51(T-1)=T51(T)+V*(
-V12(T)*T41(T)/W2-V2(T)*T51(T)/W2
```

```
          -2*V1(T)*T51(T)/W1+(1-V12(T)/W1)*T53(T)+S1(T)*T54(T)
          -T54(T)*T41(T)        )$
          T52(T-1)=T52(T)+V*(
          -V12(T)*T42(T)/W2-V2(T)*T52(T)/W2
          -2*V2(T)*T52(T)/W2-V12(T)*T*3(T)/W2+S2(T)*T55(T)
          -T54(T)*T42(T)-D5(T)/W2       )$
          T53(T-1)=T53(T)+V*(
          -V12(T)*T43(T)/W2-V2(T)*T53(T)/W2
          -2*V12(T)*T51(T)/W2+2*(1-V12(T)/W1)*T52(T)+(-V1(T)/W1-V2(T)/W2)*T53(T)
          +S2(T)*T54(T)+S1(T)*T55(T)
          -T54(T)*T43(T)-D4(T)/W2        )$
          T54(T-1)=T54(T)+V*(
          -V12(T)*T44(T)/W2-V2(T)*T54(T)/W2
          -V1(T)*T54(T)/W1+(1-V12(T)/W1)*T55(T)
          -T54(T)*T44(T)         )$
          T55(T-1)=T55(T)+V*(
          -V12(T)*T45(T)/W2-V2(T)*T55(T)/W2
          -V12(T)*T54(T)/W2-V2(T)*T55(T)/W2
          -T54(T)*T45(T)+(        )$
          H1(T-1)=H1(T)+V*(-2*V1(T)*H1(T)/W1+(1-V12(T)/W1)*H3(T)+(S1(T)-T14(T))*H4
          (T)+T14(T)*E*HX1(T)    )$
          H2(T-1)=H2(T)+V*(-2*V2(T)*H2(T)/W2-V12(T)*H3(T)/W2-T24(T)*H4(T)+
          S2(T)*H5(T)+T24(T)*E*HX1(T)      )$
          H3(T-1)=H3(T)+V*(-2*V12(T)*H1(T)/W2+2*(1-V12(T)/W1)*H2(T)+
          (-V1(T)/W1-V2(T)/W2)*H3(T)+(S2(T)-T34(T))*H4(T)+S1(T)*H5(T)
          +T34(T)*E*HX1(T)        )$
          H4(T-1)=H4(T)+V*((-V1(T)/W1-T44(T))*H4(T)+(1-V12(T)/W1)*H5(T)+T44(T)*E*H       X1(T)
N         X1(T)         )$
          H5(T-1)=H5(T)+V*((-V12(T)/W2-T54(T))*H4(T)-V2(T)*H5(T)/W2
          +T54(T)*E*HX1(T)         )$
          X1(T)=X1(T)+E*HX1(T)-(T41(T)*(V1(T)-V10(T))+T42(T)*(V2(T)-V20(T))+
          T43(T)*(V12(T)-V120(T))+T44(T)*(M1(T)-M10(T))+T45(T)*(M2(T)-M20(T))
          +H4(T)    $
B5   E5   IF J EQL 1 THEN BEGIN V10(T)=0$V20(T)=0$V120(T)=0$M10(T)=0$M20(T)=0$END
          ELSE BEGIN
B6        V10(T)=V1(T)$V20(T)=V2(T)$V120(T)=V12(T)$M10(T)=M1(T)$M20(T)=M2(T)$
E6        END$
          IF J GTR (I-1) THEN BEGIN
B7        FOR T=(TF,-1,0)  DO WRITE
          ('HX1(T),V1(T),V2(T),M1(T),M2(T),V12(T),X1(T),T','
            HX1(T),V1(T),V2(T),M1(T),M2(T),V12(T ,X1(T),T)      $
          WRITE   ('J','J)$
E7        END$
E4        B1..END        $
          E1=CLOCK$
          E3=CLOCK-E2$
          E2=CLOCK-E1$
          WRITE ('J,COMP TIME OF THE STEP,TOTAL COMP TIME','J,E3,E2)$
          [illegible] BEGIN
             [illegible] X1(T)+X1(T)+V1(T)+V2(T)+
               [illegible] (2*T/1000-M2(T))**2))      $
          [illegible] THEN WRITE  ('J,T,DEL(T)','J,T,DEL(T))          $
          [illegible]
          [illegible] WRITE  ('J,T,HX1(T),M1(T),M2(T),V1(T),V2(T)','
```

```
                J,T,HX1(T),M1(T),M2(T),V1(T),V2(T))$
E2              END$
                ET=CLOCK$WRITE(,ET,,,ET)$
                TIME=ET-BT$
                WRITE(,COMPUTING TIME,',TIME)     $
END BLOCK 1    LEVEL 1
```

APPENDIX V, PROGRAM 9

```
                        REAL T,ET,BT,TIME,E,EPS,SUM1,A1,A2,W1,W2,V,TF,I,J,N,K,A,B,C,D
BLOCK 1     LEVEL 1
B0
                        REAL ARRAY T11(-1..101),T12(-1..101),
                          T13(-1..101),  T21(-1..101),  T22(-1..101),  T23(-1..101),
                          T31(-1..101),  T32(-1..101),  T33(-1..101),  H1(-1..101),
                           H2(-1..101),   H3(-1..101),   V1(-1..101), V121(-1..101),
                           M1(-1..101),  VV2(-1..101),  MM2(-1..101),   Y1(-1..101),
                           P1(-1..101),   P2(-1..101),   P3(-1..101),  HY1(-1..101),
                         HVV2(-1..101), HMM2(-1..101),   S1(-1..101),  SS2(-1..101),
                          V10(-1..101),V1210(-1..101),  M10(-1..101),   K1(-1..101),
                           K2(-1..101),   K3(-1..101),   K4(-1..101),  P11(-1..101),
                          P12(-1..101),  P13(-1..101),  P21(-1..101),  P22(-1..101),
                          P23(-1..101),  P31(-1..101),  P32(-1..101),  P33(-1..101),
                          HH1(-1..101),  HH2(-1..101),  HH3(-1..101),   V2(-1..101),
                         V122(-1..101),   M2(-1..101),  VV1(-1..101),  MM1(-1..101),
                           Q1(-1..101),   Q2(-1..101),   Q3(-1..101), HVV1(-1..101),
                         HMM1(-1..101),   S2(-1..101),  SS1(-1..101),  V20(-1..101),
                        V1220(-1..101),  M20(-1..101),   F2(-1..101),   L1(-1..101),
                        L2(-1..101) ,U(-1..101)
                        $
                        COMMENT COORDINATION PROBLEM                  $
                        I=5$
                        N=10$
                        EPS=0.6$
                        W1=0.1$W2=0.1$A1=0.1$A2=0.1$
                        TF=50$
                         V =0.001 $
                        FOR T=(0,1,TF) DO BEGIN
B1                      K1(T)=K2(T)=K3(T)=K4(T)=1$
                        MM1(T)=-0.06*0.01$
                        VV1(T)=0.005$
                        VV2(T)=0.005$
                        X1(T)=0.018*0.0001$
                        MM2(T)=-0.02*0.001$
E1                      END$
B2      E2              FOR T=(0,1,25) DO BEGIN L1(T)=0.0004$L2(T)=0.0006*0.01$    END$
B3      E3              FOR  T=(26,1,50) DO BEGIN  L1(T)=0.0007$L2(T)=0.00002$    END$
                        FOR T=(0,1,TF) DO F2(T)=T$
                        WRITE (CLOCK) $
                        BT=CLOCK $
                        FOR R=(1,1,N)  DO BEGIN
B4                      WRITE('STEP') $
                        FOR T=(0,1,TF) DO BEGIN
```

```
[B5] K1(T)=K1(T)+EPS*(V1(T)*V1(T)-VV1(T)*VV1(T))$
K2(T)=K2(T)+EPS*(VV2(T)*VV2(T)-V2(T)*V2(T))$
K3(T)=K3(T)+EPS*(M1(T)-MM1(T))$
K4(T)=K4(T)+EPS*(MM2(T)-M2(T))$
[E5] ENDS
FOR J=(1,1,I+1) DO BEGIN
[B6] WRITE (,I,J,',I,J)        $
SUM1=0$
E=-J/I$
COMMENT FORWARD INTEGRATION OF SUB PROBLEM 1             $
FOR T=(0,1,TF) DO BEGIN
[B7] V1(0)=0$V121(0)=0$M1(0)=0$
V1(T+1)=V1(T)+V*(A1-V1(T)*V1(T)/W1-V121(T)*V121(T)/W2 )$
V121(T+1)=V121(T)+V*(V1(T)-V1(T)*V121(T)/W1-VV2(T)*V121(T)/W2    )$
M1(T+1)=M1(T)+V*(X1(T)-V1(T)*M1(T)/W1-V121(T)*MM2(T)/W2+V1(T)*L1(T)/W1
+V121(T)*L2(T)/W2)$
[E7] ENDS
COMMENT BACKWARDS INTEGRATION OF SUB PROBLEM1 $
FOR T=(TF,-1,0) DO BEGIN
[B8] P1(T-1)=P1(T)+V*(0.5-2*P1(T)*V1(T)/W1+P2(T)*(1-V121(T)/W1)+
P3(T)*S1(T)+2*K1(T)*V1(T))$
P2(T-1)=P2(T)+V*(-2*P1(T)*V121(T)/W2+P2(T)*(-V1(T)/W1-VV2(T)/W2)+
P3(T)*(SS2(T)))$
P3(T-1)=P3(T)+V*(0.5*M1(T)-P3(T)*V1(T)/W1+K3(T))*
HX1(T)=X1(T)+P3(T)$
HVV2(T)=-P2(T)*V121(T)/W2+2*K2(T)*VV2(T)   $
HMM2(T)=0.5*(MM2(T)-F2(T))-P3(T)*V121(T)/W2+K4(T)           $
[FR] IF J FOL (I+1) THEN GOTC B1 $
T11(TF)=0$T12(TF)=0$T13(TF)=0$   T21(TF)=0$T22(TF)=0$T23(TF)=0$
T31(TF)=0$T32(TF)=0$T33(TF)=2$H1(TF)=0$H2(TF)=0$H3(TF)=0*
SS2(T)=-MM2(T)/W2+L2(T)/W2*
S1(T)=-M1(T)/W1+L1(T)/W1$
T11(T-1)=T11(T)+V*(
-2*V1(T)*T11(T)/W1+(1-V121(T)/W1)*T21(T)+S1(T)*T31(T)
-2*V1(T)*T11(T)/W1+(1-V121(T)/W1)*T12(T)+S1(T)*T13(T)
-V121(T)*V121(T)*T12(T)*T21(T)/(2*K2(T)*W2*W2)+
(2*V121(T)*V121(T)/(W2*W2)-1)*T13(T)*T31(T)
-2*P1(T)/W1+2*K1(T)    )$
T12(T-1)=T12(T)+V*(
-2*V1(T)*T12(T)/W1+(1-V121(T)/W1)*T22(T)+S1(T)*T32(T)
-2*V121(T)*T11(T)/W2+(-V1(T)/W1-VV2(T)/W2-V121(T)*P2(T)/(2*K2(T)*W2*W2))
*T12(T)+(SS2(T)-2*P3(T)*V121(T)/(W2*W2))*T13(T)
-V121(T)*V121(T)*T12(T)*T22(T)/(2*K2(T)*W2*W2)+
(2*V121(T)*V121(T)/(W2*W2)-1)*T13(T)*T32(T)
-P2(T)/W1        )$
T13(T-1)=T13(T)+V*(
-2*V1(T)*T13(T)/W1+(1-V121(T)/W1)*T23(T)+S1(T)*T33(T)
-V1(T)*T13(T)/W1
-V121(T)*V121(T)*T12(T)*T23(T)/(2*K2(T)*W2*W2)+
(2*V121(T)*V121(T)/(W2*W2)-1)*T13(T)*T33(T)
-P3(T)/V1   )$
T21(T-1)=T21(T)+V*(
-2*V121(T)*T11(T)/W2+(-V1(T)/W1-VV2(T)/W2-P2(T)*V121(T)/(2*K2(T)*W2*W2))
*T21(T)+(SS2(T)-2*P3(T)*V121(T)/(W2*W2))*T31(T)
```

```
-2*V1(T)*T21(T)/W1+(1-V121(T)/W1)*T22(T)+S1(T)*T23(T)
-V121(T)*V121(T)*T22(T)*T21(T)/(2*K2(T)*W2*W2)+
(2*V121(T)*V121(T)/(W2*W2)-1)*T23(T)*T31(T)
-P2(T)/W1)$
T22(T-1)=T22(T)+V*(
-2*V121(T)*T12(T)/W2+(-V1(T)/W1-VV2(T)/W2-P2(T)*V121(T)/(2*K2(T)*W2*W2))
*T22(T)+(SS2(T)-2*P3(T)*V121(T)/(W2*W2))*T32(T)
-2*V121(T)*T21(T)/W2+(-V1(T)/W1-VV2(T)/W2-V121(T)*P2(T)/(2*K2(T)*W2*W2))
*T22(T)+(SS2(T)-2*P3(T)*V121(T)/(W2*W2))*T23(T)
-V121(T)*V121(T)*T22(T)*T22(T)/(2*K2(T)*W2*W2)+
(2*V121(T)*V121(T)/(W2*W2)-1)*T23(T)*T32(T)
-2*P1(T)/W1-P2(T)*P2(T)/(2*K2(T)*W2*W2)-2*P3(T)*P3(T)/(W2*W2) )$
T23(T-1)=T23(T)+V*(
-2*V121(T)*T13(T)/W2+(-V1(T)/W1-VV2(T)/W2-P2(T)*V121(T)/(2*K2(T)*W2*W2))
*T23(T)+(SS2(T)-2*P3(T)*V121(T)/(W2*W2))*T33(T)
-V1(T)*T23(T)/W1
-V121(T)*V121(T)*T22(T)*T23(T)/(2*K2(T)*W2*W2)+
(2*V121(T)*V121(T)/(W2*W2)-1)*T23(T)*T33(T)     )$
T31(T-1)=T31(T)+V*(
-V1(T)*T31(T)/W1
-2*V1(T)*T31(T)/W1+(1-V121(T)/W1)*T32(T)+S1(T)*T33(T)
-V121(T)*V121(T)*T32(T)*T21(T)/(2*K2(T)*W2*W2)+
(2*V121(T)*V121(T)/(W2*W2)-1)*T33(T)*T31(T)
-P3(T)/W1)$
T32(T-1)=T32(T)+V*(
-V1(T)*T32(T)/W1
-2*V121(T)*T31(T)/W2+(-V1(T)/W1-VV2(T)/W2-V121(T)*P2(T)/(2*K2(T)*W2*W2))
*T32(T)+(SS2(T)-2*P3(T)*V121(T)/(W2*W2))*T33(T)
-V121(T)*V121(T)*T32(T)*T22(T)/(2*K2(T)*W2*W2)+
(2*V121(T)*V121(T)/(W2*W2)-1)*T33(T)*T32(T)     )$
T33(T-1)=T33(T)+V*(
-V1(T)*T33(T)/W1
-V1(T)*T33(T)/W1
-V121(T)*V121(T)*T32(T)*T23(T)/(2*K2(T)*W2*W2)+
(2*V121(T)*V121(T)/(W2*W2)-1)*T33(T)*T33(T)
+0.5)$
H1(T-1)=H1(T)+V*(
-2*V1(T)*H1(T)/W1+(1-V121(T)/W1-V121(T)*V121(T)*T12(T)/(
2*K2(T)*W2*W2))*H2(T)+(S1(T)+(2*V121(T)*V121(T)/(W2*W2)-1)*T13(T))*H3(T)
-V121(T)*T12(T)*E*HVV2(T)/(2*K2(T)*W2)+(E*HX1(T)-2*V121(T)*E*HVHMM2(T)/W2)
*T13(T)        )$
H2(T-1)=H2(T)+V*(
-2*V121(T)*H2(T)/W2+(-V1(T)/W1-VV2(T)/W2-P2(T)*V121(T)/(2*K2(T)*W2*W2)-
V121(T)*V121(T)*T22(T)/(2*K2(T)*W2*W2))*H2(T)+(SS2(T)-2*P3(T)*V121(T)/(W
2*W2)+(2*V121(T)*V121(T)/(W2*W2)-1)*T23(T))*H3(T)
-V121(T)*T22(T)*E*HVV2(T)/(2*K2(T)*W2)+(E*HX1(T)-
2*V121(T)*E*HMM2(T)/W2)*T23(T)-P2(T)*E*HVV2(T)/(2*K2(T)*W2)-2*P3(T)*E*HM
M2(T)/W2)$
H3(T-1)=H3(T)+V*(
-V121(T)*V121(T)*T32(T)*H2(T)/(2*K2(T)*W2*W2)+(-V1(T)/W1+
(2*V121(T)*V121(T)/(W2*W2)-1)*T33(T))*H3(T)
-V121(T)*T32(T)*E*HVV2(T)/(2*K2(T)*W2)+(E*HX1(T)-
2*V121(T)*E*HMM2(T)/W2)*T33(T)+0.5)$
X1(T)=X1(T)+C*HX1(T)-
```

2*W2)

H2(T)/

```
            (T31(T)*(V1(T)-V10(T))+T32(T)*(V121(T)-V1210(T))+T33(T)*(M1(T)-M10(T))
            +H3(T))       $
            VV2(T)=VV2(T)+(E*HVV2(T)+P2(T)*(V121(T)-V1210(T))/W2+
            V121(T)*(T21(T)*(V1(T)-V10(T))+T22(T)*(V121(T)-V1210(T))+T23(T)*(M1(T)
            -M10(T))+H2(T))/W2)/(2*W2(T))          $
            MM2(T)=MM2(T)+2*P3(T)*(V121(T)-V1210(T))/W2+                2*E*HMM2(T)+
            (T31(T)*(V1(T)-V10(T))+T32(T)*(V121(T)-V1210(T))+T33(T)*(M1(T)-M10(T))
            +H3(T))*2*V121(T)/W2     $
B9    E9    IF J EQL 1 THEN BEGIN  V10(T)=0$V1210(T)=0$M10(T)=0$END
            ELSE BEGIN
B10         M10(T)=M1(T)$V1210(T)=V121(T)$V10(T)=V1(T)$
E10         END$
E8          B1..END$
            COMMENT FORWARDS INTEGRATION OF SUB PROBLEM 2$
            FOR T=(0,1,TF) DO BEGIN
B11         V2(0)=0$V122(0)=0$  M2(0)=0$
            V2(T+1)=V2(T)+V*(2*V122(T)+A2-V122(T)*V122(T)/W1-V2(T)*V2(T)/W2)$
            V122(T+1)=V122(T)+V*(VV1(T)-VV1(T)*V122(T)/W1-V2(T)*V122(T)/W2)  $
            M2(T+1)=M2(T)+V*(MM1(T)-V122(T)*MM1(T)/W1-V2(T)*M2(T)/W2+
            V122(T)*L1(T)/W1+V2(T)*L2(T)/W2)$
E11         END$
            COMMENT BACKWARDS INTEGRATION OF SUB PROBLEM 2 $
            FOR T=(TF,-1,0) DO BEGIN
B12         Q1(TF)=Q2(TF)=Q3(TF)=0$
            Q1(T-1)=Q1(T)+V*(0.5-2*Q1(T)*V2(T)/W2-Q2(T)*V122(T)/W2-Q3(T)*M2(T)/W2
            +Q3(T)*L2(T)/W2-2*K2(T)*V2(T)       )$
            Q2(T-1)=Q2(T)+V*(2*Q1(T)-2*Q1(T)*V122(T)/W1-Q2(T)*VV1(T)/W1
            -Q2(T)*V2(T)/W2-Q3(T)*MM1(T)/W1+Q3(T)*L1(T)/W1    )$
            Q3(T-1)=Q3(T)+V*(0.5*(M2(T)-F2(T))-Q3(T)*V2(T)/W2-K4(T))$
            HVV1(T)=Q2(T)*(1-V122(T)/W1)-2*K1(T)*VV1(T)      $
            HMM1(T)=0.5*MM1(T)+Q3(T)*(1-V122(T)/W1)-K3(T)        $
            IF J LSS I THEN SUM1=0$
            IF J EQL I THEN
            BEGIN
B13         SUM1=SUM1+(M1(T)-MM1(T))**2+(V1(T)-VV1(T))**2+(M2(T)-MM2(T))**2+
            (V2(T)-VV2(T))**2             $
            IF T EQL 0 THEN  WRITE ('SUM2','SUM1)$
E13         IF T EQL 50  THEN WRITE ('SUM1','SUM1)$END$
            IF J GTR (I-1) AND SUM1 LSS 0.01 AND
            MIN(HMM1(T)) GTR  -0.001 AND MAX(HX1(T)) LSS  0.001 AND MIN(HX1(T))
            GTR  -0.001
            THEN BEGIN
B14         FOR T=(TF,-5,0)  DO WRITE
            ('HX1(T),HMM2(T),HVV2(T),',HX1(T),HMM2(T),HVV2(T))$
            FOR T=(TF,-5,0)  DO WRITE
            ('HX1(T), X1(T),     VV1(T),  V1(T),  VV2(T),  V2(T), MM1(T), M1(T), T,'
            HX1(T),   X1(T),  VV1(T),  V1(T),  VV2(T),  V2(T),  MM1(T),  M1(T),  T)$
            FOR T=(TF,-5,0) DO WRITE
            ('SUM1,HMM1(T),HVV1(T),MM2(T),M2(T),',SUM1,HMM1(T),HVV1(T),MM2(T),M2(T))
            $
FR    E14   GOTO B4$END$
FR          IF J EQL (I+1) THEN GOTO E2      $
            P11(TF)=0$P12(TF)=0$P13(TF)=0$
            P21(TF)=0$P22(TF)=0$P23(TF)=0$
```

```
P31(TF)=P32(TF)=P33(TF)=0$
HH1(TF)=HH2(TF)=HH3(TF)=0$
SS1(T)=-MM1(T)/W1+L1(T)/W1$
S2(T)=-M2(T)/W2+L2(T)/W2$
P11(T-1)=P11(T)+V*(
-2*V2(T)*P11(T)/W2-V122(T)*P21(T)/W2+S2(T)*P31(T)
-2*V2(T)*P11(T)/W2-V122(T)*P12(T)/W2+S2(T)*P13(T)
+P12(T)*((1-V122(T)/W1)**2)*P21(T)/(2*K1(T))
-2*P13(T)*((1-V122(T)/W1)**2)*P31(T)
-2*Q1(T)/W2-2*K2(T))$
P12(T-1)=P12(T)+V*(
-2*V2(T)*P12(T)/W2-V122(T)*P22(T)/W2+S2(T)*P32(T)     +
2*(1-V122(T)/W1)*P11(T)+(-VV1(T)/W1-V2(T)/W2-Q2(T)*(1-V122(T)/W1
)/2*K1(T)*W1)*P12(T)+(SS1(T)+2*Q3(T)*(1-V122(T)/W1)/W1)*P13(T)
+P12(T)*((1-V122(T)/W1)**2)*P22(T)/(2*K1(T))-2*P13(T)*((1-V122(T)/W1)**2
)*P32(T)
-Q2(T)/W2)$
P13(T-1)=P13(T)+V*(
-2*V2(T)*P13(T)/W2-V122(T)*P23(T)/W2+S2(T)*P33(T)
-V2(T)*P13(T)/W2
+P12(T)*((1-V122(T)/W1)**2)*P23(T)/(2*K1(T))-2*P13(T)*((1-V122(T)/W1)**2
)*P33(T)
-Q3(T)/W2)$
P21(T-1)=P21(T)+V*(
2*(1-V122(T)/W1)*P11(T)+(-VV1(T)/W1-V2(T)/W2-
                                        Q2(T)*(1-V122(T)/W1)/(2*K1(T)*
W1))*P21(T)   +(SS1(T)+2*Q3(T)*(1-V122(T)/W1)/W1T*P31(T)
-2*V2(T)*P21(T)/W2-V122(T)*P22(T)/W2+S2(T)*P23(T)
+((1-V122(T)/W1)**2)*P22(T)*P21(T)/(2*K1(T))-
2*P23(T)*((1-V122(T)/W1)**2)*P31(T)
-Q2(T)/W2)$
P22(T-1)=P22(T)+V*(
2*(1-V122(T)/W1)*P12(T)+(-VV1(T)/W1-V2(T)/W2-
                                        Q2(T)*(1-V122(T)/W1)/(2*K1(T)*
W1))*P22(T)   +(SS1(T)+2*Q3(T)*(1-V122(T)/W1)/W1)*P32(T)          +
2*(1-V122(T)/W1)*P21(T)+(-VV1(T)/W1-V2(T)/W2-Q2(T)*(1-V122(T)/W1
)/2*K1(T)*W1)*P22(T)+(SS1(T)+2*Q3(T)*(1-V122(T)/W1)/W1)*P23(T)
+((1-V122(T)/W1)**2)*P22(T)*P22(T)/(2*K1(T))-
2*P23(T)*((1-V122(T)/W1)**2)*P32(T)
-2*Q1(T)/W1+Q2(T)*Q2(T)/(2*K1(T)*W1*W1)-2*Q3(T)*Q3(T)/(W1*W1))$
P23(T-1)=P23(T)+V*(
2*(1-V122(T)/W1)*P13(T)+(-VV1(T)/W1-V2(T)/W2-
                                        Q2(T)*(1-V122(T)/W1)/(2*K1(T)*
W1))*P23(T)   +(SS1(T)+2*Q3(T)*(1-V122(T)/W1)/W1)*P33(T)
-V2(T)*P23(T)/W2
+((1-V122(T)/W1)**2)*P22(T)*P23(T)/(2*K1(T))-
2*P23(T)*((1-V122(T)/W1)**2)*P33(T)     )$
P31(T-1)=P31(T)+V*(
-V2(T)*P31(T)/W2
-2*V2(T)*P31(T)/W2-V122(T)*P32(T)/W2+S2(T)*P33(T)
+P32(T)*((1-V122(T)/W1)**2)*P21(T)/(2*KITT))-
2*P33(T)*((1-V122(T)/W1)**2)*P31(T)
-Q3(T)/W2)$
P32(T-1)=P32(T)+V*(
```

```
-V2(T)*P32(T)/W2      +
2*(1-V122(T)/W1)*P31(T)+(-VV1(T)/W1-V2(T)/W2-Q2(T)*(1-V122(T)/W1
)/2*K1(T)*W1)*P32(T)+(SS1(T)+2*Q3(T)*(1-V122(T)/W1)/W1)*P33(T)
+P32(T)*((1-V122(T)/W1)**2)*P22(T)/(2*K1(T))-
2*P33(T)*((1-V122(T)/W1)**2)*P32(T)      )$
P33(T-1)=P33(T)+V*(
-V2(T)*P33(T)/W2
-V2(T)*P33(T)/W2
+P32(T)*((1-V122(T)/W1)**2)*P23(T)/(2*K1(T))-
2*P33(T)*((1-V122(T)/W1)**2)*P33(T)
+0.5)$
HH1(T-1)=HH1(T)+V*(-2*V2(T)*HH1(T)/W2+(-V122(T)/W2+
P12(T)*((1-V122(T)/W1)**2)/(2*K1(T)))*HH2(T)+
(S2(T)-2*P13(T)*((1-V122(T)/W1)**2))*HH3(T)
-P12(T)*(1-V122(T)/W1)*E*HVV1(T)/(2*K1(T))
+2*(1-V122(T)/W1)*P13(T)*E*HMM1(T)            )$
HH2(T-1)=HH2(T)+V*(2*(1-V122(T)/W1)*HH1(T)+
(-VV1(T)/W1-V2(T)/W2-Q2(T)*(1-V122(T)/W1)/(2*K1(T)*W1)+
P22(T)*((1-V122(T)/W1)**2)/(2*K1(T)))*HH2(T)+
(SS1(T)+2*Q3(T)*(1-V122(T)/W1)/W1    -2*P23(T)*((1-V122(T)/W1)**2))*HH3(T)
)  +(-P22(T)*(1-V122(T)/W1)*E*HVV1(T)/(2*K1(T))
+2*(1-V122(T)/W1)*P23(T)*E*HMM1(T))+
Q2(T)*E*HVV1(T)/(2*K1(T)*W1)-2*Q3(T)*E*HMM1(T)/W1      )$
HH3(T-1)=HH3(T)+V*(P32(T)*((1-V122(T)/W1)**2)*HH2(T)/(2*K1(T))
+(-V2(T)/W2-2*P33(T)*((1-V122(T)/W1)**2))*HH3(T)
-P32(T)*(1-V122(T)/W1)*E*HVV1(T)/(2*K1(T))
+2*(1-V122(T)/W1)*P33(T)*E*HMM1(T)            )$
VV1(T)=VV1(T)+(1-V122(T)/W1)*(P21(T)*(V2(T)-V20(T))+
P22(T)*(V122(T)-V1220(T))+P23(T)*(M2(T)-M20(T))+HH2(T))/(2*K1(T))
-Q2(T)*(V122(T)-V1220(T))/(2*K1(T)*W1) -E*HVV1(T)/(2*K1(T))           $
MM1(T)=MM1(T)+2*Q3(T)*(V122(T)-V1220(T))/W1-2*(1-V122(T)/W1)*(
P31(T)*(V2(T)-V20(T))+P32(T)*(V122(T)-V1220(T))+P33(T)*(M2(T)-M20(T))
+HH3(T))+  2*E*HMM1(T)              $
IF J EQL 1 THEN BEGIN
B15   E15   V20(T)=V1220(T)=M20(T)=0             $END
ELSE BEGIN
B16   V20(T)=V2(T)$V1220(T)=V122(T)$M20(T)=M2(T)$
E16   END$
E12   B2..END $
U(T)=V1(T)-VV1(T)$
FOR T=(0,1,TF) DO A=MAX(A,U(T))$
FOR  T=(0,1,TF) DO B=MIN(B,U(T))$
FOR T=(0,1,TF) DO  C=MAX(C,HMM2(T))$
FOR T=(0,1,TF) DO  D=MIN(D,HMM2(T))$
IF T EQL TF THEN WRITE ('A,B,C,D,',A,B,C,D)$
WRITE('NUMBER OF COORDINATION ITERATIONS,',R)$
WRITE(CLOCK)$
ET=CLOCK     $
TIME=ET-BT          $
WRITE('COMPUTING TIME,',TIME)     $
E6    B4..END$
E0    END$
```

APPENDIX V, PROGRAM 10

```
B0                        REAL T,ET,BT,TIME,TF,V,I,J,E
BLOCK 1    LEVEL 1
                          REAL ARRAY X1,X2,X1S,X2S,P1,P2,M,HM,T11,T12,T21,T22,H1,H2,
                          DEL(-1..101)      $
                          COMMENT  NON LINEAR COUPLING      $
                          COMMENT   GLOBAL METHOD     $
                          FOR I =(1,1,6)  DO BEGIN
B1                        TF=50$V=0.001$
                          FOR T=(0,1,TF) DO M(T)=0$
                          WRITE (CLOCK)$
                          BT=CLOCK$
                          FOR J=(1,1,I+1) DO BEGIN
B2                        E=-J/I$
                          COMMENT FORWARDS INTEGRATION $
                          FOR T=(0,1,TF) DO BEGIN
B3                        X1(0)=5$X2(0)=2$
                          X1(T+1)=X1(T)+V*(M(T))$
                          X2(T+1)=X2(T)+V*(X1(T)+0.1*X1(T)*X1(T))          $
                          DEL(0)=0$
                          DEL(T+1)=DEL(T)+V*(0.5*(X1(T)*X1(T)+X2(T)*X2(T)+M(T)*M(T)))        $
                          IF ( T EQL TF ) AND ( J EQL (I+1) ) THEN WRITE
                          ('DEL(T),I,T','DEL(T),I,T)   $
E3                        END$
                          COMMENT BACKWARDS INTEGRATION  $
                          FOR T=(TF,-1,0) DO BEGIN
B4                        P1(TF)=0$P2(TF)=0$
                          P1(T-1)=P1(T)+V*(X1(T)+P2(T)+0.2*P2(T)*X1(T))      $
                          P2(T-1)=P2(T)+V*X2(T)$
                          HM(T)=M(T)+P1(T)$
FR                        IF J EQL (I+1)        THEN GOTO B3$
                          T11(TF)=0$T12(TF)=0$T21(TF)=0$T22(TF)=0$H1(TF)=0$H2(TF)=0$
                          T11(T-1)=T11(T)+V*(T21(T)*(1+0.2*X1(T))+T12(T)*(1+0.2*X1(T))
                          -T11(T)*T11(T)+1+0.2*P2(T))           $
                          T12(T-1)=T12(T)+V*(T22(T)*(1+0.2*X1(T))-T11(T)*T12(T))        $
                          T21(T-1)=T21(T)+V*(T21(T)*(1+0.2*X1(T))-T21(T)*T11(T))         $
                          T22(T-1)=T22(T)+V*(-T21(T)*T12(T)+1)              $
                          H1(T-1)=H1(T)+V*(-T11(T)*H1(T)+(1+0.2*X1(T))*H2(T)+T11(T)*E*HM(T))        $
                          H2(T-1)=H2(T)+V*(-T21(T)*H1(T)+T21(T)*E*HM(T))              $
                          M(T)=M(T)+E*HM(T)-(T11(T)*(X1(T)-X1S(T))+T12(T)*(X2(T)-X2S(T))+H1(T))  $       $
                          X1S(T)=X1(T)$          X2S(T)=X2(T)$
E4                        B3..END$
                          IF ( I EQL 6) AND ( J EQL 7 ) THEN        FOR T=(0,1,TF) DO WRITE
                          ('HM(T),X1(T),X2(T),DEL(T),M(T),T','
                          HM(T),X1(T),X2(T),DEL(T),M(T),T)      $
```

```
E2          ENDS
            WRITE (CLOCK)S
            ET=CLOCKS
            TIME=ET-BTS
            WRITE ('COMPUTING TIME',TIME )S
E1          ENDS
END BLOCK 1   LEVEL 1
```

APPENDIX V, PROGRAM 11

```
B0                        REAL T,ET,BT,TIME,TE,V,I,J,E,N,R,EPS        $
BLOCK 1     LEVEL 1
                          REAL ARRAY X1,X2,T1,M,X1S,X2S,HM,HT1,P1,P2,T11,T22,H1,H2,DEL,
                          K(-1,,101)$
                          COMMENT  NON LINEAR COUPLING      $
                          TF=50$V=0.001$
                          I=7$
                          N=10$EPS=0.6$
                          WRITE (CLOCK)$
                          BT=CLOCK$
B1    F1                  FOR T=(0,1,TF) DO BEGIN  K(T)=M(T)=T1(T)=1$END$
                          FOR P=(1,1,N) DO BEGIN
B2                        WRITE (,STEP,)$
                          COMMENT COORDINATION          $
                          FOR T=(0,1,TF) DO BEGIN
B3                        K(T)=K(T)+EPS*(X1(T)+0.1*X1(T)*X1(T)-T1(T))          $
E3                        END$
                          FOR J=(1,1,I+1) DO BEGIN
B4                        E=-J/I$
                          COMMENT   SUB SYSTEM   1$
                          COMMENT FORWARDS INTEGRATION $
                          FOR T=(0,1,TF) DO BEGIN
B5                        X1(0)=5$
                          X1(T+1)=X1(T)+V*M(T)$
E5                        END$
                          COMMENT BACKWARDS INTEGRATION  $
                          FOR T=(TF,-1,0) DO BEGIN
B6                        P1(T-1)=P1(T)+V*(X1(T)+K(T)*(1+0.2*X1(T)))          $
                          HM(T)=M(T)+P1(T)$
F8                        IF J EQL (I+1)         THEN GOTO B3$
                          T11(T-1)=T11(T)-V*(T11(T)*T11(T)-1-0.2*K(T))          $
                          H1(T-1)=H1(T)-V*(T11(T)*H1(T)-T11(T)*E*HM(T))         $
                          M(T)=M(T)+E*HM(T)-(T11(T)*(X1(T)-X1S(T))+H1(T))       $
                          X1S(T)=X1(T)$
E6                        B3..END$
                          COMMENT SUB SYSTEM 2$
                          COMMENT FORWARDS INTEGRATION $
                          FOR T=(0,1,TF) DO BEGIN
B7                        X2(0)=2$
                          X2(T+1)=X2(T)+V*(T1(T)*T1(T))       $
E7                        END$
                          COMMENT BACKWARDS INTEGRATION  $
                          FOR T=(TF,-1,0) DO BEGIN
B8                        P2(TF)=0$
```

```
P2(T-1)=P2(T)+V*X2(T)      $
HT1(T)=2*(P2(T)-K(T))*T1(T)        $
IF J EQL (I+1) THEN GOTO B4$
T22(TF)=0$H2(TF)=0$
T22(T-1)=T22(T)-V*(2*T22(T)*T22(T)*T1(T)*T1(T)/(P2(T)-K(T))-1)    $
H2(T-1)=H2(T)-V*(2*T22(T)*T1(T)*T1(T)*H2(T)/(P2(T)-K(T))-
T22(T)*T1(T)*E*HT1(T)/(P2(T)-K(T))  )$
T1(T)=T1(T)+E*HT1(T)/(2*(P2(T)-K(T)))-T1(T)*(T22(T)*(X2(T)-X2S(T))+H2(T)
)/(P2(T)-K(T))              $
X2S(T)=X2(T)$
B4..END$
IF ( R EQL N )AND ( J EQL 7) THEN
FOR T=(0,1,TF) DO BEGIN
WRITE (,HM(T),HT1(T),M(T),X1(T),X2(T),',',
HM(T),HT1(T),M(T),X1(T),X2(T)   )$
END$
END$
WRITE (CLOCK)$
ET=CLOCK$
TIME=ET-BT$
WRITE (,COMPUTING TIME',TIME )$
FOR T=(0,1,TF) DO BEGIN
DEL(T+1)=DEL(T)+V*(0.5*(X1(T)*X1(T)+X2(T)*X2(T)+M(T)*M(T)))$
IF T EQL TF THEN WRITE ('DEL(T)',R,T,',DEL(T),R,T)$
END$
END$
END BLOCK 1    LEVEL 1
```

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